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MATHEMATICAL ASSOCIATION OF AMERICA

(INCORPORATED)

DEVOTED TO THE INTERESTS OF COLLEGIATE MATHEMATICS

THE AMERICAN MATHEMATICAL MONTHLY, FOUNDED IN 1894 BY BENJAMIN F. FINKEL, WAS PUBLISHED BY HIM UNTIL 1913. FROM 1913 TO 1916 IT WAS OWNED AND PUBLISHED BY REPRESENTATIVES OF FOURTEEN UNIVERSITIES AND COLLEGES IN THE MIDDLE WEST

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ON THE CHARACTERIZATION OF SOME REMARKABLE SYSTEMS OF POINTS OF INTERPOLATION BY MEANS OF CONJUGATE POINTS[1]

By LEOPOLD FEJÉR, University of Budapest

1. In the theory and practice of interpolation and mechanical quadratures problems have been considered repeatedly, which are of the following type: The

[1] A lecture delivered at several eastern and mid-western universities in the United States during April and May, 1933.

interpolation (or mechanical quadrature) abscissas are to be chosen so as to reach a certain goal, prescribed in advance. These investigations lead, as it is seen at once, far outside the familiar domain of "equidistant" interpolation; and it would be very tempting, to try to give a complete account of them (even including interpolations by complex points). Nevertheless this could hardly be done in the course of one short lecture, like the present one. So I will have to limit myself to a restricted number of problems of this kind. They will all be of recent origin, and particularly familiar to me, so that I hope to succeed in giving a short sketch of them.

2. Let us begin by considering the interpolation-formula of Lagrange

$$L(x) = y_1 l_1(x) + y_2 l_2(x) + \cdots + y_n l_n(x), \tag{1}$$

which can be found in most text-books on algebra or analysis. This formula represents explicitly the polynomial in x of degree less than or equal to $n-1$, which takes the respective values $y_1, y_2, \cdots, y_n$ at the n distinct points $x_1, x_2, \cdots, x_n$. The polynomials

$$l_1(x),\ l_2(x),\ \cdots,\ l_n(x), \tag{2}$$

which are all (exactly) of degree $n-1$, will be called *fundamental polynomials of the Lagrange interpolation* (1). They are uniquely determined by the abscissas of interpolation $x_1, x_2, \cdots, x_n$, and can be expressed in various forms with their aid. So if we define

$$\omega(x) = C(x - x_1)(x - x_2) \cdots (x - x_n), \tag{3}$$

where C is a constant, different from 0 (this notation will be used throughout this paper), one of the expressions for $l_k(x)$ is this:

$$l_k(x) = \frac{\omega(x)}{\omega'(x_k)(x - x_k)}, \qquad k = 1, 2, \cdots, n. \tag{4}$$

3. Much rarer is the use of Hermite's interpolation formula, which I quote for the simplest case only. This formula is important in the immediately following developments, and runs like this:

$$\begin{aligned} \mathrm{X}(x) &= y_1 h_1(x) + y_2 h_2(x) + \cdots + y_n h_n(x) \\ &\quad + y_1' \mathfrak{h}_1(x) + y_2' \mathfrak{h}_2(x) + \cdots + y_n' \mathfrak{h}_n(x) \\ &= \sum_{k=1}^{n} y_k h_k(x) + \sum_{k=1}^{n} y_k' \mathfrak{h}_k(x). \end{aligned} \tag{5}$$

This interpolation formula of Hermite represents explicitly the polynomial in x of degree less than or equal to $2n-1$, which takes the respective values $y_1, y_2, \cdots, y_n$ at the n distinct points $x_1, x_2, \cdots, x_n$, while its derivative takes the respective values $y_1', y_2', \cdots, y_n'$ at these points. Now the polynomials

$$h_1(x),\ h_2(x),\ \cdots,\ h_n(x), \tag{6}$$

which are all of degree $2n-1$, will be called *fundamental polynomials of the first kind*, and the polynomials

$$\mathfrak{h}_1(x), \mathfrak{h}_2(x), \cdots, \mathfrak{h}_n(x), \tag{7}$$

which are all (exactly) of degree $2n-1$, *fundamental polynomials of the second kind of the Hermite interpolation* (5). It is obvious, that the fundamental polynomials of the Hermite interpolation are also uniquely determined by the abscissas of interpolation $x_1, x_2, \cdots, x_n$. If we write again $\omega(x)=C(x-x_1)(x-x_2)\cdots(x-x_n)$ and $l_k(x)$ for the fundamental polynomial of Lagrange interpolation, connected with x_k, which has been just introduced, then one sees readily

$$h_k(x) = \left(1 - \frac{\omega''(x_k)}{\omega'(x_k)}(x - x_k)\right)(l_k(x))^2, \tag{8}$$

$$\mathfrak{h}_k(x) = (x - x_k)(l_k(x))^2, \qquad k = 1, 2, \cdots, n. \tag{9}$$

These formulas show, that $h_k(x)$, as well as $\mathfrak{h}_k(x)$, is divisible by $(l_k(x))^2$. Indeed they are obtained by multiplying $(l_k(x))^2$ respectively by the "characteristic linear function"

$$v_k(x) = 1 - \frac{\omega''(x_k)}{\omega'(x_k)}(x - x_k), \tag{10}$$

or by the linear function

$$w_k(x) = x - x_k. \tag{11}$$

4. Let us now assume that the n distinct quantities $x_1, x_2, \cdots, x_n$ are all real; we intend to consider the *sign* of the fundamental polynomials of Lagrange and Hermite.

The fundamental polynomial of Lagrange, $l_k(x)$, has exactly $n-1$ changes of sign if x runs over all real values. So the number of changes of sign of $l_k(x)$ increases to infinity if n does so.

On the other hand the behaviour of the fundamental polynomials of Hermite, $h_k(x)$ and $\mathfrak{h}_k(x)$, is essentially different in this respect. So the formula (9) shows that $\mathfrak{h}_k(x)$ changes its sign only once: at the point of interpolation x_k. And as formula (8) shows, $h_k(x)$ changes its sign at most once: at the root of the characteristic linear function $v_k(x)$, i.e. when

$$1 - \frac{\omega''(x_k)}{\omega'(x_k)}(x - x_k) = 0. \tag{12}$$

This gives for the place of change of sign, which we will denote by X_k, the value

$$X_k = x_k + \frac{\omega'(x_k)}{\omega''(x_k)}, \qquad k = 1, 2, \cdots, n. \tag{13}$$

In this way we have associated with the n distinct points

$$x_1, x_2, \cdots, x_n \tag{14}$$

the n points

$$X_1, X_2, \cdots, X_n, \tag{15}$$

the places of change of sign of the fundamental polynomials of the first kind of Hermite. The system $X_1, X_2, \cdots, X_n$ will be called *conjugate to the system* $x_1, x_2, \cdots, x_n$. In particular X_k is the conjugate to x_k.

5. Now let us consider an arbitrary interval, for instance the interval $-1 \leqq x \leqq 1$. If

$$x_1, x_2, \cdots, x_n \tag{16}$$

are n distinct points of this interval, two cases can arise.

Case 1. The interval $-1 < x < 1$ contains none of the conjugate points $X_1, X_2, \cdots, X_n$. If this is the case, the system $x_1, x_2, \cdots, x_n$ will be called *normally distributed over the interval* $-1 \leqq x \leqq 1$.

Case 2. The interval $-1 < x < 1$ is not free of conjugate points $X_1, X_2, \cdots, X_n$, but contains at least one of them. If this is the case, the system $x_1, x_2, \cdots, x_n$ will be called *anormally distributed over the interval* $-1 \leqq x \leqq 1$.

Thus we have made a sharp division of all systems $x_1, x_2, \cdots, x_n$ of n points of an interval into two distinct classes: *normal* and *anormal* ones. Let us emphasize that this classification has arisen by considering the Hermite and not the Lagrange, interpolation. Nevertheless it will be seen, that normal systems obey *more general* theorems, not only for the Hermite interpolation, but even for the classical Lagrange interpolation, and they allow *surprisingly simple proofs*. This fact of course gives the real importance to the classification we undertook for the systems of points of an interval.

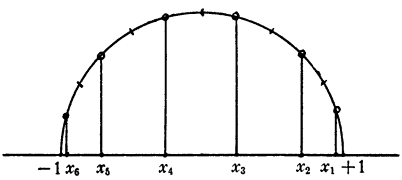

FIG. 1

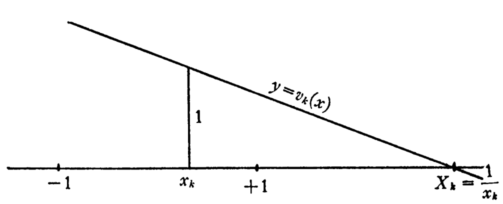

FIG. 2

6. Before we mention some general theorems on normal interpolations, let us first give four special systems of points as examples.

(*a*) Let us consider the interval from -1 to $+1$, and in it the so-called Tschebisheff point-system. This point-system $x_1, x_2, \cdots, x_n$ is obtained by taking a semicircle over the interval from -1 to $+1$ on the x-axis as diameter, subdividing it into n equal arcs, and projecting the points of bisection of these arcs on the x-axis. In this case it is easily seen that the point

$$X_k = \frac{1}{x_k} \tag{17}$$

is the *harmonic conjugate* of the interpolation point x_k, with respect to the points -1, $+1$. (See Figs. 1 and 2.) Since the harmonic conjugates to $x_1, x_2, \cdots, x_n$ all lie outside the interval $(-1, +1)$, it is clear, that the Tschebisheff abscissas $x_1, x_2, \cdots, x_n$ form a normal system.

(*b*) The Legendre-Gauss point-system is also normal; $(-1, +1)$ is again the fundamental interval, and $x_1, x_2, \cdots, x_n$ are the roots of the equation

$$P_n(x) = 0, \tag{18}$$

where $P_n(x)$ denotes the Legendre polynomial of index n. It is readily seen that in this case

$$X_k = \tfrac{1}{2}\left(x_k + \frac{1}{x_k}\right), \tag{19}$$

hence the conjugate point X_k is half-way between the point of interpolation x_k and its harmonic-conjugate. (See Fig. 3.) But the points of bisection of the

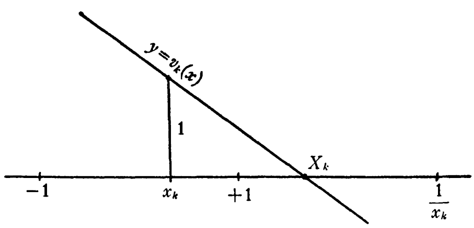

FIG. 3

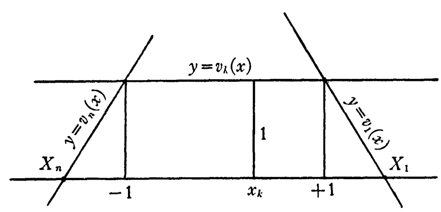

FIG. 4

interval between two conjugate harmonic points always lies outside the interval between the fundamental points. The set of Legendre-Gauss points thus is a normal system.

(*c*) Our choice for the third example of a point-system $x_1, x_2, \cdots, x_n$ will be the set of roots of the equation

$$(1 - x^2)P'_{n-1}(x) = 0, \tag{20}$$

where $P_{n-1}(x)$ denotes the Legendre polynomial of index $n-1$. This point-system has been used by the French astronomer R. Radau for the purpose of mechanical quadrature. This remarkable system is as normal as one can possibly expect. The point conjugate to $x_1 = 1$ is

$$X_1 = 1 + \frac{2}{n(n-1)}, \tag{21}$$

the point conjugate to $x_n = -1$ is

$$X_n = -1 - \frac{2}{n(n-1)}, \tag{22}$$

and all other conjugate points

$$X_2, X_3, \cdots, X_{n-1} \tag{23}$$

lie at infinity; i.e., the characteristic straight lines $y = v_k(x)$, $k = 2, 3, \cdots, n-1$, are all parallel to the x-axis. (See Fig. 4.)

(*d*). Our fourth example will be Newton's point-system $x_1, x_2, \cdots, x_n$, which is, as one may say, extremely anormal. Now the endpoints $x_1, x_2, \cdots, x_n$ of the interpolation-abscissas are

$$x_k = 1 - k\frac{2}{n+1}, \qquad k = 1, 2, \cdots, n, \tag{24}$$

i.e. equidistant. In this case all conjugate points $X_1, X_2, \cdots, X_n$ lie *in* the fundamental interval $(-1, +1)$. (One exception occurs: if n is odd, $n = 2r-1$, then exactly one conjugate point lies at infinity, namely the conjugate point to the interpolation-point $x_r = 0$.) One can even show that the totality of all conjugate systems, $X_1^{(n)}, X_2^{(n)}, \cdots, X_n^{(n)}$ $(n = 1, 2, 3, \cdots)$, covers all the fundamental interval $-1 \leqq x \leqq 1$ in the same everywhere dense way as the original totality of Newton's interpolation point-systems $x_1^{(n)}, x_2^{(n)}, \cdots, x_n^{(n)}$ $(n = 1, 2, 3, \cdots)$ itself.

7. In the previous part of my paper I have given the definition of point-systems $x_1, x_2, \cdots, x_n$ which are normally distributed over the interval $-1 \leqq x \leqq 1$, and have pointed out that certain well-known point-systems, which have often been considered, are normal. Now I will show briefly that this notion of normal point-systems can be applied with success. First I will consider the applications to the classical Lagrange interpolation theory.

I must return for an instant to the fundamental functions of the first kind $h_k(x)$ of Hermite interpolation, which have served to define normal point-systems. If we substitute $y_1 = y_2 = \cdots = y_n = 1$, $y_1' = y_2' = \cdots = y_n' = 0$ in Hermite's interpolation formula (5), we obtain the fundamental identity.

$$h_1(x) + h_2(x) + \cdots + h_n(x) = 1, \tag{25}$$

or more explicitly

$$v_1(x)(l_1(x))^2 + v_2(x)(l_2(x))^2 + \cdots + v_n(x)(l_n(x))^2 = 1. \tag{26}$$

Let us now consider, instead of the whole interval $-1 \leqq x \leqq 1$, the interior interval

$$-1 + \epsilon \leqq x \leqq 1 - \epsilon. \tag{27}$$

As the linear function $v_k(x)$ assumes the value 1 at the point $x = x_k$, and as it is everywhere positive in the interval $-1 < x < 1$, because $x_1, x_2, \cdots, x_n$ are normal, an elementary geometrical consideration shows that everywhere in the

interval (27)

$$v_k(x) \geqq \frac{\epsilon}{2}, \qquad k = 1, 2, \cdots, n, \tag{28}$$

holds. Combining this with the identity (26) we obtain

$$(l_1(x))^2 + (l_2(x))^2 + \cdots + (l_n(x))^2 \leqq \frac{2}{\epsilon}, \tag{29}$$

$$-1 + \epsilon \leqq x \leqq 1 - \epsilon.$$

So I proved at once the following theorem:

If x lies in an arbitrary, but fixed, sub-interval $-1+\epsilon \leqq x \leqq 1-\epsilon$ of the interval $(-1, +1)$ then for *any number* n of interpolation points the sum of the squares of the corresponding Lagrange fundamental polynomials

$$(l_1(x))^2 + (l_2(x))^2 + \cdots + (l_n(x))^2 \tag{30}$$

lies under a fixed limit which is independent of n, provided that the point-system $x_1, x_2, \cdots, x_n$ is *normal* with respect to the interval $(-1, +1)$.

From this the following theorem follows:

Let

$$\left\{\begin{array}{l} \overset{(1)}{x_1} \\ \overset{(2)}{x_1}, \overset{(2)}{x_2} \\ \overset{(3)}{x_1}, \overset{(3)}{x_2}, \overset{(3)}{x_3} \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ \overset{(n)}{x_1}, \overset{(n)}{x_2}, \overset{(n)}{x_3}, \cdots, \overset{(n)}{x_n} \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \end{array}\right. \tag{31}$$

be an infinite sequence of point-systems which are all *normal* with respect to the interval $a \leqq x \leqq b$, then the corresponding sequence of Lagrange interpolation polynomials, belonging to a function $f(x)$,

$$L_1(x), L_2(x), L_3(x), \cdots, L_n(x), \cdots \tag{32}$$

converge to $f(x)$ in the whole interior, $a < x < b$, of this interval, the convergence being uniform in each sub-interval $a+\epsilon \leqq x \leqq b-\epsilon$. Concerning $f(x)$, I assume that it obeys a Lipschitz condition with an exponent $>1/2$ in the whole interval $a \leqq x \leqq b$.

It is scarcely necessary to emphasize that $L_n(x)$ denotes the Lagrange interpolation polynomial belonging to the points

$$(\overset{(n)}{x_1}, y_1), (\overset{(n)}{x_2}, y_2), \cdots, (\overset{(n)}{x_n}, y_n),$$

where

$$y_1 = f(x_1^{(n)}),\ y_2 = f(x_2^{(n)}),\ \cdots,\ y_n = f(x_n^{(n)}).$$

Of course I cannot give here the proof of this theorem, although it can be obtained in a few lines if one uses a certain type of elementary argument, which is generally familiar now but which has crystallized only slowly from investigations by Lebesgue, Haar, S. Bernstein and Faber. In addition to this a theorem on the possibility of obtaining a certain degree of exactitude, when approximating $f(x)$ by polynomials of nth degree, plays a role in the proof. This problem has been highly elucidated by the work of De la Vallée Poussin, Lebesgue, Dunham Jackson and S. Bernstein.

8. The following remark may be added to the convergence theorem for arbitrary "normal Lagrange sequences," which was formulated just now.

It often occurs with important normal systems $x_1, x_2, \cdots, x_n$ of the interval $-1 \leqq x \leqq 1$ that not only

$$v_1(x) > 0,\ v_2(x) > 0,\ \cdots,\ v_n(x) > 0 \tag{33}$$

holds for $-1 < x < 1$, but even

$$v_1(x) > \rho,\ v_2(x) > \rho,\ \cdots,\ v_n(x) > \rho, \tag{34}$$

where ρ is a positive number. In this case inequality (26) implies, even for the *whole closed interval* $-1 \leqq x \leqq 1$,

$$(l_1(x))^2 + (l_2(x))^2 + \cdots + (l_n(x))^2 \leqq \frac{1}{\rho}\cdot \tag{35}$$

From this follows:

If the infinite sequence of point-systems $x_1^{(n)}, x_2^{(n)}, \cdots, x_n^{(n)}$, $n=1, 2, 3, \cdots$, in the interval $-1 \leqq x \leqq 1$, is *normal in the stronger sense*, i.e. if no $v_k(x)$ ever assumes a value $\leqq \rho$ in the interval $-1 \leqq x \leqq 1$, where ρ is a positive constant independent of x and n, then the Lagrange sequence

$$L_1(x),\ L_2(x),\ \cdots,\ L_n(x),\ \cdots \tag{36}$$

associated with $f(x)$ converges uniformly to $f(x)$ even in the whole closed interval $-1 \leqq x \leqq 1$, provided that $f(x)$ again obeys a Lipschitz condition with an exponent $> \frac{1}{2}$.

Let us remark that these convergence theorems are by no means trivial, even if $f(x)$ is analytically regular at each point of the closed interval $-1 \leqq x \leqq 1$.

9. We now pass to the application of these convergence theorems.

We have already stated that the point-systems of Tschebisheff, Legendre-Gauss and Radau are normal for the interval $-1 \leqq x \leqq 1$. More generally:

The roots $x_1, x_2, \cdots, x_n$ of the equation

$$\omega(x) = J_n(\alpha, \beta, x) = 0 \tag{37}$$

form a *normal* point-system for the interval $-1 \leqq x \leqq 1$ for any value of n, if the real parameters α, β fulfill the inequalities

$$0 \leqq \alpha \leqq \tfrac{1}{2}, \quad 0 \leqq \beta \leqq \tfrac{1}{2}. \tag{38}$$

Here $J_n(\alpha, \beta, x)$ denotes the Jacobi (hypergeometrical) polynomial of degree n corresponding to the parameter-values α, β. This polynomial of nth degree in x can be characterized, up to a constant factor which is irrelevant for the theory of interpolation, by the linear homogeneous differential equation of second order

$$\begin{aligned}(1 - x^2)\omega'' + [2(\alpha - \beta) - 2(\alpha + \beta)x]\omega' \\ + n[n + 2(\alpha + \beta) - 1]\omega = 0.\end{aligned} \tag{39}$$

But if we exclude the value $1/2$ for both α and β i.e. if we suppose

$$0 \leqq \alpha < \tfrac{1}{2}, \quad 0 \leqq \beta < \tfrac{1}{2}, \tag{40}$$

then

$$\begin{gathered}(l_1(x))^2 + (l_2(x))^2 + \cdots + (l_n(x))^2 \leqq \max\left(\frac{1}{1 - 2\alpha}, \quad \frac{1}{1 - 2\beta}\right), \\ -1 \leqq x \leqq 1, \quad n = 1, 2, 3, \cdots .\end{gathered} \tag{41}$$

So if the fixed parameters α, β satisfy the relations (40), the infinite sequence of the Jacobi point-systems is *normal even in the stronger sense.*

The decision whether *Jacobi* point-systems are normal or not, happens to be exceedingly simple. This is connected with the fact that here for a root $x = x_k$ of $\omega(x)$ the quotient $\omega'(x_k)/\omega''(x_k)$ can be immediately computed from the linear homogeneous differential equation (39) which holds for $\omega(x)$.

So the recently formulated general convergence theorem for normal and strongly normal Lagrange sequences, the proof of which was seen to be extremely simple, can really be applied here. I formulate only one theorem.

Let $x_1, x_2, \cdots, x_n$ be the roots of the equation

$$J_n(\alpha, \beta, x) = 0, \tag{42}$$

and let $L_n(x)$ be the nth Lagrange interpolation polynomial corresponding to the function $f(x)$ and to the points $x_1, x_2, \cdots, x_n$, then

$$\lim_{n\to\infty} L_n(x) = f(x) , \tag{43}$$

uniformly in the whole closed interval $-1 \leqq x \leqq 1$, if the parameters α, β of the Jacobi polynomial $J_n(\alpha, \beta, x)$ satisfy the conditions

$$0 \leqq \alpha < \tfrac{1}{2}, \quad 0 \leqq \beta < \tfrac{1}{2}; \tag{44}$$

$f(x)$ is supposed to satisfy a Lipschitz condition with an exponent $>1/2$.

For $\alpha = \beta = 0$ the Jacobi abscissas coincide with those of Radau, for $\alpha = \beta = 1/4$ with those of Tschebisheff. So for both sequences of point systems the above formulated convergence theorem is valid.

For $\alpha = \beta = 1/2$ the Jacobi point-system goes over into that of Legendre-

Gauss. Their infinite sequence is normal, but not strongly normal. Nevertheless even here general convergence theorems do hold for the corresponding Lagrange sequence.

10. So far the function $f(x)$ was assumed to be everywhere continuous in the closed interval of interpolation $-1 \leqq x \leqq 1$, and even the absolute increment $|f(x+h)-f(x)|$ was subjected to a certain Lipschitz condition.

We now will investigate the interesting case where the *continuity* of $f(x)$, and *only this*, is postulated.

After the important earlier work of Hermite, Heine, Méray, Weierstrass, Runge, Borel, Lebesgue, De la Vallée Poussin, Faber, D. Jackson, S. Bernstein and others on interpolation, the following extremely interesting question remained unsolved:

Is there any fixed infinite sequence of point-systems, all lying in the interval $-1 \leqq x \leqq 1$,

$$\left\{\begin{array}{l} x_1^{(1)} \\ x_1^{(2)}, \ x_2^{(2)}, \\ x_1^{(3)}, \ x_2^{(3)}, \ x_3^{(3)}, \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ x_1^{(n)}, \ x_2^{(n)}, \ x_3^{(n)}, \cdots, \ x_n^{(n)}, \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \end{array}\right. \tag{45}$$

such that the corresponding infinite sequence of Lagrange interpolation polynomials

$$L_1(x), L_2(x), L_3(x), \cdots, L_n(x), \cdots, \tag{46}$$

of *every* function $f(x)$, continuous in the whole interval $-1 \leqq x \leqq 1$, converges *uniformly* to this function $f(x)$, in the whole closed interval $-1 \leqq x \leqq 1$?

Faber proved, that this question must be answered in the *negative.* I have succeeded in somewhat simplifying his proof.

Now let us consider another situation. If $x_1, x_2, \cdots, x_n$ denote the nth group of points (the upper index n may again be omitted), and $y_1=f(x_1)$, $y_2=f(x_2)$, $\cdots$, $y_n=f(x_n)$ denote the corresponding values of the function $f(x)$, then the points with the coordinates

$$(x_1, y_1), (x_2, y_2), \cdots, (x_n, y_n) \tag{47}$$

lie on infinitely many *parabolas.* Lagrange's parabola $y=L_n(x)$ is that one among them, which has the *lowest degree.* (This lowest degree is $\leqq n-1$. See Fig. 5.) So far we have restricted ourselves to the consideration of the parabola of lowest degree, i.e. the Lagrange parabola which contains the points with the coordinates (x_1, y_1), (x_2, y_2), $\cdots$, (x_n, y_n). But now as we keep in view the most general continuous functions $f(x)$, we must, in the light of Faber's result, drop the restriction on the *degree* of the interpolation polynomial, if we desire again

to obtain an arbitrarily exact and uniform approximation of the general continuous function $f(x)$ by a parabolic interpolation.

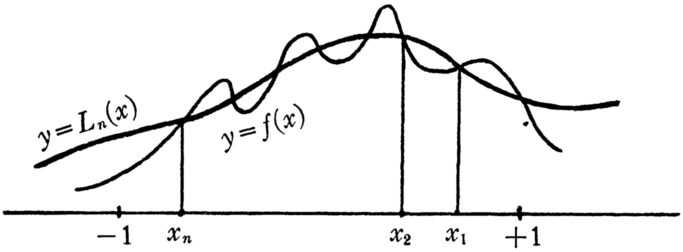

FIG. 5

If we go to the other extreme, leaving the degree of the nth interpolation polynomial *completely free*, we are lead to a triviality.

But if we require that the nth parabola, passing through the points (x_1, y_1), $(x_2, y_2), \cdots, (x_n, y_n)$, shall be *at most of degree* $2n-1$, i.e. belonging to the n-fold infinite *manifold* of parabolas of at most $2n-1$ degree passing through the n fixed points $(x_1, y_1), (x_2, y_2), \cdots, (x_n, y_n)$, we obtain interesting results.

11. First: what is Faber's problem, if we make ourselves somewhat freer with regard to the degree of the nth interpolation polynomial in the sense which has been indicated above? It is this: is there any *fixed* sequence of point-systems in the interval $-1 \leqq x \leqq 1$, $x_1^{(n)}, x_2^{(n)}, \cdots, x_n^{(n)}$ $(n=1, 2, 3, \cdots)$ such that for any n in the n-dimensional manifold of interpolation polynomials of degree $2n-1$ at most, determined by the points $x_1^{(n)}, x_2^{(n)}, \cdots, x_n^{(n)}$, and by the continuous function $f(x)$, at least one $X_n(x)$ can be found, so that the infinite sequence of interpolation polynomials

$$X_1(x), X_2(x), \cdots, X_n(x), \cdots, \tag{48}$$

converges uniformly to $f(x)$, in the whole interval $-1 \leqq x \leqq 1$?

This question can be answered in the *affirmative*. I will show that *any* sequence of *normal* point-systems possesses this property.

Let $x_1, x_2, \cdots, x_n$ be a normal point system of the interval $-1 \leqq x \leqq 1$. Moreover, let $T_{2n-1}(x)$ be the so-called Tschebisheff polynomial of degree $2n-1$ of $f(x)$ for the interval $-1 \leqq x \leqq 1$. This means that

$$\max_{-1 \leqq x \leqq 1} \left| f(x) - T_{2n-1}(x) \right| = E_{2n-1} \tag{49}$$

is less than

$$\max_{-1 \leqq x \leqq 1} \left| f(x) - g_{2n-1}(x) \right|$$

if $g_{2n-1}(x)$ denotes any other polynomial of degree $2n-1$. So $T_{2n-1}(x)$ gives, in a certain sense, the "best" approximation for $f(x)$ which can be obtained by a polynomial of degree $2n-1$.

Let us now consider that polynomial $\mathrm{X}(x)$ of at most $(2n-1)$th degree, which assumes the respective values

$$f(x_1),\ f(x_2),\ \cdots,\ f(x_n) \tag{50}$$

at the places $x_1, x_2, \cdots, x_n$, and the derivative of which assumes the respective values

$$T'_{2n-1}(x_1),\ T'_{2n-1}(x_2),\ \cdots,\ T'_{2n-1}(x_n) \tag{51}$$

at these places.

According to Hermite's interpolation-formula (5) we have then

$$\mathrm{X}(x) = \sum_{k=1}^{n} f(x_k)h_k(x) + \sum_{k=1}^{n} T'_{2n-1}(x_k)\mathfrak{h}_k(x). \tag{52}$$

But, as the same formula implies

$$T_{2n-1}(x) = \sum_{k=1}^{n} T_{2n-1}(x_k)h_k(x) + \sum_{k=1}^{n} T'_{2n-1}(x_k)\mathfrak{h}_k(x), \tag{53}$$

we have, from (52) and (53),

$$\left|T_{2n-1}(x) - \mathrm{X}(x)\right| = \left|\sum_{k=1}^{n} (T_{2n-1}(x_k) - f(x_k))h_k(x)\right|, \tag{54}$$

and therefore, in view of (49),

$$\left|T_{2n-1}(x) - \mathrm{X}(x)\right| \leqq E_{2n-1} \sum_{k=1}^{n} \left|h_k(x)\right|. \tag{55}$$

But further, the system $x_1, x_2, \cdots, x_n$ being normal, $h_k(x) \geqq 0$ for $-1 \leqq x \leqq 1$, $k=1, 2, \cdots, n$, we have

$$\sum_{k=1}^{n} \left|h_k(x)\right| = \sum_{k=1}^{n} h_k(x) = 1. \tag{56}$$

So the inequality (55) implies

$$\left|T_{2n-1}(x) - \mathrm{X}(x)\right| \leqq E_{2n-1}, \tag{57}$$

and finally, by (49) and (57),

$$\left|f(x) - \mathrm{X}(x)\right| \leqq 2E_{2n-1}, \qquad -1 \leqq x \leqq 1. \tag{58}$$

Thus I have obtained the following result for normal point-systems, which is even of some independent interest:

If $x_1, x_2, \cdots, x_n$ is a normal point-system for the interval $-1 \leqq x \leqq 1$, and if $f(x)$ is an everywhere continuous function in this interval, then in the corresponding n-dimensional manifold of polynomials of degree $2n-1$, passing through the n points $(x_1, f(x_1)), (x_2, f(x_2)), \cdots, (x_n, f(x_n))$, a parabola $y = \mathrm{X}(x)$ can be

found, whose approximation to $f(x)$ is at most twice as bad as the best approximation to $f(x)$ which can be obtained by *any* parabola of degree $2n-1$ at all.

It is obvious, that if we carry through the construction (52) for every n, we obtain an infinite sequence

$$X_1(x),\ X_2(x),\ \cdots,\ X_n(x),\ \cdots, \tag{60}$$

which possesses the required property.

12. The above given method of choice of $X_n(x)$ from the nth manifold is valid for every sequence of normal point-system. Of course it can be varied in many ways. At any rate this choice is unsatisfactory, as the *rule* which determines the "slopes"

$$X_n'(x_1),\ X_n'(x_2),\ \cdots,\ X_n'(x_n) \tag{61}$$

depends on the arbitrarily given function $f(x)$.

Now we propose to give a different rule of choice from the manifold, which is independent of the function $f(x)$: let simply

$$X_n'(x_1) = X_n'(x_2) = \cdots = X_n'(x_n) = 0, \tag{62}$$

at first for any $x_1, x_2, \cdots, x_n$.

This means, that $X(x)$ is that polynomial of at most $(2n-1)$th degree in x, which assumes the respective values $f(x_1), f(x_2), \cdots, f(x_n)$ at the places $x_1, x_2, \cdots, x_n$, and has a derivative which vanishes at all these places; in other words, $X(x)$ is a polynomial of at most $(2n-1)$th degree in x, which assumes the values $f(x_1), f(x_2), \cdots, f(x_n)$ at the places $x_1, x_2, \cdots, x_n$, each *doubly*. (See Fig. 6.) This parabola has been called by me the *step-parabola* of $f(x)$, belonging

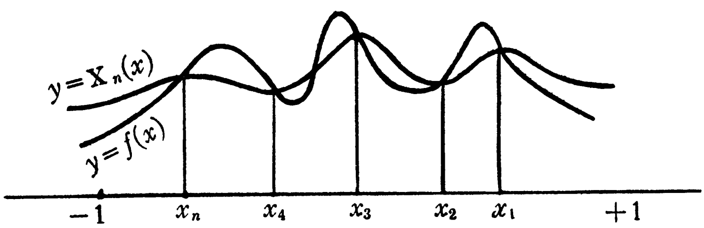

FIG. 6

to the points $x_1, x_2, \cdots, x_n$, for which Hermite's interpolation formula gives at once the representation

$$X(x) = \sum_{k=1}^{n} f(x_k)h_k(x). \tag{63}$$

Will now the sequence of the step-parabola of $f(x)$

$$X_1(x),\ X_2(x),\ \cdots,\ X_n(x),\ \cdots, \tag{64}$$

converge uniformly in the whole interval $-1 \leqq x \leqq 1$ to the continuous function $f(x)$, at least if the sequence of our point-systems is *strongly normal*?

I am not able to give an answer to this question in its complete generality. But for the special case of Jacobi's abscissas the question of the convergence of the step-parabolas is completely cleared up.

I proved the convergence of the step-parabolas to the continuous function $f(x)$ first for the sequence of the Tschebisheff abscissas, which form a strongly normal sequence, as we saw previously. But I have to state that previously Dunham Jackson found similar trigonometrical (and parabolical) interpolation polynomials. He was led to this trigonometrical interpolation by building up the interpolation formula which corresponds to the first arithmetic mean of index n in the case of Fourier series.

Moreover, I proved for the step-parabolas of the Legendre-Gauss interpolation that

$$\lim_{n\to\infty} X_n(x) = f(x) \tag{65}$$

holds for any point in $-1<x<1$, and even uniformly in each interval $-1+\epsilon \leqq x \leqq 1-\epsilon$. The limits

$$\lim_{n\to\infty} X_n(1) \text{ and } \lim_{n\to\infty} X_n(-1) \tag{66}$$

also exist but they are (generally) different from $f(1), f(-1)$, as there is always

$$\lim_{n\to\infty} X_n(1) = \lim_{n\to\infty} X_n(-1) = \tfrac{1}{2}\int_{-1}^{1} f(x)dx, \tag{67}$$

i.e. the mean-value of the function $f(x)$ for the *whole* interval $-1\leqq x\leqq 1$.

This theorem, which can be compared to Stieltjes's theorem in the theory of mechanical quadrature of Gauss, also can be proved in an exceedingly elementary way. But if one uses deeper analytical tools, corresponding for instance to Laplace's asymptotic formula for the Legendre polynomials $P_n(x)$, then the convergence question for the step-parabolas of any continuous function $f(x)$ can be solved quite generally for the Jacobi point-systems, and even for those of Laguerre and Hermite. This has been achieved by Szegö and Shohat in some important recent papers. Furthermore I would like to attract your attention to the interesting results concerning step-parabolas which have been obtained by N. Kryloff, E. Stayermann and J. Tamarkin.

THERMODYNAMICS—AN EXPOSITION

By J. E. TREVOR, Cornell University

II. *An Application of the Laws of Thermodynamics*

1. *A Formulation of the Dissipation Law.* A familiar field of application of the principles of thermodynamics is the region of the realizable thermodynamic states of any body B that is supposed subject to no force other than a uniform

and normally directed pressure. To formulate the dissipation law for this region, let B be given in any state of equilibrium that would not be stable under the arbitrary imposed pressure and temperature p and θ. B may be in a state that would be stable under another pressure and another temperature, or it may be in a state of separated parts which would be severally in equilibrium under different pressures and temperatures. Supposing B enclosed in a rigid and thermally non-conducting shell, we first allow any possible dissipative changes of its state to occur, whereby its entropy increases by a positive amount Σ_1. B now has uniform pressure and temperature p_1 and θ_1, which in general differ from p, θ. Let B now be suspended in a body of gas G of such extent[1] that relatively small transfers of work and heat to or from it do not sensibly alter its uniform pressure p and temperature θ, and let the body BG composed of B and G be enclosed in a rigid and thermally non-conducting shell.

If $\theta_1 \neq \theta$, equalization of the temperatures of B and G can be effected by the agency of a reversible cyclical change of state of an auxiliary gas, whereby a quantity of work w_2 is developed at the expense of the energy of BG. If we now establish thermal communication between B and G, reversible addition of heat w_2 to BG returns the energy of this body to its initial value, and increases the entropy of BG by the amount $w_2/\theta = \Sigma_2$. The enclosed body B is now in thermal equilibrium with G, but the pressures on the two sides of the enclosing envelope are not necessarily equal. On reversibly equalizing these pressures, heat flows from one of the bodies to the other without altering the entropy of both, and a quantity of work w_3 is developed at the expense of the energy of both. Reversible addition of heat w_3 to BG then returns this energy to its initial value, and increases the entropy of BG by the amount $w_3/\theta = \Sigma_3$. The state of the body B is now a state of stable equilibrium under the imposed pressure and temperature p, θ. In the process through which this state has been attained, the entropy of BG has increased by

$$\Sigma_1 + \Sigma_2 + \Sigma_3 = \Sigma.$$

Let the initial and final volume, entropy, and energy of B be V_1, S_1, E_1 and V_0, S_0, E_0, and let the initial and final values of the volume, entropy, and energy of G be v_1, s_1, e_1 and v_0, s_0, e_0. Since the change of state of G was reversible, we have

$$de = -\,pdv + \theta ds,$$

whence, since p and θ are constants,

$$e_0 - e_1 = -\,p(v_0 - v_1) + \theta(s_0 - s_1).$$

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[1] This is permissible, since we can imagine the employment of a body of gas of such extent that the changes of its pressure and temperature consequent on a transfer to or from it (at constant volume) of a given quantity of heat, and those consequent on a given reversible adiabatic change of its volume, are less than any assignable amount. Compare the similar comment at the close of section 6 of Part I.

In the entire process the energy and volume of BG remained unaltered, while the entropy of BG increased by the positive amount Σ,

$$e_0 + E_0 = e_1 + E_1$$
$$v_0 + V_0 = v_1 + V_1$$
$$s_0 + S_0 = s_1 + S_1 + \Sigma.$$

Eliminating e_0, v_0, s_0, e_1, v_1, s_1, between these four equations, we find

$$E_1 - E_0 + p(V_1 - V_0) - \theta(S_1 - S_0) = \theta\Sigma. \tag{1}$$

If the stable state (V_0, S_0, E_0) of B is a state of a continuous region of states all of which are stable under the pressure p at the temperature θ, any other state of the region can be attained from the state (V_0, S_0, E_0) by reversible expansion or contraction and heating or cooling of B. In this process work developed by either B or G is absorbed by the other, and heat developed by either is absorbed by the other, wherefore the energy and entropy of BG remain constant. In this process $\Sigma = 0$. Now, when the operator Δ relates to the variation of the state of B *from* any state of stable equilibrium under the pressure p at the temperature θ *to* any state whose equilibrium is not stable under these conditions, we have by (1)

$$\Delta E + p\Delta V - \theta\Delta S > 0;$$

and when Δ relates to a variation of state within a region of states whose equilibrium is stable at p, θ we have

$$\Delta E + p\Delta V - \theta\Delta S = 0.$$

It thus appears that a body supporting the uniform and normally directed pressure p at the temperature θ is in a state of stable equilibrium if all possible sets of variations ΔE, ΔV, ΔS of the energy, volume, and entropy of the body satisfy the *criterion of stability*,

$$\Delta E + p\Delta V - \theta\Delta S \geqq 0. \tag{2}$$

A useful special form of the criterion (2) is obtained by applying (2) to the transformation of each of two states of stable equilibrium into the other. If a state (V, S, E) is stable under a pressure p and temperature θ, and another state $(V+\Delta V, S+\Delta S, E+\Delta E)$ is stable under a pressure $p+\Delta p$ and temperature $\theta+\Delta\theta$, while both states do not lie in a continuous region of states that are stable under the pressure and temperature p, θ, the stability of the first state with regard to a displacement to the second requires

$$\Delta E + p\Delta V - \theta\Delta S > 0,$$

and the stability of the second state with regard to a displacement to the first requires

$$-\Delta E - (p + \Delta p)\Delta V + (\theta + \Delta\theta)\Delta S > 0.$$

Hence, by addition,

$$-\Delta p\Delta V + \Delta\theta\Delta S > 0. \tag{2a}$$

This is the formulation in question.

2. *Conditions of Equilibrium.* From this point the discussion shall be restricted to the region R of the realizable states of thermodynamic equilibrium of a *one-component* body (a body of a single chemical element or compound) that is supposed subject to no force other than a uniform and normally directed pressure. Since changes of the state in the region R are effected by reversible additions of work and heat, and thus by changes of the volume V and the entropy S of the body, the energy E of the body in R is a continuous function $E(V, S)$ of these independent variables. It must be noted that R is but a small part of the region of the realizable thermodynamic states of the body. The work-element and the heat-element for the body being—pdV and θdS, the energy equation becomes the *criterion of equilibrium*,

$$dE = -pdV + \theta dS;$$

wherefore the *conditions of equilibrium* are

$$\frac{\partial E}{\partial V} = -p, \qquad \frac{\partial E}{\partial S} = \theta. \tag{3}$$

In a given state of equilibrium a body may be an assemblage of n coexistent *phases*, which are distinct masses of solid, liquid, or vapor. For different values of n the energy $E(V, S)$ has different forms. On physical grounds we conclude that the energy E_i of the ith phase of a body is a continuous function $E_i(V_i, S_i, M_i)$ of the independent volume, entropy, and mass of the phase, and that

$$E_i(tV_i, tS_i, tM_i) = tE_i(V_i, S_i, M_i),$$

t = any positive integer. From this it can be shown to follow that E_i is a positively homogeneous function of degree one.

Let the n phases that constitute a given one-component body in a given state of a given region of n-phase states of stable or relatively stable equilibrium be severally in arbitrary states of equilibrium (obtained by redistribution of the mass between the phases, and by independent variation of the volume and entropy of each phase), whether they can then *coexist* in equilibrium or not. When p_i and θ_i are the pressure and temperature under which the equilibrium of the ith phase ($i = 1, 2, \cdots, n$) subsists, this phase has a definite volume ω_i, entropy σ_i, and mass μ_i; its energy is a definite function $\epsilon_i = E_i(\omega_i, \sigma_i, \mu_i)$ of these variables; and the $2n$ conditions of equilibrium,

$$\frac{\partial \epsilon_i}{\partial \omega_i} = -p_i, \qquad \frac{\partial \epsilon_i}{\partial \sigma_i} = \theta_i,$$

are satisfied. The $2n$ variables ω_i, σ_i are independent, as are all the phase-

masses μ_i except say μ_1, which is defined by the equation

$$M = \mu_1 + \mu_2 + \cdots + \mu_n,$$

where M is the constant mass of the given body. Whether the several phases can *coexist* in equilibrium or not, the volume ω, the entropy σ, and the energy ϵ, of the set of phases are defined as functions of the independent variables by the equations,

$$\omega = \omega_1 + \omega_2 + \cdots + \omega_n,$$
$$\sigma = \sigma_1 + \sigma_2 + \cdots + \sigma_n,$$
$$\epsilon = \epsilon_1 + \epsilon_2 + \cdots + \epsilon_n,$$

where the ϵ_i are the phase-energies $E_i(\omega_i, \sigma_i, \mu_i)$.

When the n phases coexist in the given n-phase state of stable or relatively stable equilibrium, the independent variables ω_i, σ_i, μ_j, and the functions ω, σ, ϵ, μ_1 assume values,

$$\omega_i = V_i, \quad \sigma_i = S_i, \quad \mu_j = M_j, \qquad (i = 1, 2, \cdots, n; j = 2, 3, \cdots, n)$$
$$\omega = V, \quad \sigma = S, \quad \epsilon = E, \quad \mu_1 = M_1.$$

Our task is to find the relations that connect the variables V_i, S_i, M_j, that is, to find the "conditions of phase-equilibrium" of the given n-phase state of the given body.

The change of state of the body *from* the given state of stable or relatively stable equilibrium under the imposed pressure p and temperature θ *to* the general state in which the phases are separated is subject to the criterion of stability,

$$(4) \qquad (\epsilon - E) + p(\omega - V) - \theta(\sigma - S) \geqq 0.$$

Let the ω_i, σ_i, ω_j assume all possible sets of values, including

$$(5) \qquad \omega_i = V_i, \quad \sigma_i = S_i, \quad \mu_j = M_j.$$

In this last case the phases will coexist in equilibrium if they be conjoined; the initial and final states in (4) will be identical; and the expression

$$(6) \qquad (\epsilon - E) + p(\omega - V) - \theta(\sigma - S)$$

will vanish. But zero is the least value that (6) can have. So this expression is a minimum when the conditions (5) are satisfied. Hence the first derivatives of (6) with regard to each of the independent variables ω_i, σ_i, μ_j are equal to zero for the values (5) of these variables. We thus find the desired *conditions of phase-equilibrium*,

$$(7) \qquad \begin{gathered} -p = \frac{\partial E_1}{\partial V_1} = \frac{\partial E_2}{\partial V_2} = \cdots = \frac{\partial E_n}{\partial V_n}, \quad \theta = \frac{\partial E_1}{\partial S_1} = \frac{\partial E_2}{\partial S_2} = \cdots = \frac{\partial E_n}{\partial S_n} \\ \frac{\partial E_1}{\partial M_1} = \frac{\partial E_2}{\partial M_2} = \cdots = \frac{\partial E_n}{\partial M_n}. \end{gathered}$$

It may be noted that the general conditions of equilibrium (3) can be deduced from (7). To show this, let the quantities V_1, S_1, M_1 be the functions

$$V_1 = V - \sum_2^n V_i, \qquad S_1 = S - \sum_2^n S_i, \qquad M_1 = M - \sum_2^n M_i. \tag{8}$$

Then the function $E = E_1 + E_2 + \cdots + E_n$ becomes a function of the variables that appear in the second members of (8). On differentiating with regard to the subscripted variables, and comparing with (7), it is found that E is a function of V, S alone; and on differentiating with regard to the independent V and S we find, by (7), that

$$\frac{\partial E}{\partial V} = -p, \qquad \frac{\partial E}{\partial S} = \theta,$$

which are the conditions (3).

Since the phase-energy E_i is positively homogeneous of degree one, we have

$$E_i(tV_i, tS_i, tM_i) = tE_i(V_i, S_i, M_i),$$

where t is any positive factor. When $t = 1/M_i$, this becomes

$$E_i = M_i e_i(v_i, s_i), \tag{9}$$

where the specific energy $e_i(v_i, s_i)$ is a function of the specific volume $v_i = V_i/M_i$ and the specific entropy $s_i = S_i/M_i$ of the phase. To express the conditions of phase-equilibrium (7) in terms of p, θ, v_i, s_i, we differentiate (9) with regard to each of V_i, S_i, M_i in turn, thus obtaining the conditions:

$$\begin{gathered} -p = \frac{\partial e_1}{\partial v_1} = \cdots = \frac{\partial e_n}{\partial v_n}, \qquad \theta = \frac{\partial e_1}{\partial s_1} = \cdots = \frac{\partial e_n}{\partial s_n}, \\ e_1 - \frac{\partial e_1}{\partial v_1} v_1 - \frac{\partial e_1}{\partial s_1} s_1 = \cdots = e_n - \frac{\partial e_n}{\partial v_n} v_n - \frac{\partial e_n}{\partial s_n} s_n. \end{gathered} \tag{10}$$

The conditions of equilibrium (10) are $3n-1$ equations between the $2n+2$ variables,

$$p, \theta, v_i, s_i, \qquad i = 1, 2, \cdots, n.$$

Since it is improbable that the number of equations can exceed the number of variables,—that n can exceed 3,— it is not to be expected that more than three phases can coexist in equilibrium. In fact n has never been observed to exceed 3.

3. *Conditions of Stability.* For a region of *one-phase states* of equilibrium the conditions of equilibrium are two relations between the four variables V, S, p, θ.

$$p = -\frac{\partial E}{\partial V}, \qquad \theta = \frac{\partial E}{\partial S}.$$

Taking it to be physically obvious that p and θ are independent functions, we assume outright,

Postulate I. At all points (V, S) of any continuous region of realizable one-phase states of equilibrium of any one-component body we have

$$\frac{\partial(p, \theta)}{\partial(V, S)} \neq 0.$$

It thus appears that, in the neighborhood of any point, the conditions of equilibrium determine that V and S are single-valued functions of p, θ. Hence a displacement of the state (V, S) in a field of one-phase states of equilibrium cannot occur under the conditions of constant imposed pressure and temperature. In this case, therefore, for displacements within the region, the criterion of stability (2) is

$$\Delta E + p\Delta V - \theta\Delta S > 0.$$

Since, further, by Postulate I, the hessian $\mathcal{D}$ of the function $E(V, S)$ is non-vanishing, it follows that the conditions

$$\mathcal{D} > 0, \qquad \frac{\partial^2 E}{\partial V^2} > 0, \qquad \frac{\partial^2 E}{\partial S^2} > 0,$$

are satisfied at every point (V, S) of the field of one-phase states of stable equilibrium.

For a region of *two-phase states* of equilibrium we have

$$M_1v_1 + M_2v_2 = V, \quad M_1s_1 + M_2s_2 = S, \quad M_1 + M_2 = M, \tag{11}$$

and we assume that, at any point (V, S), the phase-masses M_1, M_2 are uniquely determined by the third of these equations and either of the other two. This means that we assume the determinants

$$\begin{vmatrix} v_1 & v_2 \\ 1 & 1 \end{vmatrix}, \qquad \begin{vmatrix} s_1 & s_2 \\ 1 & 1 \end{vmatrix},$$

to be non-vanishing. Formally,

Postulate II. At all points (V, S) of any continuous region of realizable two-phase states of equilibrium of any one-component body we have

$$v_2 - v_1 \neq 0, \qquad s_2 - s_1 \neq 0.$$

For a region of two-phase states the conditions of equilibrium (10) are

$$\begin{aligned} f_1 &= p + P_1 = 0, & f_3 &= \theta - Q_1 = 0, \\ f_2 &= p + P_2 = 0, & f_4 &= \theta - Q_2 = 0, \\ f_5 &= (e_1 - P_1v_1 - Q_1s_1) - (e_2 - P_2v_2 - Q_2s_2) = 0, \end{aligned} \tag{12}$$

where

$$P_i = \frac{\partial e_i}{\partial v_i}, \qquad Q_i = \frac{\partial e_i}{\partial s_i}, \qquad i = 1, 2.$$

These are five equations in the six variables p, θ, v_1, s_1, v_2, s_2. To determine whether all the variables but p (or θ) are single-valued functions of p (or of θ) in the neighborhood of a point, we form the jacobians

$$\frac{\partial(f_1, f_2, f_3, f_4, f_5)}{\partial(\theta, v_1, s_1, v_2, s_2)} = -D_1D_2(s_2 - s_1), \quad \frac{\partial(f_1, f_2, f_3, f_4, f_5)}{\partial(p, v_1, s_1, v_2, s_2)} = +D_1D_2(v_2 - v_1).$$

Here D_i is the hessian of the specific energy $e_i(v_i, s_i)$ of the ith phase, and the second members result from reduction of the given jacobians. By Postulates I and II it appears that these jacobians are non-vanishing, and hence that all the six variables but p, or all but θ, are single-valued functions of p, or of θ, respectively, "in the small." In particular an isothermal displacement of state in a field of two-phase states of equilibrium occurs at constant pressure. In this case, therefore, for displacements within the region, the criterion of stability (2) is

$$(13) \qquad \Delta E + p\Delta V - \theta\Delta S \geqq 0,$$

where the sign of equality holds for, and only for, isothermal expansion or compression.

To obtain the corresponding conditions of stability we must determine whether $\partial^2E/\partial V^2$ or $\partial^2E/\partial S^2$ can vanish. On eliminating M_1, M_2 between the equations (11), we find

$$\begin{vmatrix} V & v_1 & v_2 \\ S & s_1 & s_2 \\ M & 1 & 1 \end{vmatrix} = 0,$$

which is a linear relation between V and S. Hence the graph of an isotherm in the V, S plane is a straight line.

The second part of Postulate II is

$$(Ms_2 - Ms_1)_p \neq 0.$$

It asserts that the ends of the isotherm in the V, S plane cannot lie in a horizontal line, and thus that isentropic expansion cannot occur at constant pressure,

$$\left(\frac{\partial p}{\partial V}\right)_S = -\frac{\partial^2 E}{\partial V^2} \neq 0.$$

Similarly, the first part of Postulate II is

$$(Mv_1 - Mv_2)_\theta \neq 0.$$

It asserts that the ends of the isotherm in the V, S plane cannot lie in a vertical line, and thus that heating at constant volume cannot occur at constant temperature,

$$\left(\frac{\partial \theta}{\partial S}\right)_V = \frac{\partial^2 E}{\partial S^2} \neq 0.$$

Since p and θ are connected by a relation, we have that the hessian $\mathcal{D}$ of the energy $E(V, S)$ vanishes identically,

$$\mathcal{D} = -\frac{\partial(p, \theta)}{\partial(V, S)} = 0.$$

Since, further, neither $\partial^2 E/\partial V^2$ nor $\partial^2 E/\partial S^2$ can vanish, it follows from the criterion (13) that the conditions of stability are

$$\mathcal{D} = 0, \qquad \frac{\partial^2 E}{\partial V^2} > 0, \qquad \frac{\partial^2 E}{\partial S^2} > 0.$$

For a region of *three-phase states* of equilibrium the conditions of equilibrium (10) are:

$$\begin{aligned} f_1 &= p + P_1 = 0, & f_4 &= \theta - Q_1 = 0, \\ f_2 &= p + P_2 = 0, & f_5 &= \theta - Q_2 = 0, \\ f_3 &= p + P_3 = 0, & f_6 &= \theta - Q_3 = 0, \end{aligned}$$
$$f_7 = (e_1 - P_1 v_1 - Q_1 s_1) - (e_2 - P_2 v_2 - Q_2 s_2) = 0,$$
$$f_8 = (e_1 - P_1 v_1 - Q_1 s_1) - (e_3 - P_3 v_3 - Q_3 s_3) = 0.$$

These are eight equations in the eight variables p, θ, v_i, s_i. On reducing the jacobian of the eight f_i, we find its value to be

$$D_1 D_2 D_3 \begin{vmatrix} v_1 & v_2 & v_3 \\ s_1 & s_2 & s_3 \\ 1 & 1 & 1 \end{vmatrix}.$$

Assuming, as a third postulate, that the phase-masses M_1, M_2, M_3 are uniquely determined by the equations

$$\begin{aligned} M_1 v_1 + M_2 v_2 + M_3 v_3 &= V \\ M_1 s_1 + M_2 s_2 + M_3 s_3 &= S \\ M_1 \quad + M_2 \quad + M_3 \quad &= M, \end{aligned}$$

it follows that the above three-rowed determinant is different from zero. By Postulate I the three D_i differ from zero. Therefore the jacobian differs from zero, wherefore the eight variables p, θ, v_i, s_i are connected by eight independent relations, and hence are constants. Accordingly a displacement of state in a field of three-phase states of equilibrium can occur only at constant pressure and temperature. In this case, therefore, the criterion of stability (2) is

$$\Delta E + p\Delta V - \theta\Delta S \doteq 0.$$

If (V_0, S_0, E_0) is an arbitrary state of reference in the region, this may be written

$$(E - E_0) + p(V - V_0) - \theta(S - S_0) = 0,$$

or

$$E = -pV + \theta S + (E_0 + pV_0 - \theta S_0). \tag{14}$$

Thus the energy $E(V, S)$ for the region is linear in V, S, with constant coefficients.

4. *Implicit Functions Defined by the Conditions of Equilibrium.* For a region of *one-phase states* the conditions of equilibrium (10) are

$$f_1 = p + \frac{\partial e}{\partial v} = 0, \qquad f_2 = \theta - \frac{\partial e}{\partial s} = 0.$$

These are two equations between the four variables v, s, p, θ. To determine whether any two of these variables are single-valued functions of the other two we form the jacobians:

$$\left.\begin{aligned} \frac{\partial(f_1, f_2)}{\partial(v, s)} &= -D \\ \frac{\partial(f_1, f_2)}{\partial(v, \theta)} &= +\frac{\partial^2 e}{\partial v^2} \\ \frac{\partial(f_1, f_2)}{\partial(p, s)} &= -\frac{\partial^2 e}{\partial s^2} \end{aligned}\right| \quad \begin{aligned} \frac{\partial(f_1, f_2)}{\partial(p, \theta)} &= +1 \\ \frac{\partial(f_1, f_2)}{\partial(p, v)} &= -\frac{\partial^2 e}{\partial v \partial s} \\ \frac{\partial(f_1, f_2)}{\partial(\theta, s)} &= -\frac{\partial^2 e}{\partial v \partial s}. \end{aligned}$$

Here D is the hessian of the specific energy $e(v, s)$. The conditions of stability are $D>0$, $\partial^2 e/\partial v^2>0$, $\partial^2 e/\partial s^2>0$. Hence, in the neighborhood of any point in the region of stable equilibria, the respective quantities

$$v, s, \qquad v, \theta, \qquad p, s, \qquad p, \theta, \tag{15}$$

are single-valued functions of the respective variables

$$p, \theta, \qquad p, s, \qquad v, \theta, \qquad v, s.$$

It appears also that p and v are such functions of θ, s, and conversely, when $\partial^2 e/\partial v \partial s \neq 0$. About the value of $\partial^2 e/\partial v \partial s$ the conditions of stability afford no information.

It may now be shown that the functions (15) are single-valued "in the large." Since the form (2a) of the criterion of stability,

$$-\Delta p \Delta v + \Delta\theta \Delta s > 0,$$

is not satisfied by the successive sets of conditions:

$$\begin{aligned} \Delta p &= \Delta\theta = 0, \text{ and } \Delta v \neq 0 \text{ or } \Delta s \neq 0; \\ \Delta p &= \Delta s = 0, \text{ and } \Delta v \neq 0 \text{ or } \Delta\theta \neq 0; \\ \Delta v &= \Delta\theta = 0, \text{ and } \Delta p \neq 0 \text{ or } \Delta s \neq 0; \\ \Delta v &= \Delta s = 0, \text{ and } \Delta p \neq 0 \text{ or } \Delta\theta \neq 0; \end{aligned}$$

it follows that the functions (15) are single-valued in the large. On applying the same reasoning to p and v considered as functions of θ, s, and conversely, we observe that the sets of conditions:

$$\Delta\theta = \Delta s = 0, \text{ and } \Delta p \neq 0 \text{ or } \Delta v \neq 0;$$
$$\Delta p = \Delta v = 0, \text{ and } \Delta\theta \neq 0 \text{ or } \Delta s \neq 0;$$

respectively reduce (2a) to

$$-\Delta p\Delta v > 0, \qquad \Delta\theta\Delta s > 0,$$

which assert merely that the corresponding changes of p and v have opposite signs, respectively that the corresponding changes of θ and s have the same sign. The functions in question are therefore not necessarily single-valued in the large. Cases of multiple valuedness of these functions are well known.

By the usual procedure for calculating the derivatives of implicit functions, we find the following equations for the first derivatives of the eight functions (15). The double subscript notation indicates second derivatives of the function $e(v, s)$.

$$\begin{array}{llll}
v, s \text{ indep.} & \dfrac{\partial p}{\partial v} = -e_{11}, & -\dfrac{\partial p}{\partial s} = +\dfrac{\partial \theta}{\partial v} = +e_{12}, & \dfrac{\partial \theta}{\partial s} = +e_{22}; \\
v, \theta \quad `` & \dfrac{\partial p}{\partial v} = -\dfrac{\Delta}{e_{22}}, & \dfrac{\partial p}{\partial \theta} = +\dfrac{\partial s}{\partial v} = -\dfrac{e_{12}}{e_{22}}, & \dfrac{\partial s}{\partial \theta} = +\dfrac{1}{e_{22}}; \\
p, s \quad `` & \dfrac{\partial v}{\partial p} = -\dfrac{1}{e_{11}}, & \dfrac{\partial v}{\partial s} = +\dfrac{\partial \theta}{\partial p} = -\dfrac{e_{12}}{e_{11}}, & \dfrac{\partial \theta}{\partial s} = +\dfrac{\Delta}{e_{11}}; \\
p, \theta \quad `` & \dfrac{\partial v}{\partial p} = -\dfrac{e_{22}}{\Delta}, & \dfrac{\partial v}{\partial \theta} = -\dfrac{\partial s}{\partial p} = -\dfrac{e_{12}}{\Delta}, & \dfrac{\partial s}{\partial \theta} = +\dfrac{e_{11}}{\Delta}.
\end{array}$$

For a region of *two-phase states* the conditions of equilibrium are the five relations (12)

$$f_j = 0, \qquad j = 1, 2, \cdots, 5,$$

between the six variables p, θ, v_1, s_1, v_2, s_2. To determine whether all these variables but any one (call it x) are single-valued functions of x in the neighborhood of a point, we require to determine whether the jacobian J_x of the f_j with regard to all the variables but x is non-vanishing there. On reducing the J_x thus formed when we choose p, θ, v_i, s_i $(i=1, 2)$ successively as independent variable, we find their values to be:

$$(16) \qquad \begin{array}{l|l} J_p = -D_1D_2(s_2 - s_1) & J_{v_i} = +D_1D_2\psi_i \\ J_\theta = +D_1D_2(v_2 - v_1) & J_{s_i} = +D_1D_2\phi_i \end{array}$$

where D_i is the hessian of the specific energy $e_i(v_i, s_i)$ of the ith phase, and the ymbols ϕ_i, ψ_i denote:

$$\phi_i = \frac{1}{D_i}\left(\frac{\partial^2 e_i}{\partial v_i^2}(v_2 - v_1) + \frac{\partial^2 e_i}{\partial v_i \partial s_i}(s_2 - s_1)\right),$$

$$\psi_i = \frac{1}{D_i}\left(\frac{\partial^2 e_i}{\partial v_i \partial s_i}(v_2 - v_1) + \frac{\partial^2 e_i}{\partial s_i^2}(s_2 - s_1)\right).$$

It has already been observed that J_p and J_θ are non-vanishing, and hence that all the six variables but p, or all but θ, are single-valued functions of p, or of θ, respectively, in the small. The first derivatives of these functions, obtained from (16) by the theorem on the jacobians of implicit functions, are:

$$(17)\qquad \begin{aligned} &\frac{d\theta}{dp} = \frac{v_2 - v_1}{s_2 - s_1}, \qquad \frac{dv_i}{dp} = -\frac{\psi_i}{s_2 - s_1}, \qquad \frac{ds_i}{dp} = \frac{\phi_i}{s_2 - s_1};\\ &\frac{dp}{d\theta} = \frac{s_2 - s_1}{v_2 - v_1}, \qquad \frac{dv_i}{d\theta} = -\frac{\psi_i}{v_2 - v_1}, \qquad \frac{ds_i}{d\theta} = \frac{\phi_i}{v_2 - v_1}. \end{aligned}$$

The graph of θ as a function of p, and of p as a function of θ, is a curve whose slope, by (17) and Postulate II, maintains its sign. The function is therefore single-valued in the large.

A similar conclusion concerning v_i, s_i regarded as functions of p or of θ can be drawn. Within a given field of two-phase states of stable equilibrium take any two states having equal pressures, and hence equal temperatures. Let the quantity v_1 have the respective distinct values v_1 and $v_1+\Delta v_1$ in the two states. Then the states of unit mass of the first phase are states of the field of stable one-phase states for unit mass of this phase, and hence are subject to the criterion of stability (2a),

$$-\Delta p \Delta v_1 + \Delta\theta\Delta s_1 > 0.$$

But this inequality is not satisfied by $\Delta v_1 \neq 0$, $\Delta p = \Delta\theta = 0$. Hence the two states of the first phase are not distinct—a transformation of either into the other is not a physical change of state. The function v_1 of p, or of θ, is therefore single-valued in the large. If for v_1 we read s_1 or v_2 or s_2, the reasoning is the same. The functions v_i, s_i of p, or of θ, are single-valued in the large.

The concluding equations (16),

$$J_{v_i} = D_1 D_2 \psi_i, \qquad J_{s_i} = D_1 D_2 \phi_i,$$

remain for consideration. By certain of the equations (12) it appears that $\psi_i = 0$ when and only when dv_i/dp and $dv_i/d\theta$ vanish, and that $\phi_i = 0$ when and only when ds_i/dp and $ds_i/d\theta$ vanish. In familiar cases these conditions are satisfied. So p, θ, s_1, v_2, s_2 are not always single-valued functions of v_1, and p, θ, v_1, v_2, s_2 are not always single-valued functions of s_1.

For a region of *three-phase states* our information is already complete. The discussion in the preceding section has established that the quantities

$$p, \quad \theta, \quad v_1, \quad s_1, \quad v_2, \quad s_2, \quad v_3, \quad s_3$$

are constants, and that the energy $E(V, S)$ of the body is a linear function of the independent V, S, with experimentally determinable constant coefficients.

5. *Energy Surfaces.* An interesting graphic representation of the states of stable equilibrium of a body of a given one-component substance is afforded by the "energy surface" for the substance, which is the surface

$$e = e(v, s),$$

where v, s, e are the volume, entropy, and energy of a *body of unit mass.* When the point representing a state of equilibrium of this body is displaced from any point on the surface to any other attainable point, whether on the surface or off it, the stability of the equilibrium of the initial state requires that

$$\Delta e + p\Delta v - \theta\Delta s \geqq 0.$$

This asserts that the e-intercept of the plane tangent to the surface at the initial point is less than the e-intercept of a parallel plane through the displaced point. Since stability for constant v, s is assured when

$$(\Delta e)_{v,s} > 0,$$

it appears that any attainable point not on the surface must lie above it.

For a region of one-phase states the conditions of stability are

$$D_e > 0, \qquad e_{11} > 0, \qquad e_{22} > 0,$$

where D_e is the hessian of $e(v, s)$ and the double-subscript notation indicates second derivatives. These conditions determine that the surface is dome-shaped and convex downward. By the conditions of equilibrium its isentropic and isometric (for constant v) sections are equal to $-p$ and θ. For a region of two-phase states the conditions of stability

$$D_e = 0, \qquad e_{11} > 0, \qquad e_{22} > 0,$$

determine a developable surface convex downward, having isothermal generators, and again with slopes of section equal to $-p$ and θ. For a region of three-phase states the surface (14),

$$e = -pv + \theta s + (e_0 + pv_0 - \theta s_0),$$

is a plane, having the constant slopes of section $-p$ and θ, and the intercept $e_0 + pv_0 - \theta s_0$ on the e-axis.

In this way we obtain an assemblage of energy surfaces, whose points are in continuous one-to-one correspondence with the states of stable equilibrium of the body. The assemblage is a continuous surface, which is convex downward everywhere except where it is plane.

6. *Transformations.* In important special cases we observe that stability is assured:

With unchanging v, s, when $e=e$ is a minimum;
" " v, θ, " $f=e-\theta s$ " " "
" " p, s, " $g=e+pv$ " " "
" " p, θ, " $h=e+pv-\theta s$ " " "

By differentiating the functions f, g, h, and comparing with the criterion of equilibrium (18), we obtain

(18) $$de = -pdv + \theta ds$$

(19) $$df = -pdv - sd\theta$$

(20) $$dg = vdp + \theta ds$$

(21) $$dh = vdp - sd\theta.$$

In a clumsy and inhomogeneous nomenclature, the important functions f, g, h are termed the *free energy*, the *total heat*, and the *thermodynamic potential* of the body.

For a region of *one-phase states* we have that s is a single-valued function of v, θ, that v is a single-valued function of p, s, and that v, s are single-valued functions of p, θ. Hence f, g, and h may be considered to be functions of v, θ, of p, s, and of p, θ, respectively. Hence, further, the first derivatives of these functions can be equated with the corresponding coefficients in the equations (19) to (21), thus yielding the conditions of equilibrium in the three new sets of independent variables.

We proceed to seek the conditions of stability in the new variables. The captions of the following table are the successive pairs of independent variables x, y. The first row contains the jacobians of these pairs with regard to the energy variable v, s, together with the signs of these jacobians as determined by the conditions of stability. The next row contains the jacobians of the same pairs with regard to the free energy variables v, θ, and so on.

x, y:	v, s	v, θ	p, s	p, θ
$\dfrac{\partial(x, y)}{\partial(v, s)}$:	$1 > 0$	$\dfrac{\partial\theta}{\partial s} > 0$	$\dfrac{\partial p}{\partial v} < 0$	$\dfrac{\partial(p, \theta)}{\partial(v, s)} < 0$
$\dfrac{\partial(x, y)}{\partial(v, \theta)}$:	$\dfrac{\partial s}{\partial\theta} > 0$	$1 > 0$	$\dfrac{\partial(p, s)}{\partial(v, \theta)} < 0$	$\dfrac{\partial p}{\partial v} < 0$
$\dfrac{\partial(x, y)}{\partial(p, s)}$:	$\dfrac{\partial v}{\partial p} < 0$	$\dfrac{\partial(v, \theta)}{\partial(p, s)} < 0$	$1 > 0$	$\dfrac{\partial\theta}{\partial s} > 0$
$\dfrac{\partial(x, y)}{\partial(p, \theta)}$:	$\dfrac{\partial(v, s)}{\partial(p, \theta)} < 0$	$\dfrac{\partial v}{\partial p} < 0$	$\dfrac{\partial s}{\partial\theta} > 0$	$1 > 0$

Any jacobian $\partial(x, y)/\partial(v, s)$ in the first row is connected with each of the jacobians below it by the corresponding relation,

$$\frac{\partial(x, y)}{\partial(v, s)} = \frac{\partial(x, y)}{\partial(v, \theta)} \frac{\partial(v, \theta)}{\partial(v, s)} = \frac{\partial(x, y)}{\partial(p, s)} \frac{\partial(p, s)}{\partial(v, s)} = \frac{\partial(x, y)}{\partial(p, \theta)} \frac{\partial(p, \theta)}{\partial(v, s)}.$$

By conditions of stability, the last factor in each member after the first is respectively positive, negative, and negative. Hence the jacobians in the second, third, and fourth rows of the table have the signs that are entered there. Writing D_ϕ for the hessian of a function ϕ, the conditions of stability thus found are:

$$\begin{array}{lll} D_e > 0, & e_{11} > 0, & e_{22} > 0; \\ D_f < 0, & f_{11} > 0, & f_{22} < 0; \\ D_g < 0, & g_{11} < 0, & g_{22} > 0; \\ D_h > 0, & h_{11} < 0, & h_{22} < 0. \end{array}$$

These conditions determine that the energy surface $e=e(v, s)$ for the region is dome-shaped and convex downward; that the free energy and total heat surfaces $f=f(v, \theta)$ and $g=g(p, s)$ are saddle-shaped; and that the thermodynamic potential surface $h=h(p, \theta)$ is dome-shaped and concave downward.

In the case of a region of *two-phase states*, the conditions of stability $e_{11}>0$, $e_{22}>0$ remain, but the condition on the hessian D_e becomes $D_e=0$. Hence the foregoing conclusions remain valid, except that the elements of the last column of the table vanish, and that the last row, since p and θ are no longer independent, must be abandoned. In this case p is a function of θ. Integrating the condition of equilibrium $\partial f/\partial v = -p(\theta)$, we find

$$f = -pv + \psi(\theta),$$

or, by the definitions of f and h,

$$h = \psi(\theta).$$

For a two-phase region the conditions of stability thus become:

$$\begin{array}{lll} D_e = 0, & e_{11} > 0, & e_{22} > 0; \\ D_f < 0, & f_{11} = 0, & f_{22} < 0; \\ D_g < 0, & g_{11} < 0, & g_{22} = 0; \\ p = \phi(\theta), & h = \psi(\theta). & \end{array}$$

These conditions determine that the energy surface for the region is a developable surface, convex downward; that the free energy and total heat surfaces are skew surfaces, concave downward; and that the locus of the thermodynamic potential in the p, θ, h space is a space curve.

For a region of *three-phase states*, by (14), the graph of the energy $e(v, s)$ is a plane,

$$e = -p_0 v + \theta_0 s + h_0,$$

where p_0, θ_0, h_0 are constants. Hence the loci of the free energy in the v, θ, f

space, and of the total heat in the p, s, g space, are the straight lines,

$$f = -p_0 v + h_0, \qquad \theta = \theta_0;$$
$$g = \theta_0 s + h_0, \qquad p = p_0;$$

and the locus of the thermodynamic potential in the p, θ, h space is the point,

$$p = p_0, \qquad \theta = \theta_0, \qquad h = h_0.$$

7. *Conclusion.* The foundation is now laid for a detailed study of the thermodynamic equilibria of any one-component body supposed subject to no force other than a uniform and normally directed pressure. By application of the conditions of equilibrium and of stability, in any one of the four possible choices of independent variables, the detailed theory of the thermodynamic behavior of bodies in given regions of n-phase states ($n=1, 2, 3$) of equilibrium can readily be developed in a systematic and comprehensive way.

QUESTIONS, DISCUSSIONS, AND NOTES

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The Department of Questions and Discussions in the Monthly is open to all forms of activity in collegiate mathematics, including the teaching of mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.

EVALUATION OF $\int_0^1 \frac{\log u}{u} \log^2 (1+u)du$ AND RELATED DEFINITE INTEGRALS

By GEORGE RUTLEDGE and R. D. DOUGLASS, Massachusetts Institute of Technology

Introduction. It is well known and easily apparent that the identity

$$\prod_{n=1}^{\infty}\left(1-\frac{x^2}{n^2}\right) = 1 - \frac{\pi^2}{3!}x^2 + \frac{\pi^4}{5!}x^4 - \cdots, \tag{1}$$

obtained by equating expansions of $\sin \pi x$ and removing the common factor πx, affords a convenient means of summing certain positive-term series, namely:

$$\frac{\pi^2}{3!} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots,$$

$$\frac{\pi^4}{5!} = \left(\frac{1}{1^2}\right)\frac{1}{2^2} + \left(\frac{1}{1^2}+\frac{1}{2^2}\right)\frac{1}{3^2} + \left(\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}\right)\frac{1}{4^2} + \cdots, \tag{2}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\frac{\pi^{2i}}{(2i+1)!} = \alpha_{i\,i} + \alpha_{i\,i+1} + \alpha_{i\,i+2} + \cdots,$$

where α_{ij} indicates the sum of all those products of the squared reciprocals of the first j integers, taken i at a time, which involve the integer j. An attempt to obtain an evaluation also of the sums of the corresponding alternating series (beyond the first)

$$
\begin{aligned}
A_2 &= \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \cdots = \frac{\pi^2}{12}, \\
A_4 &= \left(\frac{1}{1^2}\right)\frac{1}{2^2} - \left(\frac{1}{1^2} + \frac{1}{2^2}\right)\frac{1}{3^2} + \left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2}\right)\frac{1}{4^2} - \cdots, \\
& \cdots\cdots\cdots\cdots\cdots\cdots \\
A_{2i} &= \alpha_{i\,i} - \alpha_{i\,i+1} + \alpha_{i\,i+2} - \cdots,
\end{aligned}
\tag{3}
$$

has led the authors to various definite integrals which may be evaluated in terms of the constant A_4, in particular.

The principal results obtained are the following evaluations:

$$\int_0^1 \frac{\log u}{u} \log^2 (1 + u)du = A_4 - \frac{\pi^4}{288}, \tag{4}$$

$$\int_0^\pi \phi \log^2 \left(2 \cos \frac{\phi}{2}\right)d\phi = A_4 + \frac{31\pi^4}{480} \cdot \tag{5}$$

The formula (5) is derived from (4) by an integration in the complex plane. Attempts to express the constant A_4 in terms of known constants have not been successful. However, the numerical value of A_4 may be computed from the series which defines A_4. The first five digits are as follows:

$$A_4 = .16265 \cdots . \tag{6}$$

The integrals (4) and (5) are not to be found for example in the tables of Bierens de Haan, or elsewhere in the literature, so far as the authors have been able to discover. A paper by F. Morley[1] is devoted to integrals of the general form

$$\int_0^\pi \phi^n \log^m \left(2 \cos \frac{\phi}{2}\right)d\phi, \tag{7}$$

where m is any integer, but n is restricted to even integral values. In addition to the integrals (4) and (5) several closely related integrals will be evaluated.

1. *Evaluations Obtained from Products of Series.* The product

$$\left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots\right)\left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots\right) \tag{8}$$

may be written

[1] Bull. Amer. Math. Soc. (2), vol. 7 (1901), p. 390.

9)
$$\sum_{n=2}^{\infty}\sum_{p=1}^{n-1}\frac{1}{p^2(n-p)^2}.$$

Since

$$\frac{1}{p^2(n-p)^2}=\frac{2}{n^3}\left[\frac{1}{p}+\frac{1}{n-p}\right]+\frac{1}{n^2}\left[\frac{1}{p^2}+\frac{1}{(n-p)^2}\right], \tag{10}$$

and consequently

$$\sum_{p=1}^{n-1}\frac{1}{p^2(n-p)^2}=\frac{4}{n^3}\sum_{p=1}^{n-1}\frac{1}{p}+\frac{2}{n^2}\sum_{p=1}^{n-1}\frac{1}{p^2}, \tag{11}$$

it results that

$$\begin{aligned}&4\left[\frac{1}{2^3}+\left(1+\frac{1}{2}\right)\frac{1}{3^3}+\left(1+\frac{1}{2}+\frac{1}{3}\right)\frac{1}{4^3}+\cdots\right]\\ &+2\left[\frac{1}{2^2}+\left(1+\frac{1}{2^2}\right)\frac{1}{3^2}+\left(1+\frac{1}{2^2}+\frac{1}{3^2}\right)\frac{1}{4^2}+\cdots\right]=\frac{\pi^4}{36}.\end{aligned} \tag{12}$$

From (2) and (12) it then follows that

$$\frac{1}{2^3}+\left(1+\frac{1}{2}\right)\frac{1}{3^3}+\left(1+\frac{1}{2}+\frac{1}{3}\right)\frac{1}{4^3}+\cdots=\frac{\pi^4}{360}. \tag{13}$$

The sum of the series (13) has been previously found[1] in a very different manner. From the power series

$$\log^2(1-u)=2\left[\frac{1}{2}u^2+\left(1+\frac{1}{2}\right)\frac{1}{3}u^3+\cdots\right] \tag{14}$$

we now obtain

$$\begin{aligned}&\int_0^1\frac{\log u}{u}\log^2(1-u)du\\ &=-2\left[\frac{1}{2^3}+\left(1+\frac{1}{2}\right)\frac{1}{3^3}+\left(1+\frac{1}{2}+\frac{1}{3}\right)\frac{1}{4^3}+\cdots\right]=-\frac{\pi^4}{180}.\end{aligned} \tag{15}$$

From the product

$$\left(1-\frac{1}{2^2}+\frac{1}{3^2}-\cdots\right)\left(1-\frac{1}{2^2}+\frac{1}{3^2}-\cdots\right) \tag{16}$$

there results in like manner the identity

[1] Morley, Proc. London Math. Soc., vol. 34 (1902), pp. 397–402.

$$
(17)\qquad \begin{aligned} &4\left[\frac{1}{2^3} - \left(1+\frac{1}{2}\right)\frac{1}{3^3} + \left(1+\frac{1}{2}+\frac{1}{3}\right)\frac{1}{4^3} - \cdots\right] \\ &+ 2\left[\frac{1}{2^2} - \left(1+\frac{1}{2^2}\right)\frac{1}{3^2} + \left(1+\frac{1}{2^2}+\frac{1}{3^2}\right)\frac{1}{4^2} - \cdots\right] = \frac{\pi^4}{144}\cdot \end{aligned}
$$

From (3) and (17) it then follows that

$$
(18)\qquad \frac{1}{2^3} - \left(1+\frac{1}{2}\right)\frac{1}{3^3} + \left(1+\frac{1}{2}+\frac{1}{3}\right)\frac{1}{4^3} - \cdots = -\frac{1}{2}A_4 + \frac{\pi^4}{576}\cdot
$$

From the power series

$$
(19)\qquad \log^2(1+u) = 2\left[\frac{1}{2}u^2 - \left(1+\frac{1}{2}\right)\frac{1}{3}u^3 + \cdots\right]
$$

we then have

$$
(20)\qquad \begin{aligned} &\int_0^1 \frac{\log u}{u}\log^2(1+u)du \\ &= -2\left[\frac{1}{2^3} - \left(1+\frac{1}{2}\right)\frac{1}{3^3} + \left(1+\frac{1}{2}+\frac{1}{3}\right)\frac{1}{4^3} - \cdots\right] = A_4 - \frac{\pi^4}{288}\cdot \end{aligned}
$$

2. *Evaluations Obtained from Linear Relations.* From the identity

$$
(21)\qquad \begin{aligned} &\int_0^1 \frac{\log u}{u}\log^2\frac{1+u}{1-u}du + 2\int_0^1 \frac{\log u}{u}\log(1+u)\log(1-u)du \\ &= \int_0^1 \frac{\log u}{u}\log^2(1+u)du + \int_0^1 \frac{\log u}{u}\log^2(1-u)du \end{aligned}
$$

we have, by use of (15) and (20), the linear relation

$$
(22)\qquad \begin{aligned} &\int_0^1 \frac{\log u}{u}\log^2\frac{1+u}{1-u}du + 2\int_0^1 \frac{\log u}{u}\log(1+u)\log(1-u)du \\ &= A_4 - \frac{13\pi^4}{1440}\cdot \end{aligned}
$$

A second linear relation between these two integrals will now be obtained. From the power series

$$
(23)\qquad \log^2\frac{1+u}{1-u} = 8\left[\frac{1}{2}u^2 + \left(1+\frac{1}{3}\right)\frac{1}{4}u^4 + \left(1+\frac{1}{3}+\frac{1}{5}\right)\frac{1}{6}u^6 + \cdots\right],
$$

which is easily obtained as the term-by-term integral of the derivative of the function in question, we have

$$(24)\qquad \begin{aligned}&\int_0^1 \frac{\log u}{u}\log^2\frac{1+u}{1-u}\,du\\ &\qquad = -\left[1+\left(1+\frac{1}{3}\right)\frac{1}{2^3}+\left(1+\frac{1}{3}+\frac{1}{5}\right)\frac{1}{3^3}+\cdots\right];\end{aligned}$$

and from the power series

$$(25)\qquad \log(1+u)\log(1-u) = -2\left[\frac{1}{2}u^2+\left(1-\frac{1}{2}+\frac{1}{3}\right)\frac{1}{4}u^4+\cdots\right],$$

which is also obtained as the term-by-term integral of the derivative,[1] we have

$$(26)\qquad \begin{aligned}&\int_0^1 \frac{\log u}{u}\log(1+u)\log(1-u)du\\ &\qquad = \frac{1}{4}\left[1+\left(1-\frac{1}{2}+\frac{1}{3}\right)\frac{1}{2^3}+\cdots\right].\end{aligned}$$

From (24) and (26) there results the relation

$$(27)\qquad \begin{aligned}&\int_0^1 \frac{\log u}{u}\log^2\frac{1+u}{1-u}\,du + 4\int_0^1 \frac{\log u}{u}\log(1+u)\log(1-u)\,du\\ &= -1-\left(1+\frac{1}{3}\right)\frac{1}{2^3}-\left(1+\frac{1}{3}+\frac{1}{5}\right)\frac{1}{3^3}-\cdots\\ &\quad +1+\left(1-\frac{1}{2}+\frac{1}{3}\right)\frac{1}{2^3}+\left(1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}\right)\frac{1}{3^3}+\cdots\\ &= -\frac{1}{2}\left[\frac{1}{2^3}+\left(1+\frac{1}{2}\right)\frac{1}{3^3}+\left(1+\frac{1}{2}+\frac{1}{3}\right)\frac{1}{4^3}+\cdots\right]=\frac{-\pi^4}{720}\cdot\end{aligned}$$

From the independent linear relations (22) and (27) we now obtain the evaluations

$$(28)\qquad \int_0^1 \frac{\log u}{u}\log^2\frac{1+u}{1-u}\,du = 2A_4-\frac{\pi^4}{60},$$

$$(29)\qquad \int_0^1 \frac{\log u}{u}\log(1+u)\log(1-u)\,du = -\frac{1}{2}A_4+\frac{11\pi^4}{2880}\cdot$$

3. *Evaluations Obtained by Contour Integration.* If we evaluate the integral (20) over a path in the complex plane consisting of the negative real-axis from 0 to -1 and the half of the unit circle in the upper half-plane from -1 to $+1$, we obtain for the value of this integral

---

[1] See Bromwich, *Introduction to the Theory of Infinite Series*, Second Edition, p. 191.

$$(30)\qquad \begin{aligned}&\int_0^1 \frac{\log t}{t}\log^2(1-t)dt + \int_0^\pi \phi\left[\log^2 2\cos\frac{\phi}{2} - \frac{\phi^2}{4}\right]d\phi \\ &+ i\left\{\pi\int_0^1 \frac{1}{t}\log^2(1-t)dt + \int_0^\pi \phi^2\log 2\cos\frac{\phi}{2}\,d\phi\right\}.\end{aligned}$$

Now[1]

$$(31)\qquad \int_0^\pi \phi^2 \log 2\cos\frac{\phi}{2}\,d\phi = -2\pi\left[1+\frac{1}{2^3}+\frac{1}{3^3}+\cdots\right],$$

and from the power series (14) we obtain

$$(32)\qquad \begin{aligned}&\pi\int_0^1 \frac{1}{t}\log^2(1-t)dt \\ &= 2\pi\left[\frac{1}{2^2}+\left(1+\frac{1}{2}\right)\frac{1}{3^2}+\left(1+\frac{1}{2}+\frac{1}{3}\right)\frac{1}{4^2}+\cdots\right].\end{aligned}$$

The series on the right of (32) is numerically equal[2] to the series on the right of (31), and hence the imaginary part of the integral vanishes. This is of course mere verification, since the contour integral under consideration is equal to the real integral (20). Then from (30), by use of (15) and (20) we have

$$(33)\qquad \int_0^\pi \phi\log^2 2\cos\frac{\phi}{2}\,d\phi = A_4 + \frac{31\pi^4}{480}\cdot$$

If we evaluate the integral (15) over this same contour we obtain in a wholly analogous manner

$$(34)\qquad \int_0^\pi \phi\log^2 2\sin\frac{\phi}{2}\,d\phi = -A_4 + \frac{3\pi^4}{160}\cdot$$

However it is simpler to obtain (34) from (33) by writing

$$(35)\qquad \int_0^\pi \phi\log^2 2\sin\frac{\phi}{2}\,d\phi = \int_0^\pi \phi\log^2 2\cos\frac{\phi-\pi}{2}\,d\phi.$$

Then by use in succession of the substitutions[3], $\phi-\pi=\psi$, $\psi=-\phi$, we have

$$(36)\qquad \begin{aligned}&\int_0^\pi \phi\log^2 2\sin\frac{\phi}{2}\,d\phi = \pi\int_0^\pi \log^2 2\cos\frac{\phi}{2}\,d\phi - \int_0^\pi \phi\log^2 2\cos\frac{\phi}{2}\,d\phi \\ &= \frac{\pi^4}{12} - A_4 - \frac{31\pi^4}{480} = -A_4 + \frac{3\pi^4}{160}\cdot\end{aligned}$$

---

[1] Morley, loc. cit., Bulletin, p. 392.

[2] R. D. Douglass, Journal of Math. and Physics of Mass. Inst. of Tech., vol. 10 (1931), p. 142.

[3] Morley, loc. cit., Bulletin, p. 392; also, Bromwich, p. 520.

Finally it may be pointed out that the results (33), (34) are also obtainable by integration of the trigonometric series[1]

$$
\begin{aligned}
\phi \log^2 2 \cos \frac{\phi}{2} &= \frac{\phi^3}{4} + 2\phi\Big[\frac{1}{2}\cos 2\phi - \Big(1+\frac{1}{2}\Big)\frac{1}{3}\cos 3\phi \\
&\quad + \Big(1+\frac{1}{2}+\frac{1}{3}\Big)\frac{1}{4}\cos 4\phi - \cdots\Big], \quad 0 \leqq \phi < \pi;
\end{aligned} \tag{37}
$$

$$
\begin{aligned}
\phi \log^2 2 \sin \frac{\phi}{2} &= \frac{\phi(\phi-\pi)^2}{4} + 2\phi\Big[\frac{1}{2}\cos 2\phi + \Big(1+\frac{1}{2}\Big)\frac{1}{3}\cos 3\phi \\
&\quad + \Big(1+\frac{1}{2}+\frac{1}{3}\Big)\frac{1}{4}\cos 4\phi + \cdots\Big], \quad 0 < \phi \leqq \pi.
\end{aligned} \tag{38}
$$

From (37) we obtain

$$
\begin{aligned}
\int_0^\pi \phi \log^2 2 \cos \frac{\phi}{2}\, d\phi &= \frac{\pi^4}{16} \\
&\quad + 4\Big[\Big(1+\frac{1}{2}\Big)\frac{1}{3^3} + \Big(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}\Big)\frac{1}{5^3} + \cdots\Big] \\
&= \frac{\pi^4}{16} + \Big\{-2\Big[\frac{1}{2^3} - \Big(1+\frac{1}{2}\Big)\frac{1}{3^3} + \Big(1+\frac{1}{2}+\frac{1}{3}\Big)\frac{1}{4^3} - \cdots\Big]\Big\} \\
&\quad - \Big\{-2\Big[\frac{1}{2^3} + \Big(1+\frac{1}{2}\Big)\frac{1}{3^3} + \Big(1+\frac{1}{2}+\frac{1}{3}\Big)\frac{1}{4^3} + \cdots\Big]\Big\} \\
&= \frac{\pi^4}{16} + \Big\{A_4 - \frac{\pi^4}{288}\Big\} - \Big\{-\frac{\pi^4}{180}\Big\} = A_4 + \frac{31\pi^4}{480}.
\end{aligned} \tag{39}
$$

From (38) we obtain in like manner

$$
\begin{aligned}
\int_0^\pi &\phi \log^2 2 \sin \frac{\phi}{2}\, d\phi \\
&= \frac{\pi^4}{48} - 4\Big[\Big(1+\frac{1}{2}\Big)\frac{1}{3^3} + \Big(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}\Big)\frac{1}{5^3} + \cdots\Big] \\
&= \frac{\pi^4}{48} - \Big\{A_4 - \frac{\pi^4}{288}\Big\} + \Big\{-\frac{\pi^4}{180}\Big\} = -A_4 + \frac{3\pi^4}{160}.
\end{aligned} \tag{40}
$$

It is worthy of comment that, since the results (39), (40) are in agreement with the results (33), (34) of contour integration, the validity of term-by-term integration of the trigonometric series (37), (38) between the limits 0 and π is established without further consideration.

[1] See Bromwich, p. 529.

4. *Evaluations Obtained from the Riemann ζ-Function.* We conclude with an observation regarding the relation[1]

$$(41)\qquad \begin{aligned} 2\int_0^\infty \frac{x^{s-1}e^x}{e^{2x}-1}\,dx &= 2\Gamma(s)\,\frac{2^s-1}{2^s}\left\{1+\frac{1}{2^s}+\frac{1}{3^s}+\cdots\right\} \\ &= 2\Gamma(s)\left\{1+\frac{1}{3^s}+\frac{1}{5^s}+\cdots\right\}, \quad \text{real part of } s>1. \end{aligned}$$

By use of the substitution

$$(42)\qquad x = \log\frac{1+u}{1-u},$$

the relation (41) becomes

$$(43)\qquad \int_0^1 \frac{1}{u}\log^{s-1}\frac{1+u}{1-u}\,du = 2\Gamma(s)\left\{1+\frac{1}{3^s}+\frac{1}{5^s}+\cdots\right\},$$

where, in particular, s may have the values 2, 3, 4, $\cdots$. It is improbable that the form (43) of the relation (41) has not been previously observed, although the authors have not been able to find the value of the integral (43) recorded (except for the case $s=2$). This observation regarding the integral (43) is included in this paper because of the close relation of this integral to the integrals evaluated in Sec. 2 of the paper.

We remark finally that the nature of the constant A_4 in terms of which the various integrals discussed in this paper are evaluated remains undetermined. It is with the hope of finding an answer to this question that this paper is submitted to the department of Questions, Discussions, and Notes.

Simplification of the Equations of Conics

By H. B. Thornton, Sumner Junior College, Kansas City, Kansas

The process given in most textbooks on analytic geometry for simplifying the equation of a conic by means of substitutions from trigonometric formulae is usually long and laborious. By means of the formulae derived below (which as far as I know are not given in any text) the simplification can be accomplished easily and quickly. Let

$$(1)\qquad \begin{aligned} x &= x_0 + \lambda t \\ y &= y_0 + \mu t \end{aligned}$$

be the parametric equations of a line intersecting the conic

$$ax^2 + by^2 + 2hxy + 2gx + 2fy + c = 0.$$

Then the roots of the equation

[1] Whittaker and Watson, *Modern Analysis*, Third Edition, p. 267.

$$t^2(a\lambda^2 + b\mu^2 + 2h\lambda\mu) + t(2a\lambda x_0 + 2b\mu y_0 + 2h\lambda y_0 + 2h\mu x_0 + 2g\lambda + 2f\mu)$$
$$+ ax_0^2 + by_0^2 + 2hx_0y_0 + 2gx_0 + 2fy_0 + c = 0$$

are, except for the factor $(\lambda^2+\mu^2)^{1/2}$, the distances from the point (x_0, y_0) on the line to the points of intersection of the line with the conic. The condition that the roots of this equation be numerically equal but opposite in sign is

$$\lambda(ax_0 + hy_0 + g) + \mu(hx_0 + by_0 + f) = 0.$$

If x_0, y_0 are taken as the variables this is the equation of a system of diameters. The condition that any line of this system be perpendicular to the system of chords (1) is

$$\frac{a\lambda + h\mu}{\lambda} = \frac{h\lambda + b\mu}{\mu}.$$

Set each of these ratios equal to β. Then the condition that the equations

$$a\lambda + h\mu = \lambda\beta$$
$$h\lambda + b\mu = \mu\beta$$

have a common solution other than 0, 0 is

$$\begin{vmatrix} a-\beta & h \\ h & b-\beta \end{vmatrix} = 0,$$

or

$$\beta^2 - (a+b)\beta - h^2 + ab = 0,$$

whence

$$\beta_1, \beta_2 = \frac{a + b \pm \sqrt{(a-b)^2 + 4h^2}}{2}.$$

Then, upon making the suitable transformation, the simplified equation of the conic will take the form

$$\beta_1 x^2 + \beta_2 y^2 - \frac{D}{d} = 0, \tag{2}$$

where

$$D = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$$

and $d = h^2 - ab$. If $d = 0$ (a parabola), the simplified form becomes

$$\beta_1 x^2 + 2\sqrt{-D/\beta_1}\, y = 0. \tag{3}$$

RECENT PUBLICATIONS

EDITED BY R. A. JOHNSON, Brooklyn College of the City of New York

All books for review should be sent directly to the editor of this department, at Brooklyn College, 66 Court Street, Brooklyn, N. Y., and not to any of the other editors or officers of the Association.

REVIEWS

Geschichte der Elementar-Mathematik in systematischer Darstellung mit besonderer Berücksichtung der Fachwörter. By Johannes Tropfke. Zweiter Band, Allgemeine Arithmetik. Dritte Auflage. Berlin, W. De Gruyter & Co., 1933. iv+266 pages. 12 RM.

A third of a century has gone by since the first edition of Dr. Tropfke's *Geschichte* appeared, and it is interesting to observe the change. The original edition consisted of only two volumes, the first covering the history of elementary arithmetic and algebra, and the second being concerned with Euclidean geometry, trigonometry, analytic geometry, conic sections, series, and a few minor topics. The second edition contained seven volumes and the third will probably have the same number. The volume here under review is devoted to general arithmetic, including numbers of various types, elementary algebra, and logarithms, and therefore an entire volume is now required to cover the ground of only a part of the original one of thirty years ago. The second volume of the second edition contained 221 pages, whereas the present one has 41 more. These particulars are mentioned for the purpose of showing how the labors of Dr. Tropfke have extended, the general nature of the work and the topics considered having remained substantially the same.

The added material consists largely but not wholly of new or expanded notes, and since these are made up of bibliographical references they are of special value to the student. They show a careful re-reading of the authorities cited in the earlier editions and often carry with them an extension of the text itself. They also show considerable study of the literature of recent date, as in the case of Neugebauer's and Struve's articles in *Quellen und Studien* and of Datta in the AMERICAN MATHEMATICAL MONTHLY. The number of these notes was 1157 in the preceding edition, and this has been increased to 1460 in the present one, but the references to new contributions in the English language are far from being complete. To make no reference to Dr. Chace's edition of the Ahmes Papyrus for example, although referring to Peet's memoir (not his edition of 1923) is one of many evidences of the latter fact.

As to the minor details of the work it should be said that Dr. Tropfke is in general a rather careful writer—more careful than when he was preparing the earlier editions. It is practically impossible, however, to write a book containing thousands of dates, page references, names, and statements without

The Physical Significance of the Quantum Theory. By F. A. Lindemann. Oxford, The Clarendon Press, 1932. 148 pages. $2.50.

This interesting book represents an attempt to discuss the principal features of the quantum theory on the basis of what may be termed simple physical considerations with considerably less emphasis on the mathematical development than is usual in books on this subject. The author expresses the feeling that while the mathematical analysis of quantum mechanics gives accurate results (in the sense of general agreement with experimental observations) it does not convey a clear physical meaning and hence another type of presentation is essential.

After a brief historical survey of the quantum theory which hardly does justice to the epoch-making nature of Bohr's foundation of the quantum theory of atomic structure in 1913 and the years immediately following, the author proceeds to develop his principal thesis that all the difficulties that have been encountered in constructing theories of atomic phenomena are attributable to our persistence in the use of the naive space-time concepts of classical physics. This view, of course, leads directly to the celebrated *indeterminacy principle* of Heisenberg, according to which it is impossible to measure at the same time with the same degree of accuracy the dynamical quantity *momentum* and spatial quantity *displacement*; a similar situation exists with respect to the dynamical quantity *energy* and the *time*. In each case the product of the uncertainties in the measurement of the dynamical and spatio-temporal quantities respectively is numerically not less than the Planck constant of action "h" (6.547×10^{-27} erg sec). For large scale objects under ordinary conditions the indeterminacy is negligible, but it can become very appreciable in the case of atomic phenomena. The principle has been the subject of much discussion in contemporary physical literature for it marks a definite departure from determinism in physical theories. Consequently its significance for the future of physical theory can scarcely be exaggerated.

The task which the author of the present book has set himself is essentially to derive with a minimum of analysis all the physical results of the quantum theory from the indeterminacy principle. For example, the attempt is made to deduce in this way the characteristics of the new types of statistics known by the names of Bose-Einstein (the form which works best apparently in the study of the behavior of photons or light particles) and Fermi-Dirac (that which has been very successful in interpreting the behavior of aggregates of electrons). There is a certain plausibility about the method, but it strikes the reviewer as being of doubtful validity. Certainly it will carry no conviction to those accustomed to the logical, analytical reasoning by which definite results are derived from definite physical theories. From the interpretative standpoint the method and illustrations used are of considerable interest. Nevertheless there exists the danger that the incautious reader not particularly well acquainted with contemporary physics may get the impression that the modern quantum physicists have made their theories unnecessarily difficult and that the whole prob-

lem if looked at from the author's point of view is really much simpler than any one had previously supposed. Would that this were really true! It does not appear to the reviewer, however, that the author has succeeded in achieving this result. It would be a matter of the greatest interest to science if he could. It seems, however, that the careful working out of his ideas would demand much more elaborate analysis than he presents in most of the present work.

Among other topics to which the author applies his ideas are the quantum conditions which serve to describe the stationary states of atoms, the periodic system of the elements and radiation. There is also a chapter on the relation between the author's point of view and wave mechanics. His analysis here is more detailed and follows in the main the usual treatment found in texts on quantum mechanics.

R. B. LINDSAY

MATHEMATICS CLUBS

EDITED BY F. M. WEIDA, The George Washington University, Washington, D. C.

All reports of club activities, suggestions and topics for club programs, and material of interest to clubs should be sent to F. M. Weida, The George Washington University, Washington, D. C. All manuscripts should be typewritten, with double spacing, and with margins at least one inch wide.

CLUB ACTIVITIES

1932–33

THE PI MU EPSILON MATHEMATICAL FRATERNITY

Pi Mu Epsilon is an academic fraternity in institutions of university grade. Its primary aim is the advancement of mathematics and scholarship. It is a living, active, working fraternity of scholars in which the members are engaged in study and research and the exchange of ideas in the field of mathematical science.

Pi Mu Epsilon of the University of Missouri

The officers for the year 1932–1933 were: Ralph Traber, Director; Stuart Haynes, Vice Director; Richard Kilpatrick, Secretary; O. T. Snodgrass, Treasurer; George Ewing, Corresponding Secretary.

The meetings and programs were as follows:

September 27, 1932: "Homogeneous coordinates" by Professor Louis Ingold.
October 11, 1932: "Group theory" by Ralph Traber.
October 25, 1932: "Number fields" by Professor G. E. Wahlin.
November 8, 1932: "Some problems in the calculus of variations" by George Ewing.
November 22, 1932: "Fourier series" by Norman Beers.
December 2, 1932: Election of new members.
December 13, 1932: Informal initiation.
December 16, 1932: Formal initiation of three new members.
January 10, 1933: "Interpolation" by Richard Emberson.
February 14, 1933: "Volume and surface integrals in n dimensions" by Professor Herman Betz.
February 28, 1933: A ciphering match in which all of the members present took part.
March 14, 1933: "Forecasting the stock market" by Dr. H. P. Hartkemeier of the department of accounting and statistics.

March 28, 1933: "Mathematical infinity" by Professor Louis Ingold.
April 11, 1933: "Congruences" by Russell Michel.
April 25, 1933: "Some observations in the factors which affect the teaching of mathematics in our High Schools" by Dr. C. H. Butler of the University High School.
April 28, 1933: Election of new members.
May 9, 1933: Informal initiation.
May 13, 1933: Formal initiation of 13 new members and the annual banquet.
May 23, 1933: Business meeting and the election of officers.

GEORGE M. EWING, *Corresponding Secretary*

Pi Mu Epsilon of Lehigh University

The year was devoted to a series of programs on mapping by complex functions. A public lecture on March 31st by Professor Arnold Dresden of Swarthmore College on "How sure are we of our mathematics?" was sponsored by the chapter.

At the meeting of May 3rd, the following officers were elected for the ensuing year: Dr. C. A. Shook, Director; D. C. Bomberger, First Vice Director; K. L. Honeyman, Secretary. The chapter had 48 active members during the year.

The meetings and programs were as follows:
November 1, 1932: "Ordinary algebra of complex numbers" by R. J. Myers; "Complex numbers as number pairs" by W. B. Coleman; "Mapping by means of the linear function of a complex variable" by J. W. Langhaar.
December 14, 1932: Three papers on "Mapping by certain special non-linear functions of a complex variable" by D. C. Bomberger, J. G. Williams, F. P. Shannon.
February 23, 1933: "Construction of Riemann surfaces for some simple functions" by Dr. L. L. Smail.
May 3, 1933: "Geographical maps" by Melvin Dresher; Business meeting for election of officers and new members.

Thirteen new members were publicly pledged in the Lehigh Chapel and initiated at a banquet in May.

WALTER C. BACHMAN, *Secretary*

LOCAL MATHEMATICS CLUBS

The Case Mathematics Club

The Case Mathematics Club was organized in the spring of 1930 for the mutual improvement of the members in the study of mathematics and for the encouragement of the study of mathematics throughout the entire college. Since then, the club has been meeting regularly, listening to mathematical papers on both historical and analytic topics and discussing them afterward. Membership is open to all interested members of the student body and of the faculty.

The officers for 1932–1933 were: Leo P. Tarasov, President; Briggs H. Napier, Vice President; Bruce Thompson, Treasurer and Corresponding Secretary; Robert D. LaGanke, Recording Secretary.

The meetings and programs were as follows:
October 14, 1932: "Mathematicians of Europe" by Professor Simon of Western Reserve University. This meeting was held at the home of Dean T. M. Focke of Case.
October 28, 1932: "The philosophic significance of mathematics" by Rufus Oldenburger of the Case mathematics faculty.
November 17, 1932: "Some problems of mathematical astronomy" by Mr. K. D. Kelly. This meeting was held at the Warner and Swasey Observatory.
December 16, 1932: "Ternary cubics" by Richard S. Burington of the Case faculty.
February 10, 1933: "The contributions of Gauss" by Robert A. Harrington.
March 24, 1933: "The geometry of the complex field" by Louis G. Henyey.
April 14, 1933: "Euler and eighteenth century mathematics" by Bruce Thompson.

April 28, 1933: "The complex variable and various types of circular motion" by Robert S. LaGanke.
May 12, 1933: "Generalized derivatives and integrals" by Professor F. G. Jonah of Western Reserve University. This meeting was held at the home of Dean T. M. Focke.

All the meetings, except as otherwise mentioned, were held in the lecture room of the Case Physics Building.

In addition to the above activities, the club awarded a competitive prize of thirty-five dollars to the member of the club who had submitted the best mathematical paper during the year. This award was delivered at the Case commencement.

ROBERT S. LAGANKE, *Secretary*

The Mathematics Club of the George Washington University

The officers for the year 1932–1933 were: Dr. F. E. Johnston, President; Abraham Sinkov, Secretary. The officers are elected annually at the first meeting of the academic year.

The aim of the club is to stimulate a creative interest in mathematics. Membership is open to all persons having a genuine interest in the subject. This year we had 30 members.

The meetings and programs were as follows:

October 19, 1932: "The principles of life insurance" by Dr. F. M. Weida.
November 2, 1932: "Lewis Carroll as a mathematician" by Dr. E. W. Woolard.
November 16, 1932: "Inverse probability" by Dr. W. E. Deming.
November 30, 1932: "Characteristic functions" by S. Kullback.
December 14, 1932: "Finite geometries" by Michael Goldberg.
January 11, 1933: "On skew cubics" by Dr. Tobias Dantzig of the University of Maryland.
February 15, 1933: "Groups generated by two operators" by Abraham Sinkov.
March 1, 1933: "On the theory of equations in generalized number systems" by Archie Blake.
March 15, 1933: "Some problems in nomography" by Albert Wertheimer.
March 29, 1933: "The mechanical description of plane curves" by Dr. R. C. Yates of the University of Maryland.
April 26, 1933: "Mechanical proofs of the law of the lever" by J. H. Edmonston.
May 12, 1933: "New properties of the arithmetic means of the partial sums of the Fourier series" by Dr. Leopold Fejér of the University of Budapest, Budapest, Hungary.

A. SINKOV, *Secretary*

Chi Upsilon Zeta of Elmira College

The officers for 1932–1933 were: May O'Connor, President; Jane Brewer, Vice President; Mary Obuhanych, Secretary-Treasurer. The President is nominated by the members of the club and voted upon by ballot at the April meeting of the club. The other officers are elected at the regular meeting of the club in October. There are 15 active members.

The object of the club is to bring the professors and students of mathematics into closer contact, and to promote interest in the more general phases of mathematics, and to create interest in the subject for its own sake. Any student who is taking or has completed a course in mathematics may become a member of the club by paying fifty cents a year.

The meetings and programs were as follows:

October 26, 1932: "A dramatic skit, 'Mechanics'" by members of the club.
November 29, 1932: "Leonardo da Vinci, the mathematician" by Helen Barnes.
February 15, 1933: "Chess problems" by Edith Woolsey; "The Pythagorean theorem" by Norma Fletcher; "Ferrari's solution of the Biquadratic equation" by Mary Obuhanych.
March 17, 1933: "Ptolemaic theory" by Lillian Hay; "Contributions of Copernicus" by Lillian Peters; "Cube root" by Clare Daley.
April 12, 1933: "Should women study mathematics in college?" by Professor Frank W. Lindsley.
April 26, 1933: "Ciphers and Cryptograms" by Maybelle Boesen; "Brain teasers" by Jane Brewer.
April 29, 1933: "Life of Newton" by Dorothy Cate.
May 10, 1933: Annual picnic.

MARY OBUHANYCH, *Secretary-Treasurer*

The Mathematics Club of Hunter College

In spite of the depression the Mathematics Club at Hunter College has just closed a busy and happy year. During the year, Miss Jeanette Homan, Actuarial Statistician at the American Telephone and Telegraph Company spoke on "The woman mathematician in business"; Professor Simons gave a non-technical report, supplemented by lantern slides, on the International Congress of Mathematicians held in Zürich; Professor Bushey reported on the meetings of the American Mathematical Society held in California last summer; and Dr. Michels, of the Social Science Department at Hunter College, spoke on the topic "Some aspects of the present banking situation" at the time of the national bank holiday. At the remaining program meetings, students reported on the following topics: Prime numbers; Perfect numbers; The Peruvian Quipu; The prismatoid; The sun dial; Special devices in integration; Conic sections from the point of view of projective geometry. The average attendance at the program meetings was about thirty.

The social side of the Mathematics Club life was by no means neglected. At the beginning of each term a party was given for new members: last Fall the Club sponsored a theater party; and on April 1, the Club held a "Puzzle party." Plans for a hike on Columbus Day were cancelled because of inclement weather. At a home talent Chapel held during the Spring term, the Club gave the skit "Flatlanders" under the direction of Professor Whelan.

The officers for the year were: Helen Schroeder, President; Gertrude Stern, Vice President; Pearl Zudeck, Secretary; Lillian McNulty, Treasurer; Gertrude Grodinsky, Publicity Manager; Laura Guggenbühl, Faculty Adviser.

LAURA GUGGENBÜHL, *Faculty Adviser*

PROBLEMS AND SOLUTIONS

EDITED BY OTTO DUNKEL, H. L. OLSON, AND W. F. CHENEY, JR.

ELEMENTARY PROBLEMS

Send all communications about Elementary Problems and Solutions to W. F. Cheney, Jr. Dept. Box 35, Storrs, Connecticut.

The Department of Elementary Problems welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. Problems may be submitted unaccompanied by their solutions.

PROBLEMS FOR SOLUTION

E70. *Proposed by Roy MacKay, Albuquerque High School, Albuquerque, N.M.*

Show that the area of a right triangle in terms of the bisector of the right angle, t, and the median to the hypotenuse, m, is given by the formulas,

$$K = 2m^2t/\left\{(t^2 + 8m^2)^{1/2} \mp t\right\},$$

where the upper or lower sign is to be used according as t is the bisector of the interior or exterior angle at the right angle vertex.

E71. *Proposed by W. F. Cheney, Jr., Connecticut State College.*

In a simple example of addition, each digit was replaced by a code letter, and the result was

$$\begin{array}{cccccc} N & E & W & T & O & N \\ & K & L & E & I & N \\ \hline K & E & P & L & E & R \end{array}$$

Identify the letters and show that there are just two solutions, which differ only through the interchange of the values determined for O and I.

E72. *Proposed by J. M. West, Pennsylvania State College.*

Given that $A+B+C=180°$, prove that

$$\sin A \cos^2 A \sin (B-C) + \sin B \cos^2 B \sin (C-A) + \sin C \cos^2 C \sin (A-B) = 0.$$

E73. *Proposed by W. F. Cheney, Jr., Connecticut State College.*

Show that there is just one right triangle whose three sides are relatively prime integers between 2000 and 3000.

E74. *Proposed by J. E. Trevor, Cornell University.*

A vertical sheet of horizontal rays of light falls upon the outside of a horizontal reflecting circular cylinder, the axis of which meets the incident sheet at an arbitrary angle. The reflected rays form an illuminated curve on a dark screen parallel to the incident sheet. Find the equation of this curve.

SOLUTIONS

E40 [1933, 296]. *Proposed by Maud Willey, Long Beach, Mississippi.*

Let $C_i=0$ $(i=1, 2, 3)$ be the equations of three circles. Prove that the three circles, $\sum_{i=1}^{3} K_{ij}C_i=0$ $(j=1, 2, 3)$, have the same radical center as the three circles $C_i=0$.

Prove that the four spheres, $\sum_{i=1}^{4} K_{ij}S_i=0$ $(j=1, 2, 3, 4)$, have the same radical center as the four spheres whose equations are $S_i=0$ $(i=1, 2, 3, 4)$.

Prove that the n hyperspheres, $\sum_{i=1}^{n} K_{ij}S_i=0$ $(j=1, 2, \cdots, n)$ in space of $n-1$ dimensions, have the same radical center as the n hyperspheres $S_i=0$ $(i=1, 2, \cdots, n)$.

Solution by Simon Vatriquant, Athênée Royale d'Ixelles, Brussels.

The first and second parts of the statement are particular cases of the third one, and it will be sufficient to prove this third part.

The equation

$$(1)\quad \sum_{i=1}^{n} K_{ij}S_i \equiv \left(\sum_{i=1}^{n} K_{ij}\right)(x^2+y^2+z^2+\cdots) - 2\left(\sum_{i=1}^{n} K_{ij}a_i\right)x - \cdots = 0,$$

represents a hypersphere (except when $\sum^{n}_{=1} K_{ij}=0$), whose center is the center of gravity of the centers of the hyperspheres $S_i=0$ $(i=1, 2, 3, \cdots, n)$, with the masses K_{ij} $(i=1, 2, 3, \cdots, n)$.

Let P be the radical center of the hyperspheres $S_i=0$, and p its power with respect to these hyperspheres. The power of P with respect to the hypersphere

(1) may be written under the form, $(\sum^n_{=1} K_{ij} S_i)/(\sum^n_{i=1} K_{ij})$, the coordinates $(x, y, z, \cdots)$ being those of P. But for these values of the coordinates $(x, y, z, \cdots)$, we have $S_1 = S_2 = \cdots = S_n = p$, and the value of the power of P is also p. [If we suppose $\sum^n_{=1} K_{ij} = 0$, (1) represents hyperplanes through P, for it is a linear equation, satisfied by the coordinates of P.]

For the n values of j, the power of P is the same, and P is the radical center of the n hyperspheres (1), $(j = 1, 2, 3, \cdots, n)$.

E42 [1933, 360]. *Proposed by C. A. Rupp, Pennsylvania State College.*

If *abcde* and *baced* are squares, and $c+d$ and $b+e$ are successive primes, determine the five distinct digits a, b, c, d and e, and show that the solution is unique.

Solution by E. P. Starke, Rutgers University.

Since the formula $(50x \pm y)^2$, with y running from 0 to 25 inclusive, will represent any possible square, we find that the last two digits of a square must be 00, 01, 04, 09, 16, 25, 36, 49, 64, 81, 21, 44, 69, 96, 56, 89, 24, 61, 41, 84, 29 or 76. The digits are given distinct, and must still terminate a square when reversed. Hence the only possible values for (d, e) are $(i)(1, 6)$, $(ii)(6, 1)$, $(iii)(6,9)$ and $(iv)(9, 6)$.

Since $(c+d)$ and $(b+e)$ are successive primes, we have in case (i) that $(c+d)$ must be 3 or 5, and then $(b+e)$ has to be 2 or 5, or 3 or 7, respectively. But since no value of b exists to accomplish this, case (i) is ruled out.

In case (ii), $(c+d)$ must be 11 or 13, and $(b+e)$ must then be 7 or 13, or 11 or 17, which again demands the impossible of b.

In case (iii), $(c+d)$ may be 7 or 11 or 13, and then $(b+e)$ will be 5 or 11, or 7 or 13, or 11 or 17, respectively, giving $b = 2, 4, 2$ or 8, respectively. We now have $a2169$, $a4569$, $a8769$ and $a2769$. The only value of a which leads to a square is $a = 1$, which makes 12769 and 18769, each squares. But $baced = 21796$ is not a square, so case (iii) gives the one solution, $abcde = 18769$.

In case (iv), we find b's leading to $a1296$, $a7296$, $a5496$ and $a7896$, but no value of a will make any of these a square.

Hence the solution, $abcde = 18796$, is unique.

Solved also by W. E. Buker, C. W. Trigg, Simon Vatriquant, and the proposer.

E43 [1933, 360]. *Proposed by Arthur Haas, Thomas Jefferson High School, Brooklyn, N. Y.*

In the following simple multiplication the x's represent unknown digits to be determined. Show that there are just two solutions.

$$\begin{array}{rrrrr}
 & & x & x & x \\
 & & & x & x \\
\hline
 & & x & x & x \\
 & x & x & 4 & \\
\hline
x & x & x & 1 & 7
\end{array}$$

Solution by B. C. Zimmerman, Orange Walk, British Honduras.

Since the first and second partial products are obviously $x77$ and $9x4$ respectively, the right-hand digit of the multiplier is odd and the left-hand digit of the multiplier is even and larger than the other. When we set the right-hand digit of the multiplier equal to 1, the right-hand digit of the multiplicand must be 7 and the left-hand digit of the multiplier can only be 2. This multiplier 21 leads to a multiplicand of 477 and a product of 10017, which satisfies the conditions of the problem.

If we set the right-hand digit of the multiplier equal to 3, the right-hand digit of the multiplicand must be 9 and the left-hand digit of the multiplier must be 6. This multiplier of 63 can only go with a multiplicand of 159 to give the same product as before, 10017.

If we set the right-hand digit of the multiplier equal to 5 or 7 or 9, we are led to a contradiction in every case, so the only possible solutions are

$$\begin{array}{r} 477 \\ 21 \\ \hline 477 \\ 954 \\ \hline 10017 \end{array} \qquad \text{and} \qquad \begin{array}{r} 159 \\ 63 \\ \hline 477 \\ 954 \\ \hline 10017 \end{array}$$

with the same product and partial products in each case, and the two multiplicands and multipliers differing only in the switching of a factor 3.

Solved also by W. E. Buker, M. L. Constable, Arnold Court, W. C. Eells, H. R. Leifer, Theodore Lindquist, C. W. Munshower, E. P. Starke, C. W. Trigg, Simon Vatriquant and the proposer.

E44 [1933, 360]. *Proposed by Mannis Charosh, New Utrecht High School, Brooklyn, N. Y.*

ABC is an isosceles triangle with $AB=AC$. ADB is a right triangle with D, the vertex of the right angle, on the opposite side of AB from C. Angle DAB is equal to angle BAC, and DF and CE are perpendicular to AB and AD at F and E respectively. Prove that AF and FB differ by AE.

Solution by Roy MacKay, Albuquerque High School, Albuquerque, N.M.

Let $AB=AC=a$, and angle $DAB=$angle $BAC=x$. Then

$$|AF-FB| = |AD\cos x - BD\sin x| = |a\cos^2 x - a\sin^2 x| = a|\cos 2x| = AE.$$

Solved also by Frank Ayres, Jr., Erna Jonas, C. W. Munshower, E. P. Starke, C. W. Trigg, Simon Vatriquant, Maud Willey and the proposer.

E45 [1933, 360]. *Proposed by W. R. Ransom, Tufts College.*

The ellipse of minimum area which can be circumscribed about a pair of equal, tangent circles, passes through the centers of its largest circles of curvature, and these centers and the two foci are the vertices of a square.

Solution by W. B. Campbell, Rangoon, Burma.

If we solve the equations, $(x+c)^2+y^2=c^2$ and $b^2x^2+a^2y^2=a^2b^2$, for equal roots, assuming $y\neq 0$, we find the real points of common tangency exist at $(\pm b^2/c,\ \pm b[2c^2-b^2]^{1/2}/c)$, and no other common points, provided that

$$\text{(I)} \qquad a^2 = b^4/(b^2 - c^2), \qquad \text{with } c^2 < b^2 < 2c^2.$$

For $y=0$, $a=2c$ and any value of b gives common tangents at $(\pm 2c,\ 0)$, but this ellipse also cuts the circles if $b^2<2c^2$.

Since the function, $F=a^2b^2$, is a minimum when the area, πab, is, we examine F subject to the conditions (I) and find it to be a minimum when $b^2=3c^2/2$. Then $a^2=9c^2/2$ and the area is $(3/2)\pi c^2\sqrt{3}$. For any ellipse, the largest radius of curvature is a^2/b, occurring at $(0,\ \pm b)$, with the centers of the circles of curvature at $(0,\ \pm[a^2-b^2]/b)$, which points are *not* on the ellipse when the area of the latter is a minimum.

If the circumscribing ellipse is required to be tangent to the circles at $(\pm 2c,\ 0)$, then $a^2=4c^2$ and the minimum area of the ellipse is $2\pi c^2\sqrt{2}$, when $b^2=2c^2$. The centers of the largest circles of curvature for *this* ellipse are at $(0,\ \pm c\sqrt{2})$, on the ellipse, and the foci are at $(\pm c\sqrt{2},\ 0)$, which four points are the vertices of a square.

Solved also by H. T. R. Aude, Lazarus Medveson, Jr., E. P. Starke and the proposer.

Editor's Note: As each of the solvers of this problem has pointed out, the original statement contains a defect which was unfortunately overlooked when it was published. The proposer intended to present for consideration the smallest ellipse circumscribing two equal tangent circles and tangent to them on their center-line, but neglected to insert the latter requirement. Although the theorem as proposed is false, "proofs" of it were nevertheless received from several of our readers.

E46 [1933, 360]. *Proposed by B. H. Brown, Dartmouth College.*

Show that $\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\cdots+1/N$ is never an integer for any N.

Solution by Benjamin Rosenbaum, Milford High School, Milford, Connecticut

Let that one of the denominators in the given series which contains the highest power of 2 as a factor be $2^k\cdot a$, where a is an odd integer. Then $a=1$, since if any larger odd multiple of 2^k were present among the denominators, then a would be greater than 2, and 2^{k+1} would be among the denominators. Then the least common denominator of all the given fractions is $2^k\cdot b$, where b is an odd integer. When each of the fractions is reduced to this common denominator, each one will have an even numerator excepting only $1/2^k=b/(2^kb)$. Hence the sum of all the numerators after this reduction must be odd, and hence not divisible by the denominator, $2^k\cdot b$. Therefore the sum of the series can never be an integer.

Mr. C. N. Haskins of Lebanon, New Hampshire, points out that several solutions of this problem have been published in foreign languages, as for example, by

L. Theisinger, Monatshefte für Mathematik u. Physik, vol. 26(1915), pp. 132–133;

R. Obláth, Mathematikai és Physikai Lapok, vol. 27 (1918), pp. 93–94;

J. Kürschák, Ibid., pp. 299–300;

Polyá-Szegö, Aufgaben u. Lehrsätze aus der Analysis, vol. 2 (1925), pp. 159, 381.

Solved also by C. E. Buell, M. J. Macphail, Raphael Robinson, J. Rosenbaum, E. P. Starke, Simon Vatriquant and the proposer.

Advanced Problems

Send all communications about Advanced Problems and Solutions to Otto Dunkel, Washington University, St. Louis, Mo. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide.

Problems containing results believed to be new, or extensions of old results are especially sought. The editorial work would be greatly facilitated, if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known textbooks or results found in readily accessible sources, will not be proposed as problems for solution in this department. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

PROBLEMS FOR SOLUTION

3654. *Proposed by N. A. Court, University of Oklahoma.*

The tangent and the normal at a variable point of a certain curve determine a pair of conjugate points of a given involution on a given straight line. Find the curve.

3655. *Proposed by Thurman Andrew, Quincy, Mass.*

The distributions of reds and blues (errors or objects) are each defined by the equation:

$$y = che^{-h^2x^2};$$

where c and h are constants and x may take values $0, \pm 1, \pm 2, \cdots$ only.

a) What is the probability that a red, chosen at random, will be exactly n greater than a blue, chosen at random?

b) Given $c=113$, $h=\frac{1}{2}$; find the probability that the red will be 3 larger than the blue.

3656. *Proposed by Raphael Robinson, University of California at Berkeley.*

If the vertices of a simplex S in n dimensions are numbered from 0 to n, the lengths of the edges are $e_{01}, e_{02}, \cdots, e_{n-1,n}$, and the lengths of the medians $m_0, m_1, \cdots, m_n$, then

$$m_0 = \frac{1}{n}\sqrt{n(e_{01}^2 + e_{02}^2 + \cdots + e_{0n}^2) - (e_{12}^2 + e_{13}^2 + \cdots + e_{n-1,n}^2)},$$

and $m_1, \cdots, m_n$ are given by similar formulas.

3657. *Proposed by W. B. Campbell, Rangoon, Burma.*

A hollow circular disk, of radius a, with a row of small apertures distributed along its curved surface, rotates in a vertical plane with peripheral velocity v. It is supplied with water from a reservoir in which the height of the free surface above the center of the disk is b. Assuming that the initial velocity of each drop ejected is the vector sum of the tangential velocity of the aperture and a radial velocity corresponding to the head, find the envelope of the paths of the drops. Classify the resulting loci.

3658. *Proposed by J. M. Feld, Brooklyn College of the City of New York.*

The Simpson line of a point P on the circumcircle of a triangle ABC is the tangent at the vertex of a parabola tangent to the sides of ABC and having its focus at P. See the solution of 3535 [1933, 56].

3659. *Proposed by R. Goormaghtigh, Bruges, Belgium.*

Let H_1, H_2, H_3, H_4 be the feet of the altitudes of a tetrahedron $A_1A_2A_3A_4$ and $A_{12}A_{13}A_{14}$, $A_{23}A_{24}A_{21}$, $A_{34}A_{31}A_{32}$, $A_{41}A_{42}A_{43}$ the antipedal triangles of $A_2A_3A_4$, $A_3A_4A_1$, $A_4A_1A_2$, $A_1A_2A_3$ with respect to H_1, H_2, H_3, H_4 respectively.

Prove that the perpendiculars drawn from A_{12}, A_{13}, A_{14} on $A_1A_3A_4$, $A_1A_4A_2$, $A_1A_2A_3$ are concurrent at a point α_1: and that, if α_2, α_3, α_4 are the points similar to α_1, $A_1A_2A_3A_4$ and the tetrahedron formed by the mid-points of $H_1\alpha_1$, $H_2\alpha_2$, $H_3\alpha_3$, $H_4\alpha_4$ are orthologic.

3660. *Proposed by Lulu Hofman, New York City.*

Given a definite projective transformation T of a primitive one-dimensional form into itself, expressed with reference to a given particular anharmonic ratio coordinate system C. To characterize all other anharmonic ratio coordinate systems C' such that with reference to them T has the same analytic expression as with reference to C.

3661. *Proposed by Maud Willey, Long Beach, Miss.*

What is the order of the group of movements into itself, in space of n dimensions, of the regular n-dimensional solid whose 2^n vertices have the coordinates $(\pm 1, \pm 1, \cdots, \pm 1)$?

SOLUTIONS

3589 [1933, 52]. *Proposed by R. E. Gaines, University of Richmond.*

If a tangent to the cardioid $\rho = a(1+\cos\theta)$ at the point P_1 cuts the curve again in P_2 and P_3, the area of the segment cut off by the chord P_2P_3 is $\frac{3}{4}a^2(\phi - \sin\phi)$, where ϕ is the angle which the chord subtends at the origin.

Solution by the Proposer.

The area desired is the difference between the area of the sector of the cardioid and the area of the enclosed triangle, and it is easily found to be

$$(1)\qquad A = \tfrac{1}{2}a^2\left[\frac{3}{2}\phi + 2(\sin\theta_3 - \sin\theta_2) + \tfrac{1}{4}(\sin 2\theta_3 - \sin 2\theta_2) - \frac{\rho_2\rho_3}{a^2}\sin\phi\right],$$
$$\phi = \theta_3 - \theta_2.$$

In order to reduce the right side the following results, which appear in the solution of 3585 [1933, 612], will be used:

$$(2)\qquad \begin{gathered} \rho_2 + \rho_3 = 2(a - \rho_1), \qquad 2\rho_2\rho_3 = a\rho_1, \qquad \cos\phi = \frac{4\rho_1}{a} - 1, \\ \sin\theta_2 \sin\theta_3 = \frac{3\rho_1}{2a}, \qquad \cos\theta_2\cos\theta_3 = \frac{5\rho_1}{2a} - 1. \end{gathered}$$

Thus

$$(3)\qquad \cos(\theta_2 + \theta_3) = \frac{\rho_1}{a} - 1, \qquad \frac{\rho_1}{a} = \tfrac{1}{2}\cos^2\tfrac{1}{2}\phi.$$

For $\rho_1 = \frac{1}{2}a$, $\theta_2 = \theta_3 = 4\pi/3$, $\cos(\theta_2+\theta_3) = -\frac{1}{2}$. As ρ_1 decreases from $\frac{1}{2}a$ to zero, the first equation of (3) shows that $\theta_2+\theta_3$ increases from $8\pi/3$ to 3π, or $\frac{1}{2}(\theta_2+\theta_3)$ increases from $4\pi/3$ to $3\pi/2$. Hence $\cos\frac{1}{2}(\theta_2+\theta_3)$ is negative. From (2) it follows that ϕ increases from zero to π.

We have the following reductions in terms of (1):

$$\sin 2\theta_3 - \sin 2\theta_2 = 2\cos(\theta_3 + \theta_2)\sin\phi = 2\left(\frac{\rho_1}{a} - 1\right)\sin\phi,$$
$$\sin\theta_3 - \sin\theta_2 = 2\cos\tfrac{1}{2}(\theta_3 + \theta_2)\sin\tfrac{1}{2}\phi,$$
$$\frac{\rho_1}{a} - 1 = \cos(\theta_3 + \theta_2) = 2\cos^2\tfrac{1}{2}(\theta_3 + \theta_2) - 1, \text{ or}$$
$$\cos^2\tfrac{1}{2}(\theta_3 + \theta_2) = \frac{\rho_1}{2a} = \tfrac{1}{4}\cos^2\tfrac{1}{2}\phi, \qquad \cos\tfrac{1}{2}(\theta_3 + \theta_2) = -\tfrac{1}{2}\cos\tfrac{1}{2}\phi.$$
$$\sin\theta_3 - \sin\theta_2 = -\tfrac{1}{2}\sin\phi.$$

Hence

$$\begin{aligned} A &= \frac{a^2}{2}\left[\frac{3}{2}\phi - \sin\phi + \tfrac{1}{2}\left(\frac{\rho_1}{a} - 1\right)\sin\phi - \tfrac{1}{2}\frac{\rho_1}{a}\sin\phi\right], \\ &= \tfrac{3}{4}a^2[\phi - \sin\phi]. \end{aligned}$$

3590. [1933, 52] *Proposed by G. E. Raynor, Lehigh University.*

Prove that Simpson's (one-third) rule gives the correct value of the integrals

$$\int_0^\pi \sin^{2m}\theta d\theta, \qquad \int_0^\pi \cos^{2m}\theta d\theta,$$

where m is a positive integer, if the number of intervals used is greater than two.

Solution by J. A. Bullard, University of Vermont.

It is well known that

$$\text{(1)}\qquad \begin{aligned} \cos^{2m}\theta &= 2^{-2m}\left[{}_{2m}C_m + 2\sum_{p=1}^{m} {}_{2m}C_{m-p}\cos 2p\theta\right],\\ \sin^{2m}\theta &= 2^{-2m}\left[{}_{2m}C_m + 2\sum_{p=1}^{m} (-1)^p\, {}_{2m}C_{m-p}\cos 2p\theta\right]. \end{aligned}$$

The values of the two given integrals may be found by integrating each term on the right in the equations (1). Each term after the summation sign yields zero, and hence the common value of the two integrals is

$$\text{(2)}\qquad 2^{-2m}\pi\ {}_{2m}C_m = B\left(\frac{2m+1}{2}, \frac{1}{2}\right).$$

If we now apply Simpson's rule to each term on the right, it is clear that the constant term on the right gives the result (2) regardless of the number of intervals. Hence, in order to give the correct result by this rule, the sum of the results for the cosine terms must be zero. It will be shown that this is the case if the number of divisions, $2k$, exceeds $2m$; for in this case $k>m\geqq p$, and it will be shown that each cosine term then gives zero by the rule. With $2kh=\pi$, Simpson's rule applied to $\cos 2p\theta$ gives

$$\text{(3)}\qquad \begin{aligned} &\tfrac{1}{3}h\left[\cos 0 + \cos 2p\pi + 4\sum_{i=1}^{k}\cos 2ph(2i-1) + 2\sum_{i=1}^{k-1}\cos 4phi\right],\\ &= \tfrac{1}{3}h[2 + 2\sin 4pkh \csc 2ph + 2\sin 2p(k-1)h\cos 2pkh\csc 2ph]\\ &= \tfrac{1}{3}h[2+0-2] = 0, \end{aligned}$$

since $2ph=p\pi/k$ is not an integral multiple of π, and $k>m\geqq 1$.

Solved also by L. S. Johnston, J. W. Hurst, and the proposer.

3592 [1933, 53]. *Proposed by V. Thébault, Le Mans, France.*

Given the tetrahedron $ABCD$; construct five equal spheres so that one of them shall be tangent to the remaining four, and each of these shall touch three faces of the tetrahedron.

Solution by A. D. Bradley, Hunter College.

Construct T, the incenter of $ABCD$. Let R be the circum-radius and r the in-radius of $ABCD$. Construct

$$\text{(1)}\qquad x = \frac{Rr}{2r+R}\cdot$$

On CT determine C' such that its distance from each of the three faces of the trihedral angle C is x; then a sphere of radius x and center C' is tangent to the

faces of the trihedral angle C. Determine D', B' and A' in the same way on the lines TD, TB and TA; then A', B', C' and D' are the centers of spheres of radius x each tangent to three faces of $ABCD$.

The plane of $A'C'D'$ is parallel to the plane of ACD. Let N be the foot of the perpendicular from T to the plane ACD, and let TN cut the plane $A'C'D'$ in M. Then

$$\frac{TC'}{TC} = \frac{TM}{TN} = \frac{r-x}{r},$$

and there are three other similar equations. Hence T is the center of similitude of $A'B'C'D'$ and $ABCD$. Let R' be the circum-radius of $A'B'C'D'$; then

$$\frac{R'}{R} = \frac{r-x}{r} = \frac{2x}{R}, \quad \text{or} \quad R' = 2x,$$

by aid of (1). Hence a sphere of radius x and with its center at the circumcenter of $A'B'C'D'$ is tangent to the spheres A', B', C' and D'.

Note: This solution was obtained by considering a corresponding problem in the plane: Given the triangle ABC; construct four equal circles so that one of them shall be tangent to the remaining three, and each of these shall touch two sides of the triangle.

Solved also by B. Hoffman and the proposer.

3594 [1933, 115]. *Proposed by H. T. R. Aude, Colgate University.*

Find sets of integers for rational right triangles which, as the numbers increase, approach a $30°-60°$ right triangle.

Solution by Hugh Hamilton, Brown University

The general integral right triangle, which is equivalent to the general rational right triangle, has for its hypotenuse the length m^2+n^2, and for its legs the lengths m^2-n^2, $2mn$, where m and n are positive integers. The legs are essentially interchangeable by the substitution $M=m+n$, $N=m-n$, where M and N are positive integers. If m and n are both odd or both even, one may set $2M=m+n$, $2N=m-n$. The right triangles with sides in the corresponding order

$$\begin{array}{ccc} m^2+n^2, & m^2-n^2, & 2mn \\ M^2+N^2, & 2MN, & M^2-N^2 \end{array}$$

are then similar. Hence one need consider as the angle θ which approaches $60°$ only the angle between m^2-n^2 and m^2+n^2. It follows that

$$\tan\frac{\theta}{2} = \frac{\sin\theta}{\cos\theta+1} = \frac{n}{m};$$

and m and n must be chosen so that m/n approaches $\sqrt{3}$. The development of this radical in a continued fraction is

$$\sqrt{3} = 1 + \frac{1}{1+}\,\frac{1}{2+}\,\frac{1}{1+}\,\frac{1}{2+}\cdots;$$

and m and n may be selected as the numerator and denominator, respectively, of the convergents m_i/n_i of this continued fraction. One has thus the infinite sequence of pairs of values

$$m_i = 1,\ 2,\ 5,\ 7,\ 19,\ \cdots$$
$$n_i = 1,\ 1,\ 3,\ 4,\ 11,\ \cdots,$$

and from these values the lengths of the sides of the triangle may be computed.

Solved also by Norman Anning, A. S. Householder, A. Pelletier, F. Underwood, S. Vatriquant, and the proposer.

Note by Otto Dunkel. In the series of right triangles

$$m_i^2 + n_i^2, \qquad m_i^2 - n_i^2, \qquad 2m_in_i, \tag{1}$$

the three sides will have no common factor for i even, but for i odd there will be a highest common factor 2. For since m_i, n_i are prime to each other, if there is a highest common factor of the set of integers (1) it must be 2. The integers m_i, n_i satisfy the relations

$$\begin{aligned} m_{2j+1} &= 3n_{2j} + m_{2j}, \\ n_{2j+1} &= n_{2j} + m_{2j}. \end{aligned} \tag{2}$$

Hence m_{2j}, n_{2j} cannot both be odd, for if they were both odd m_{2j+1} and n_{2j+1} would have the common factor 2. It follows from this also that the two integers m_{2j+1}, n_{2j+1} must be both odd. This proves the above statement.

We also have

$$\begin{aligned} m_{2j+2} &= 3n_{2j} + 2m_{2j}, \\ n_{2j+2} &= 2n_{2j} + m_{2j}; \end{aligned} \tag{3}$$

and thus, if m_{2j} is odd and n_{2j} is even, m_{2j+2} is even and n_{2j+2} is odd. If m_{2j} is even and n_{2j} is odd, then m_{2j+2} is odd and n_{2j+2} is even.

Associated with (2) are the formulae

$$\begin{aligned} 2m_{2j} &= 3n_{2j-1} + m_{2j-1} \\ 2n_{2j} &= n_{2j-1} + m_{2j-1}. \end{aligned} \tag{2'}$$

A simple computation with (2) gives

$$\begin{aligned} \tfrac{1}{2}(m_{2j+1}^2 + n_{2j+1}^2) &= (2n_{2j} + m_{2j})^2 + n_{2j}^2, \\ \tfrac{1}{2}(m_{2j+1}^2 - n_{2j+1}^2) &= 2(2n_{2j} + m_{2j})n_{2j}, \\ \tfrac{1}{2}(2m_{2j+1}n_{2j+1}) &= (2n_{2j} + m_{2j})^2 - n_{2j}^2. \end{aligned} \tag{4}$$

If the equations (2′) are used in the same way the fraction $\frac{1}{2}$ must be transferred to the right side. Thus we have another substitution which may be used in place of the two of the solution above

$$M_i = 2n_i + m_i, \qquad N_i = n_i,$$

$$\lim_{i\to\infty} \frac{M_i}{N_i} = 2 + \lim_{i\to\infty} \frac{m_i}{n_i} = 2 + \sqrt{3}. \tag{5}$$

This tells us that if θ is the angle between $M_i^2 - N_i^2$ and $M_i^2 + N_i^2$, then θ approaches 30°.

From (2), (2′), and (5) we find

$$M_{2j} + N_{2j} = m_{2j+1}, \qquad M_{2j-1} + N_{2j-1} = 2m_{2j},$$

$$M_{2j} - N_{2j} = n_{2j+1}, \qquad M_{2j-1} - N_{2j-1} = 2n_{2j},$$

and these are the substitutions in the above solution.

3595 [1933, 115]. *Proposed by Nathan Altshiller-Court, University of Oklahoma.*

Through the edges a, b, c, of a trihedral angle planes are drawn perpendicular to the faces bc, ca, ab and cutting these faces along the lines e, f, g, respectively. Show that the faces of the given trihedral angle bisect the dihedral angles of the trihedral angle formed by the lines e, f, g.

Solution by J. Rosenbaum, Milford, Conn.

Let us first assume that no edge of the trihedral angle is perpendicular to the opposite face. Let O be the vertex of the given trihedral angle. From any point E, not the vertex O, on e draw, in the plane of bc, BC perpendicular to OE, cutting b and c in B and C respectively; draw from C in the plane ca the perpendicular to f, cutting f and a in F and A respectively. Since the plane COE is perpendicular to AOE, CE is perpendicular to AOE and also to AE. Similarly, CF is perpendicular to BF. Let AE and BF meet in L, which must be the orthocenter of ABC. Let CL meet AB in G; then CG is perpendicular to AB. Since CE is perpendicular to AOE, it is perpendicular to OL. In the same way CF is seen to be perpendicular to OL; and thus OL is perpendicular to ABC. Since CG is perpendicular to AB, and OL is perpendicular to ABC, OCG is perpendicular to OAB. Hence G lies on g.

It is well known that the altitudes of ABC bisect internally or externally the angles of triangle EFG; and hence L is equidistant from the sides of this triangle. Let M and N be the feet of the perpendiculars from L to GE and GF. Then from the symmetry of the figures $OGLM$ and $OGLN$ we see that the plane OGL bisects the dihedral angle $OG-PE$, and OAB bisects the same angle. There are similar results for the other faces, and this proves the theorem.

Solved also by A. Pelletier.

Editorial Note. Following the above solution, the solver points out that if

one edge, say a, of the trihedral angle is perpendicular to the opposite face, the line e is not determined: but that in this case, if the plane through a perpendicular to bc is taken so as to be coaxial with the two similar planes through b and c, the conclusion of the theorem will still hold. This is interesting additional information; but the important thing is that there is nothing in the hypotheses of the theorem as stated to compel us to take the plane through a in this way, that if we take it in some other way the conclusion will not follow, and that therefore the theorem, as stated in the problem, is *not true*. A theorem states that a certain conclusion follows from certain hypotheses. If a single example, however special, can be found for which the hypotheses of the theorem hold while the conclusion does not, then the theorem is a false theorem. There is, of course, a still more special case, not mentioned by the solver, namely that of the trirectangular trihedral angle. The problem proposed should have had the words "of whose face angles not more than one is a right angle" inserted after "trihedral angle" in the first line. Otherwise it is not possible to "show" that the conclusion will follow.

3596 [1933, 115]. *Proposed by L. S. Johnston, University of Detroit.*

Given

$$ {}_sH_r = \frac{s(s+1)(s+2)\cdots(s+r-1)}{r!}, \qquad {}_sC_r = \frac{s(s-1)(s-2)\cdots(s-r+1)}{r!}, $$

where s and r are any positive integers; and

$$ {}_sH_0 = {}_sC_0 = 1 $$

where s is any positive integer: prove that

$$ \phi(s, p) = \sum_{k=0}^{p}(-1)^k {}_sH_{p-k}\, {}_sC_k = 0 $$

for all positive integral values of s and p.

Solution by Jewell C. Hughes, Hunter College.

If ${}_sH_{-1} = {}_{s-2}C_{-1} = 0$,

$$ \phi(s, p-1) = \sum_{k=0}^{p}(-1)^k {}_sC_k\, {}_sH_{p-1-k}, $$

$$ (1) \quad \phi(s, p) - \phi(s, p-1) = \sum_{k=0}^{p}(-1)^k {}_sC_k\, {}_{s-1}H_{p-k}, $$

$$ (2) \quad = \sum_{k=0}^{p}(-1)^k {}_{s-1}C_{k-1}\, {}_{s-1}H_{p-k} + \sum_{k=0}^{p}(-1)^k {}_{s-1}C_k\, {}_{s-1}H_{p-k}, $$

where (1) follows from ${}_sH_r - {}_sH_{r-1} = {}_{s-1}H_r$, and (2) results from ${}_sC_k = {}_{s-1}C_{k-1} + {}_{s-1}C_k$. In the first summation on the right of (2) the term for $k=0$ drops out,

and if we replace k by $k+1$ this term reduces to $-\phi(s-1,\ p-1)$. Hence (2) may be written

$$(3) \qquad \phi(s,\ p) - \phi(s,\ p - 1) = \phi(s - 1,\ p) - \phi(s - 1,\ p - 1).$$

By a continued application of (3) we have

$$(4) \qquad \phi(s,\ p) - \phi(s,\ p - 1) = \phi(1,\ p) - \phi(1,\ p - 1) = 0,$$

and by repeated application of (4) we have

$$\phi(s,\ p) = \phi(s,\ 1) = s - s = 0,$$

which is the desired result.

Solved also by Frank Ayres, Jr., J. A. Bullard, P. S. Dwyer, M. F. Rosskopf, W. P. Udinski, F. Underwood, Morgan Ward, and M. Weinberger (two solutions).

Note by Otto Dunkel. We may write

$$(-1)^p \frac{p!}{s} \phi(s, p) = \sum_{k=0}^{p}(-1)^{p-k}{}_pC_k f(k) = \Delta^p f(0) = 0,$$

where $f(k)=(s-k+1)(s-k+2) \cdots (s-k+p-1)$, a polynomial of degree $p-1$ in k, and where Δ is the difference operator.

3598 [1933, 116]. *Proposed by V. Thébault, Le Mans, France.*

On the lines PA, PB, PC, PD, joining a point P in space to the vertices of the *orthocentric* tetrahedron $ABCD$ are marked the inverse points A_1, B_1, C_1, D_1, of the vertices in the inversion $(P,\ k)$. The planes perpendicular at these points to the lines PA_1, PB_1, PC_1, PD_1, form a tetrahedron $\alpha\beta\gamma\delta$. Show that the planes perpendicular at P to the lines $P\alpha$, $P\beta$, $P\gamma$, $P\delta$, cut the planes $\beta\gamma\delta$, $\gamma\delta\alpha$, $\delta\alpha\beta$, $\alpha\beta\gamma$ along four coplanar lines, and that the plane of these lines is perpendicular to the line joining P to the orthocenter of the tetrahedron $ABCD$.

Solution by R. Goormaghtigh, Bruges, Belgium.

Let α', β', γ', δ' be the projections of P on the faces of the tetrahedron. The planes perpendicular at B_1, C_1, D_1 to PB_1, PC_1, PD_1 are the inverses of the spheres having PB, PC, PD as diameters and α is the inverse of α'. The plane $\beta\gamma\delta$ is the inverse of the sphere $A\beta'\gamma'\delta'$ having AP as diameter, and the inverse of the intersection of the plane $\beta\gamma\delta$ and the plane π_a perpendicular at P to $P\alpha$ is a circle Γ_a passing through P and the projection of P on the altitude from A, and placed in a plane parallel to BCD. But Γ_a and the three similar circles belong to the sphere ω having as diameter the distance from P to the orthocenter H of the tetrahedron. Hence the planes $\beta\gamma\delta$, $\gamma\delta\alpha$, $\delta\alpha\beta$, $\alpha\beta\gamma$ cut the planes π_a, π_b, π_c, π_d respectively, along four lines belonging to the plane, inverse of ω and therefore perpendicular to PH.

Solved also by the proposer.

Note by Otto Dunkel. The following details may be of aid in reading the above. Angle $PB_1\alpha$ is right, since the perpendicular plane at B_1 to PB contains the vertices α, γ, δ of the second tetrahedron. Thus $PB_1\alpha$, $PC_1\alpha$, $PD_1\alpha$ are right angles, and the sphere $(P\alpha)$ with $P\alpha$ as diameter passes through B_1, C_1, D_1. The inverse of $(P\alpha)$ is thus the plane BCD, and hence α', the inverse of α, is the intersection of the diameter $P\alpha$ with the face BCD.

Similarly, since $P\beta'$ is perpendicular to the face ACD, $P\beta'A$ is a right angle. There are two other right angles. $P\gamma'A$ and $P\delta'A$; and the sphere (PA) with diameter PA passes through β', γ', δ'. Hence the inverse of (PA) is the plane of A_1, β, γ, δ.

We now use the fact that the altitudes of $ABCD$ meet in a point H. Since AH is perpendicular to the face BCD, and π_a is the plane perpendicular to $P\alpha$ at P, π_a is parallel to the face BCD. Let $\bar{A}$ be the intersection of AH and π_a. The sphere (PH) with diameter PH passes through $\bar{A}$, since $H\bar{A}P$ is a right angle, and for similar reasons it passes through $\bar{B}$, $\bar{C}$, $\bar{D}$, where these three points are defined in the same manner as $\bar{A}$. The sphere (PA) also contains $\bar{A}$, and it intersects $(PH)=\omega$ in a circle Γ_a in the plane π_a and with the diameter $P\bar{A}$, since π_a is perpendicular to the straight line $AH\bar{A}$ and PA and PH are diameters. Thus ω cuts (PA), (PB), (PC), (PD) in the circles Γ_a, Γ_b, Γ_c, Γ_d with planes π_a, π_b, π_c, π_d parallel to the faces of $ABCD$. The inverse of Γ_a is l_a, the straight line intersection of $\beta\gamma\delta$ and π_a, for (PA) inverts into $\beta\gamma\delta$ and π_a inverts into itself. But Γ_a lies also upon (PH) and hence l_a, its inverse, lies in that plane π perpendicular to PH which is the inverse of (PH). Hence l_a, l_b, l_c, l_d lie in a plane π perpendicular to PH.

NEWS AND NOTICES

Readers are invited to contribute to the general interest of this department by sending items to Professor J. H. Weaver, Ohio State University, Columbus, Ohio.

The Southern Intercollegiate Mathematics Association was organized on October 21, 1933, at a meeting held at Centenary College, Shreveport, La. The objects of the association are to promote and support contests in mathematics and to encourage mutual and intellectual fellowship among its members. The institutions represented at the meeting were: Centenary College, Louisiana Polytechnic Institute, Louisiana State Normal College, Loyola University, McMurry College, Millsaps College, Mississippi Woman's College, and Simmons University. Annual contests will be held in algebra, trigonometry, analytical geometry, calculus, and a comprehensive examination in all these subjects. Officers were elected as follows: President, Professor Israel Maizlish, Centenary College; Vice-president, Professor J. E. Burnam, Simmons University; Secretary-Treasurer, Frances White, Louisiana Polytechnic Institute. The first annual meeting is scheduled for May 5, 1934, at Louisiana State Normal College, Natchitoches, La.

The following German mathematicians have been appointed visiting professors or lecturers in American colleges or universities: Professor Felix Bernstein of Göttingen at Columbia University; Professor S. Bochner of Munich at Princeton; Professor Hans Lewy of Göttingen at Brown University; Professor Emmy Noether of Göttingen at Bryn Mawr; Professor Otto Szasz of Frankfurt at the Massachusetts Institute of Technology.

The Abbé Georges Lemaître, professor of astrophysics at the University of Louvain, will be visiting professor at the Catholic University of America in Washington, D.C. He will lecture on the astronomical applications of the theory of relativity and will conduct a seminar for advanced students of physics and mathematics in the graduate school of that institution.

Professor Nathan Altshiller-Court of the University of Oklahoma delivered an address before the mathematics section of the Kansas State Teachers Association on November 4, 1933, on the subject "Modern solid geometry."

The new science building at Radcliffe College has been named William Elwood Byerly Hall in honor of W. E. Byerly, Perkins professor of mathematics emeritus at Harvard University, who served for more than thirty years as Chairman of the Academic Board at Radcliffe.

Professor Louis Karpinski is lecturing in Europe during the current year. In November he lectured at the University of Athens on the topic "The connections between the development of the early Greek mathematics and the progress of science and civilization." During Thanksgiving week he lectured at Cairo, Egypt, on the topic "The importance of the ancient Egyptian mathematics." He also lectured at Beirut, Jerusalem, and at the American College in Athens.

Professor Edward Kasner of Columbia University is giving a course of lectures on the concepts of modern mathematics at the New School for Social Research. His subjects are "Infinity," "The fourth dimension," "Einstein."

David Eugene Smith, Professor Emeritus at Columbia University, spent four months during last winter and spring in Persia, Iraq, and Syria. During his stay he secured more than one hundred and fifty manuscripts, mostly Persian, Arabic, and Hebrew. Among them were several Arabic translations of the Greek classics and a fourteenth century manuscript of Omar Khayyâm's Algebra. Among the translations from the Greek are two copies of Euclid's "Elements," Archimedes' "On the circle," and Aristarchus' "On astronomy."

Among the American mathematicians who have been elected fellows of the Econometric Society are the following: G. C. Evans, Harold Hotelling, C. F. Roos, and E. B. Wilson.

Professor Frank Schlesinger of Yale has been elected correspondent of the Paris Academy of Sciences in the section of astronomy.

Professor Wayne Dancer represented the Mathematical Association at the inauguration of Philip C. Nash as president of the University of the City of Toledo on October 16, 1933.

Assistant professor M. A. Basoco of the University of Nebraska has been promoted to an associate professorship of mathematics.

Dr. L. J. Briggs, head of the division of mechanics and acoustics of the United States Bureau of Standards, has been appointed director of that bureau.

Dr. E. H. C. Hildebrandt has been appointed to an instructorship at Brooklyn College.

Dr. H. L. Krall has been appointed to an instructorship at Pennsylvania State College.

Dr. C. W. Mendel of the University of Chicago has been appointed to an instructorship at the University of Cincinnati.

Dr. C. B. Morrey has been appointed to an instructorship at the University of California at Berkeley.

Dr. L. T. Moston of Harvard University has been appointed to the chair of mathematics at Waynesburg College.

Dr. W. M. Rust has been appointed to an instructorship at Harvard University.

Assistant Professor T. O. Walton of Kalamazoo College has been promoted to a professorship.

Professor E. S. Crawley, professor emeritus of mathematics at the University of Pennsylvania, died October 18, 1933, at the age of 71. He was a charter member of the Mathematical Association.

Professor J. T. Erwin, retired professor of mathematics at the George Washington University, died October 15, 1933, at the age of 63.

On the death of the late Dr. J. C. Glashan, the distinguished Canadian mathematician, his extensive library was left to his widow who now wishes to dispose of it. An estimate, based upon a hastily prepared catalogue, shows that it consists of about 5000 volumes, chiefly of standard modern treatises on pure and applied mathematics. It would be an excellent library for a college or a private collection and we are assured on high authority that it is well worth an examination by anyone interested. Contact with the owner may be made through Professor James T. Shotwell, of the Carnegie Endowment for International Peace, 405 West 117th Street, New York City.

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BOOKS FOR REVIEW should be addressed to R. A. JOHNSON, Brooklyn College, 66 Court Street, Brooklyn, N.Y.

BUSINESS CORRESPONDENCE should be addressed to the SECRETARY-TREASURER of the Association, W. D. CAIRNS, 33 Peters Hall, Oberlin, Ohio.

CHANGE OF ADDRESS: Members should send notice of any change of address to the SECRETARY-TREASURER, W. D. CAIRNS, Oberlin, Ohio, before the 10th of each month.

MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Eighteenth Summer Meeting of the Association, Williamstown, Mass., Sept. 3-4, 1934.

The following is a list of the sections of the Association, with dates of those section meetings which have been scheduled for 1934 and reported to the Secretary.

ALLEGHENY MOUNTAIN, Pittsburgh, Pa., Feb. 10; Washington, Pa., May 5.
ILLINOIS, Jacksonville, May 4-5.
INDIANA, La Fayette, May 11-12.
IOWA, Des Moines, April 20-21.
KANSAS, Topeka, Mar. 17.
KENTUCKY, May.
LOUISIANA-MISSISSIPPI, Jackson, Miss., Mar. 23-24.
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Williamsburg, Va., May.
MICHIGAN, Ann Arbor, Mar. 17.
MINNESOTA, Northfield, May 12.
MISSOURI.
NEBRASKA.
OHIO, Columbus, Apr. 5.
OKLAHOMA, Oklahoma City, Feb. 9.
PHILADELPHIA, Philadelphia, Dec. 1.
ROCKY MOUNTAIN.
SOUTHEASTERN, University, Ala., Mar. 30-31.
SOUTHERN CALIFORNIA, Riverside, Mar. 3.
TEXAS.
WISCONSIN, Oshkosh, May 5.

AFFILIATED ORGANIZATIONS: THE NEW ENGLAND ASSOCIATION OF TEACHERS OF MATHEMATICS. THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS.

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THE EIGHTH ANNUAL MEETING OF THE PHILADELPHIA SECTION

The eighth annual meeting of the Philadelphia Section of the Mathematical Association of America was held at the New Jersey College for Women, Rutgers University, on Saturday, December 2, 1933, Professors Kline and Smith presiding.

The attendance was sixty-five, including the following twenty-nine members of the Association: H. W. Brinkmann, L. H. Bunyan, P. A. Caris, J. W. Clawson, E. H. Cutler, Arnold Dresden, Edward Kasner, J. R. Kline, P. A. Knedler, P. V. Kunkel, V. V. Latshaw, D. L. McDonough, A. E. Meder, Jr., H. H. Mitchell, Richard Morris, C. A. Nelson, T. S. Peterson, G. E. Raynor, J. B. Reynolds, George Rosengarten, J. A. Roulton, J. A. Shohat, C. A. Shook, L. L. Smail, W. M. Smith, R. M. Walter, R. L. Wilder, A. H. Wilson, C. R. Wilson.

At the business meeting, the following officers were chosen for next year: Chairman, W. M. Smith, Lafayette College; Secretary, P. A. Caris, University of Pennsylvania; Program Committee, Professors Morris, Clawson, Brinkmann.

The next meeting will be held on Saturday, December 1, 1934, at Philadelphia.

The following papers were presented:

1. "Binomial congruences" by Professor E. P. Starke, Rutgers University.

2. "The interpretation of imaginaries in projective geometry" by Professor H. W. Brinkmann, Swarthmore College.

3. "Connectivity of spaces" by Professor R. L. Wilder, University of Michigan and Institute for Advanced Study.

4. "Polygons and groups" by Professor Edward Kasner, Columbia University.

Abstracts of the papers follow:

1. In an attempt to render the elementary theory of numbers more appropriate for undergraduate instruction, the theory of binomial congruences is developed on an elementary basis, employing nothing more advanced than the notion of a primitive root, modulo m. The objectives gained in the paper are: the necessary and sufficient conditions for the existence of a solution of a binomial congruence to a general modulus; the determination of the number of such solutions; the development of a method for computing the solutions.

2. Professor Brinkmann discussed the following topics: harmonic ranges with especial attention to imaginary point-pairs; involutions whose double points are imaginary; pairing the conics of a pencil of conics intersecting in distinct points. Brief mention was also made of some of the properties of a conic tangent to the line at infinity at one of the circular points.

3. Professor Wilder's paper was of an expository nature. Commencing with the definition of Riemann for "$(m+1)$-fold connectedness" of surfaces, he outlined the development of modern notions of connectivities of higher dimensional polyhedrals. He then gave a brief description of recent extensions of these

ideas by topologists to arbitrary point sets on general spaces, indicating various connectivity groups and the corresponding connectivity numbers.

4. The first part of Professor Kasner's paper dealt with material published in the MONTHLY for March 1903. The second part extended that material to bipoint transformations connected with a given polygon. Brief mention was also made of $(r-1)$-point transformations and their associated groups. It is the intention of the author to publish the general theory in detail.

P. A. CARIS, *Secretary*

THE ANNUAL MEETING OF THE MINNESOTA SECTION

The annual meeting of the Minnesota Section of the Mathematical Association of America was held at the College of St. Thomas, St. Paul, Minnesota, on Saturday, May 13, 1933. Sessions were held at 11:00 o'clock and at 2:30 o'clock with a luncheon at 1:00 o'clock.

Professor F. J. Taylor, chairman of the Section, presided at the morning session, with Professor C. J. Blackall presiding at the afternoon session. Fifty-five persons attended the meeting including the following twenty-five members of the Association: Sister Mary Aloysius, C. J. Blackall, Jessie W. Boyce, W. E. Brooke, W. H. Bussey, Elizabeth Carlson, J. O. Chellevold, Sister Claudette, C. H. Fischer, Gladys Gibbens, C. H. Gingrich, Borghild Gunstad, Solomon Guttman, W. L. Hart, H. E. Hartig, Dunham Jackson, C. M. Jensen, W. H. Kirchner, Marie M. Ness, L. J. Quaid, H. I. Tanjerd, F. J. Taylor, Ella Thorp, A. L. Underhill, Marian A. Wilder, G. L. Winkelmann.

At the afternoon session a vote of thanks was adopted as a sign of appreciation of the cordial hospitality of the College of St. Thomas, and the efforts of its department of mathematics. Officers for the following year were elected as follows: Chairman, C. J. Carlson, St. Olaf College; Secretary, A. L. Underhill, University of Minnesota; Members of the Executive Committee: H. E. Hartig, University of Minnesota; Inez Rundstrom, Gustavus Adolphus College.

The following eight papers were presented:

1. "A type of sine series" by Mary Elveback, University of Minnesota, by invitation.

2. "Cyclic numbers and cyclic equations" by Solomon Guttman, Minneapolis.

3. "A comparison of methods of teaching high school algebra" by R. M. Drake, University High School, Minneapolis, by invitation.

4. "The adequacy of student's Z test" by Marian A. Wilder, University of Minnesota.

5. "A new elementary course in college mathematics" by Professor W. L. Hart, University of Minnesota.

6. "The study of a new number system" by Reverend G. L. Winkelmann, St. John's University, Collegeville.

7. "Certain needed reorganization in the field of secondary school mathematics" by Professor L. B. Kinney, University High School, Minneapolis, by invitation.

8. "A proof of Weierstrass's theorem" by Professor Dunham Jackson, University of Minnesota.

Abstracts of these papers follow:

1. The paper is concerned with the representation of a function $f(x)$ by a series of the form $\sum c_k \sin (k-\frac{1}{2})x$ in which the coefficients have the values

$$c_k = \frac{2}{\pi}\int_0^\pi f(x) \sin (k - \tfrac{1}{2})x dx.$$

This series is proved to be equivalent to the Fourier series for the representation of an arbitrary summable function by showing that as an immediate consequence of Riemann's theorem the difference between the respective partial sums approaches zero uniformly as n becomes infinite. In this way it is shown simply not only that all theorems on convergence and uniformity of convergence true for one series are true for the other, but also that the series are equivalent to a certain extent in the degree of their convergence.

2. In this paper were shown the principal characteristics of cyclic numbers and the method of multiplication and division of ordinary numbers by any such number in any scale of notation. It was also pointed out that by cyclic permutations of the coefficients of an algebraic equation a series of equations are obtained where all have one common root.

3. This is a report of an experiment being conducted in the classes in ninth grade algebra in the University High School at the University of Minnesota. One group is being taught by an individual method of instruction and a second group, matched student for student with the first, is being taught by a group method. Individual differences in the former group are supposedly taken care of by differences in rate of progress; in the latter, by different levels of assignment.

Conclusions: Because the number of cases is comparatively small and the time limited to nine months, it would be impossible to generalize from the results of this one experiment. But if a number of such experiments were conducted and the results were, in most cases, in the same direction, then it would be possible to conclude that students are more likely to achieve up to their capacity for achieving under the group method of instruction than they are under the individual method.

4. The equation for the correlation surface of $y=(\bar{x}-m)/\sigma$ and $z=(\bar{x}-m)/s$, where m and σ are the parameters of the population and $\bar{x}$ and s the statistics of the sample, has been given by Karl Pearson (Biometrika, vol. 23, Nov. 1931, pp. 1–9). In the present investigation the volume under this surface beyond the 5% point for each distribution has been evaluated for $n=3$ to 29 and also for $n=99$. From this volume, the percentage of error in calling a significant z ratio a significant deviation of the mean has been determined. The error is 74% for $n=3$, decreases to 26% for $n=25$, but is still 14% for $n=99$. When the plane

of truncation for the z variable is moved to the 1% point of the distribution, the y plane remaining at the 5% point, the error reduces to 72% for $n=3$ and for $n \geqq 25$ we have an error of less than 5%.

5. Professor Hart discussed a course entitled "An Introduction to the Mathematics of Business and Current Affairs" (three hours per week for one year; prerequisites: one year of high school algebra). This course is being given in the newly formed General College of the University of Minnesota as a part of a *two year* curriculum which does not primarily aim at the preparation of students for the *senior* colleges of the University of Minnesota. The new course described by Professor Hart consists of the following parts: (1) a brief review of the most useful parts of ninth grade algebra; (2) a treatment of logarithms; (3) the merest elements of the theory of progressions; (4) a fairly complete treatment of the mathematics of investment with the material restricted to problems involving simple data; (5) an elementary treatment of permutations, combinations and probability; (6) a discussion of the simplest features of the mathematical theory of life insurance and life annuities; (7) an introduction to statistics.

6. The general laws for changing a number from one radix to another were given and demonstrated. The comparison of radices 10 and 17 led to the magic squares of orders 3 and 4. Original devices for changing a magic square of order 4 to another square of the same order were explained and exemplified. These same devices proved the inter-relationship of the historic fourth-order magic squares. A working model of an original magic cube of order 4 with numbers from 1 to 16, which has the property that any horizontal, frontal, or profile cutting plane produces a magic square, was manipulated. Perfect numbers were expressed as an equilateral progression of circles or as an arithmetic progression having $a=1$, $l=n$, $d=1$. The four periods of the primitive roots of unity of $x^{17}=1$ were converted into a magic square. Finally it was observed that the sum of the numerical coefficients of the nth power of the simple binomial is 2^n.

7. The purpose of this paper is to raise the question of the desirability and possibility of bringing the content and organization of senior high school mathematics in the secondary schools into line with that in the junior high school. There are a number of advantages in the correlation and continuity of the various fields of mathematics as found in the junior high school courses. A plan was suggested for organizing the senior high school courses along the same lines.

8. This paper will be published in full in the MONTHLY.

A. L. UNDERHILL, *Secretary*

THE APRIL MEETING OF THE SOUTHEASTERN SECTION

The eleventh annual meeting of the Southeastern Section of the Mathematical Association of America was held at the University of Georgia, Athens, Georgia, on Friday and Saturday, April 7–8, 1933. Sessions were held in the afternoon and evening of the 7th, and on the morning of the 8th. The chairman of the Section, Professor T. M. Simpson presided, except Friday evening.

The attendance was sixty-eight including the following sixteen members of the Association: D. F. Barrow, Iris Callaway, J. B. Coleman, Forrest Cumming, U. P. Davis, B. F. Dostal, M. D. Earle, Floyd Field, H. K. Fulmer, Leslie J. Gaylord, P. R. Hill, F. W. Kokomoor, Z. M. Pirenian, H. A. Robinson, T. M. Simpson, R. P. Stephens.

On the evening of the 7th, a dinner was held in honor of the visiting speaker, Professor Frank Morley. At this time Dean R. P. Stephens presided, and President S. V. Sanford of the University of Georgia gave the address of welcome.

At a business session on the 8th the following officers were chosen for 1933–34: Chairman, D. F. Barrow, University of Georgia; Vice-Chairman, Floyd Field, Georgia School of Technology; Secretary-Treasurer, H. A. Robinson, Agnes Scott College.

The following ten papers were read:

1. "Arabic mathematics in the dark ages" by Professor F. W. Kokomoor, University of Florida.

2. "Methods of proof in geometry" by Superintendent Herman Usher, Buena Vista (Georgia) Public Schools, by invitation.

3. "Subfreshman mathematics" by Professor C. G. Phipps, University of Florida, by invitation.

4. "Importance of mathematics in basic education" by Dean Floyd Field, Georgia School of Technology.

5. "The old order changeth" by Professor Frank Morley, Johns Hopkins University, by invitation.

6. "A study in probability" by Professor P. R. Hill, University of Georgia.

7. "The Jacobian algorithm for periodic continued fractions as representing a cubic irrationality" by Professor J. B. Coleman, University of South Carolina.

8. "Regions and paths" by Professor Frank Morley, by invitation.

9. "Convergence of infinite exponentials" by Professor D. F. Barrow, University of Georgia.

10. "A time integral in the calculus of variations" by Professor W. S. Beckwith, Georgia State Teachers College, by invitation.

Abstracts of some of these papers follow, the numbers corresponding to the numbers in the list of titles:

3. The University of Florida is trying out a new plan to handle the student who is very poorly prepared in mathematics. At the end of the second week of school a simple test on algebra and arithmetic is given to all freshman classes. Those who are unable to score a certain percentage are put into a no-credit course where they are taught the fundamentals they should have learned earlier.

6. This is a preliminary account of an experiment designed to test the fundamental laws of probability. One hundred pennies dated 1919 and one hundred dated 1920 are mixed. One is drawn and its date recorded. It is returned and the two hundred are thoroughly mixed. Forty thousand drawings have already been made. The extreme accuracy of materials and procedure is attested by the

fact that at the end of forty thousand trials the total weight of the 1919 pennies differed from the total weight of the 1920 pennies by not more than two milligrams. The method has been devised to secure any desired degree of accuracy, however great.

7. In a previous paper conditions were found for the reducibility of the characteristic equation for periodic ternary continued fractions. Under the conditions imposed upon the partial quotients, in expanding by the Jacobian algorithm, it is here proved that the characteristic equation is always irreducible, so that the limits of the ratios of the convergents are always cubic irrationalities. The result is obtained by means of two types of continuants in terms of which the coefficients of the characteristic cubic are expressed.

8. By the axioms of Euclid there exists a Euclidean plane with straight lines on it. These we may say all meet at infinity which we regard as a discontinuity—say a puncture in the plane. Under inversions the plane is a sphere with a puncture: it is the puncture which makes the sphere a plane.

The puncture is a degenerate kind of infinity, and a better one was found by cutting the plane along a circle. If we consider the region inside a circle Ω we have a hyperbolic, non-Euclidean geometry where we have dropped the axiom of the unique parallel, and replaced the straight line by that arc of a circle orthogonal to Ω which is inside Ω. To call the arc a straight line is to invite confusion, for we have a Euclidean background, the plane. We call the arc a *path* or a geodesic.

The third stage in this sketch of natural geometry is conformal mapping. Here we take any simply-connected region of the plane. The boundary of the region is now infinity. The paths of the region are the maps of the paths for the circular region.

Thus for any simply-connected region we have natural paths or geodesics. To draw them we may use the idea of the flow of a perfect fluid say electricity. We make at a point of the region a point-charge—say a small hole e through which the fluid enters, flows over the region and passes out over the boundary, in general, orthogonally. The essential is that the flow is along the paths which meet at e. The boundary determines the paths, and the paths determine the flow. This way of drawing the paths from a point e to the boundary is explained in introductions to physics. It clearly can be used in cases where instead of a point e we have a second boundary: that is, when we have a doubly-connected region of which we seek the paths. In this way then one can approach on the one hand the ideas of Riemann, Klein, Schottky, Schwarz and others, on the other hand the study of electricity: and get a glimpse of "the flaming bounds of space and time" as being what is called matter.

9. Doctor Barrow presented a theorem concerning the upper limit of convergence of infinite exponentials more accurate than earlier theorems.

10. Professor Beckwith considered a time integral in a time-space-force field, assuming the velocity of a particle to be a function of its position. His problem was to determine certain necessary conditions on the path of a particle in motion

in the field and passing through two points such that the time of motion between the two points is a minimum. His work led to 3-dimensional equations which must be simultaneously satisfied, similar to the 2-dimensional Euler's differential equation, Weierstrass's minimizing arc condition and Legendre's.

H. A. ROBINSON, *Secretary*

THE CONVERGENCE OF FOURIER SERIES[1]

By DUNHAM JACKSON, University of Minnesota

1. *Introduction.* Considerable parts of the theory of Fourier series have an interest, both for their mathematical content and by reason of the importance of their applications, for students whose experience of mathematics in general is only moderately advanced. The writer has had occasion more than once to give an introductory course on Fourier series and related topics for classes whose mathematical preparation was not assumed to extend beyond a first course in the calculus. The question has arisen each time how far it is possible to go beyond merely formal relationships, and to give such a class a genuine appreciation of some of the properties of convergence, even the most elementary of which are so characteristic of the type of series in question and have had so profound an influence on the course of modern mathematical development. This paper is an outline of the writer's most recent attempt in that direction. No part of the treatment is new, and most parts have been used long since for purposes of elementary exposition. The object of this account is merely to suggest one way of cutting the whole picture into pieces of convenient size, and arranging them in order so that at any stage the next piece is not far to seek.[2]

Specifically, the essential technicalities are brought within easy reach by the following devices:

(a) The main convergence proof is made to depend on nothing more abstruse than the fact that the general term of a convergent series approaches zero.

(b) Certain theorems relating to an arbitrary continuous function are made to depend on acceptance of the proposition that the graph of such a function can be approximated with any desired accuracy by a broken line. Elsewhere a reader unfamiliar with the precise definition of the word "continuous" may take it as self-explanatory, as far as an understanding of the main features of the argument is concerned.

(c) The phrase "uniformly convergent" is introduced as naturally descriptive of a type of convergence already characterized in quantitative terms.

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[1] Presented to the Mathematical Association of America at New Orleans, December 31, 1931.

[2] Among the numerous more or less extensive presentations of the theory of Fourier series the following may be specially mentioned in connection with the present paper: M. Bôcher, *Introduction to the Theory of Fourier's Series*, Annals of Mathematics, (2), vol. 7 (1905–06), pp. 81–152; H. Lebesgue, *Leçons sur les séries trigonométriques*, Paris, 1906.

Such facts as the integrability of a continuous function, or of a function which is continuous except for a finite number of finite jumps, will be considered "obvious" on the basis of the interpretations that are customary in a first study of the calculus. An attempt is made throughout to give an exposition to which more critical study may have something to add, but from which it will have nothing to retract. In actual presentation to a class the less familiar ideas are naturally explained and illustrated at greater length than in the text.

2. *Formulas for the coefficients.* The Fourier series for a given function has the form

$$\begin{aligned} &a_0/2 + a_1 \cos x + a_2 \cos 2x + \cdots \\ &\quad + b_1 \sin x + b_2 \sin 2x + \cdots, \end{aligned} \tag{1}$$

in which the coefficients are given by the formulas

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos kt\,dt, \qquad b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin kt\,dt. \tag{2}$$

These are found by setting $f(t)$ equal to the series (1), written in terms of the variable t, multiplying by $\cos kt$ or $\sin kt$, and integrating term by term, with the use of the relations

$$\begin{aligned} &\int_{-\pi}^{\pi}\cos jt\cos kt\,dt = 0, \qquad \int_{-\pi}^{\pi}\sin jt\sin kt\,dt = 0 \qquad (j \neq k), \\ &\int_{-\pi}^{\pi}\sin jt\cos kt\,dt = 0 \qquad \text{(for all } j \text{ and } k), \\ &\int_{-\pi}^{\pi}\cos^2 kt\,dt = \int_{-\pi}^{\pi}\sin^2 kt\,dt = \pi \qquad (k \geqq 1). \end{aligned} \tag{3}$$

The representation of the constant term by $a_0/2$ rather than a_0 is an artifice to make the general formula for a_k applicable without change when $k=0$.

If the calculation leading to the equations (2) were to be regarded as a "proof," some inquiry would be necessary as to the validity of the processes involved. In the discussion to be presented here the formulas (2), which are suggested by the calculation as at least presumptively important, and which have a meaning whenever $f(t)$ is an integrable function, will be taken outright as starting point. A series (1) will be written down with these coefficients, and it will be inquired whether the series does in fact converge and represent $f(x)$. It will be assumed throughout that $f(x)$ is periodic, with period 2π.

3. *Order of magnitude of the coefficients in the case of certain continuous functions.* If $f(t)$ has a continuous derivative, integration by parts gives

$$\pi a_k = \left[\frac{1}{k} f(t)\sin kt\right]_{-\pi}^{\pi} - \frac{1}{k}\int_{-\pi}^{\pi} f'(t)\sin kt\,dt = -\frac{1}{k}\int_{-\pi}^{\pi} f'(t)\sin kt\,dt.$$

If there is a continuous second derivative,

$$\int_{-\pi}^{\pi} f'(t) \sin kt\,dt = \left[-\frac{1}{k} f'(t) \cos kt \right]_{-\pi}^{\pi} + \frac{1}{k}\int_{-\pi}^{\pi} f''(t) \cos kt\,dt$$
$$= \frac{1}{k}\int_{-\pi}^{\pi} f''(t) \cos kt\,dt.$$

Let M be the maximum of $|f''(t)|$. Then

$$|f''(t) \cos kt| \leqq M, \qquad \left|\int_{-\pi}^{\pi} f''(t) \cos kt\,dt\right| \leqq 2M\pi,$$

and $|a_k| \leqq 2M/k^2$. Similarly $|b_k| \leqq 2M/k^2$. It follows that

$$|a_k \cos kx + b_k \sin kx| \leqq 4M/k^2,$$

and as $1/k^2$ is the general term of a well known convergent series, the series (1) is also certainly convergent.

The same conclusion can be reached with a somewhat less restrictive hypothesis on $f(t)$, and this will be important for an application later. Let $f(t)$ still be continuous everywhere, and let it be supposed that any period interval can be divided into a finite number of subintervals throughout each of which $f(t)$ has continuous first and second derivatives, but that the derivatives may not be continuous in passing from one subinterval to the next. The graph of $f(t)$ over a period is then made up of a finite number of pieces, each having continuous curvature, but there may be corners (or, as an admissible alternative, abrupt changes of curvature without change of direction) at the points where two pieces come together. Let the successive points of division marking the subintervals of the period from $-\pi$ to π be $x_1, x_2, \cdots, x_{p-1}$, and for uniformity of notation let $x_0=-\pi$, $x_p=\pi$. The derivatives may have different values from the right and from the left at these points, but the function $f(t)$ itself has a determinate value at each of them. For each value of i from 0 to $p-1$,

$$\int_{x_i}^{x_{i+1}} f(t) \cos kt\,dt = \left[\frac{1}{k} f(t) \sin kt \right]_{x_i}^{x_{i+1}} - \frac{1}{k}\int_{x_i}^{x_{i+1}} f'(t) \sin kt\,dt.$$

When equations of this form are written for all p subintervals and added, the terms $(1/k)f(x_i) \sin kx_i$ cancel, each occurring once with a plus and once with a minus sign, and

$$\int_{-\pi}^{\pi} f(t) \cos kt\,dt = -\frac{1}{k}\sum_{i=0}^{p-1} \int_{x_i}^{x_{i+1}} f'(t) \sin kt\,dt.$$

Another integration by parts gives

$$\int_{x_i}^{x_{i+1}} f'(t) \sin kt\,dt = \left[-\frac{1}{k} f'(t) \cos kt \right]_{x_i}^{x_{i+1}} + \frac{1}{k}\int_{x_i}^{x_{i+1}} f''(t) \cos kt\,dt.$$

When these expressions are added for the various intervals the terms outside the signs of integration do not cancel, since $f'(x_i)$ at the left-hand end of one interval does not in general mean the same thing as $f'(x_i)$ at the right-hand end of the preceding interval. But under the hypotheses $f'(t)$ and $f''(t)$ remain finite everywhere, in spite of their discontinuities; if M and M_1 are numbers such that $|f''(t)| \leqq M$, $|f'(t)| \leqq M_1$, for all values of t, then

$$|f'(x_{i+1}) \cos kx_{i+1} - f'(x_i) \cos kx_i| \leqq 2M_1$$

in each case, and

$$\left| \int_{x_i}^{x_{i+1}} f''(t) \cos kt\, dt \right| \leqq M(x_{i+1} - x_i).$$

Hence

$$|\pi a_k| = \left| \int_{-\pi}^{\pi} f(t) \cos kt\, dt \right| \leqq \frac{2pM_1}{k^2} + \frac{2\pi M}{k^2}.$$

A similar calculation applies to b_k. With a readily intelligible abbreviation of the hypothesis, the result may be stated as follows:

THEOREM I. *If $f(x)$ is a function which has a continuous second derivative except for a finite number of corners in a period, and if a_k, b_k are the coefficients in its Fourier series, there is a number C, independent of k, such that*

$$|a_k| \leqq \frac{C}{k^2}, \qquad |b_k| \leqq \frac{C}{k^2}.$$

The convergence of the series is an immediate corollary. The special importance of the generalized hypothesis is that it applies in particular to a function whose graph over any period is made up of a finite number of straight line segments of finite slope joined end to end, or, as it may be described for brevity, *a function whose graph is a broken line.*

It must be recognized however that the series has not yet been proved to converge *to the value* $f(x)$. The preceding convergence proof would apply equally well, as far as it goes, to a series of cosines alone, without sine terms; but a cosine series is not adequate for the representation of an arbitrary periodic function. From the point of view of completeness of demonstration the question still remains whether the cosines and sines together are sufficient in all cases, or whether still other terms may sometimes be needed. An answer to this question will be found later, after some further preliminaries.

4. *Approach of the coefficients to zero in general (Riemann's Theorem).* Let $f(x)$ now be any function of period 2π which (with sufficient generality for the purposes of this paper) is continuous except for a finite number of finite jumps in a period. Let a_k, b_k be its Fourier coefficients (2), and let $S_n(x)$ be the partial sum of the series (1) through terms of the nth order:

$$S_n(x) \equiv a_0/2 + a_1 \cos x + \cdots + a_n \cos nx + b_1 \sin x + \cdots + b_n \sin nx. \tag{4}$$

By the use of the relations (3) it is seen that

$$\int_{-\pi}^{\pi} S_n(t) \cos kt\, dt = \pi a_k = \int_{-\pi}^{\pi} f(t) \cos kt\, dt,$$

$$\int_{-\pi}^{\pi} S_n(t) \sin kt\, dt = \pi b_k = \int_{-\pi}^{\pi} f(t) \sin kt\, dt$$

for values of $k \leqq n$. If these equations, read from right to left, are multiplied by $a_0/2, a_1, \cdots, a_n, b_1, \cdots, b_n$ for the successive values of k respectively and added, it is found that

$$\int_{-\pi}^{\pi} f(t)S_n(t)dt = \pi\left[\frac{a_0^2}{2} + \sum_{k=1}^{n} (a_k^2 + b_k^2)\right] = \int_{-\pi}^{\pi} [S_n(t)]^2 dt.$$

Hence

$$\begin{aligned}\int_{-\pi}^{\pi} [f(t) - S_n(t)]^2 dt &= \int_{-\pi}^{\pi} [f(t)]^2 dt - 2\int_{-\pi}^{\pi} f(t)S_n(t)dt + \int_{-\pi}^{\pi} [S_n(t)]^2 dt \\ &= \int_{-\pi}^{\pi} [f(t)]^2 dt - \int_{-\pi}^{\pi} [S_n(t)]^2 dt \\ &= \int_{-\pi}^{\pi} [f(t)]^2 dt - \pi\left[\frac{a_0^2}{2} + \sum_{k=1}^{n} (a_k^2 + b_k^2)\right].\end{aligned}$$

As the first member can not be negative, it must be that

$$\frac{a_0^2}{2} + \sum_{k=1}^{n} (a_k^2 + b_k^2) \leqq \frac{1}{\pi}\int_{-\pi}^{\pi} [f(t)]^2 dt.$$

The fact that this is true for all values of n, while the last integral does not depend on n, means that the infinite series

$$\frac{a_0^2}{2} + \sum_{k=1}^{\infty} (a_k^2 + b_k^2)$$

is convergent. And since the general term of a convergent series approaches zero it must be that

$$\lim_{k\to\infty} a_k = 0, \quad \lim_{k\to\infty} b_k = 0.$$

It will be convenient to have this result stated for reference with another notation for the arbitrary function, and with the index k replaced by n:

THEOREM II. *If $\phi(t)$ is any function of period 2π which is continuous except for a finite number of finite jumps in a period,*

$$\lim_{n\to\infty}\int_{-\pi}^{\pi}\phi(t)\cos nt\,dt = \lim_{n\to\infty}\int_{-\pi}^{\pi}\phi(t)\sin nt\,dt = 0.$$

A simple consequence of this theorem, of no conspicuous interest in itself, will be required later as a lemma. It is clear that the hypothesis of periodicity is not essential, since the integrals to which the theorem relates do not involve values of the function outside the interval $(-\pi, \pi)$. If any function whatever is given over this interval, a periodic function can be constructed from it by suitable repetition of its values in successive intervals of length 2π. If the original function as defined from $-\pi$ to π approaches different limits at the two ends of this interval the corresponding periodic function will have a finite jump, to be sure, in passing from one interval to the next, but such a discontinuity is admissible under the hypothesis. If $\phi(x)$ is any function continuous from $-\pi$ to π except for a finite number of finite jumps the same will be true of the functions $\phi(x)\cos(x/2)$ and $\phi(x)\sin(x/2)$, and the theorem can be applied to these functions, regardless of the fact that $\cos(x/2)$ and $\sin(x/2)$ do not of themselves have the period 2π when considered for unrestricted values of x. With a change of notation for the independent variable, application of the theorem to the combination

$$\phi(u)\sin(n+\tfrac{1}{2})u \equiv [\phi(u)\sin\tfrac{1}{2}u]\cos nu + [\phi(u)\cos\tfrac{1}{2}u]\sin nu$$

gives the

COROLLARY. *If $\phi(u)$ is any function which is continuous from $-\pi$ to π except for a finite number of finite jumps,*

$$\lim_{n\to\infty}\int_{-\pi}^{\pi}\phi(u)\sin(n+\tfrac{1}{2})u\,du = 0.$$

5. *Integral formula for the partial sum of the series.* Further study of convergence depends on a trigonometric identity. Since

$$\begin{aligned}\sin\tfrac{1}{2}v[\tfrac{1}{2} &+ \cos v + \cos 2v + \cdots + \cos nv]\\ &= \tfrac{1}{2}\sin\tfrac{1}{2}v + \sum_{k=1}^{n}\sin\tfrac{1}{2}v\cos kv\\ &= \tfrac{1}{2}\sin\tfrac{1}{2}v + \tfrac{1}{2}\sum_{k=1}^{n}[\sin(k+\tfrac{1}{2})v - \sin(k-\tfrac{1}{2})v]\\ &= \tfrac{1}{2}\sin(n+\tfrac{1}{2})v\end{aligned}$$

it appears that

$$\tfrac{1}{2} + \cos v + \cdots + \cos nv = \frac{\sin(n+\tfrac{1}{2})v}{2\sin\tfrac{1}{2}v}. \tag{5}$$

If the values of a_k, b_k given by (2) are substituted explicitly in (4) and the resulting expression written out at length it is found that

$$S_n(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\left[\tfrac{1}{2} + \sum_{k=1}^{n} (\cos kt \cos kx + \sin kt \sin kx)\right]dt$$
$$= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\left[\tfrac{1}{2} + \sum_{k=1}^{n} \cos k(t - x)\right]dt,$$

which by the identity just obtained reduces to

$$S_n(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\,\frac{\sin (n + \frac{1}{2})(t - x)}{2 \sin \frac{1}{2}(t - x)}\,dt. \tag{6}$$

With the substitution $t-x=u$ this becomes

$$S_n(x) = \frac{1}{\pi}\int_{-\pi-x}^{\pi-x} f(x + u)\,\frac{\sin (n + \frac{1}{2})u}{2 \sin \frac{1}{2}u}\,du.$$

The integrand has the period 2π when considered as a function of u. (Addition of 2π to u reverses the signs of numerator and denominator in the fraction, but leaves the fraction as a whole unchanged.) It is a general fact that the integral of a periodic function over any interval whose length is a period is the same as the integral over any other interval of equal length. This is evident from the interpretation of the integral as the area under a curve, and is readily proved analytically with the aid of a suitable change of variable. In the present instance the integral from $-\pi-x$ to $\pi-x$ is the same as that from $-\pi$ to π, so that

$$S_n(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x + u)\,\frac{\sin (n + \frac{1}{2})u}{2 \sin \frac{1}{2}u}\,du. \tag{7}$$

By integration of the identity (5) it is seen that

$$1 = \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{\sin (n + \frac{1}{2})u}{2 \sin \frac{1}{2}u}\,du. \tag{8}$$

(This is in fact merely the form taken by the general expression (7) for the special case $f(x) \equiv 1$, since the Fourier series for any constant reduces to the constant itself.) If (8) is multiplied by $f(x)$, for any particular value of x, the factor $f(x)$, being independent of the variable of integration, may be placed under the integral sign:

$$f(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,\frac{\sin (n + \frac{1}{2})u}{2 \sin \frac{1}{2}u}\,du.$$

Hence

$$S_n(x) - f(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} [f(x + u) - f(x)]\,\frac{\sin (n + \frac{1}{2})u}{2 \sin \frac{1}{2}u}\,du. \tag{9}$$

The problem of convergence of the series is thus reduced to the problem of showing that the last integral approaches zero, under suitable hypotheses with regard to the function $f(x)$ under consideration.

6. *Convergence at a point of continuity.* Attention will be restricted for the present to the question of convergence at a point where $f(x)$ is continuous. *Let it be supposed that $f(x)$ is continuous everywhere, or continuous except for a finite number of finite jumps in a period, and that at the particular point where convergence is to be proved it is continuous and has a finite right-hand derivative and a finite left-hand derivative*, which may or may not be equal. This means that its graph is continuous at the point in question, and may be smooth or may have a corner there, with finite slopes from both sides.

Analytically the hypothesis implies that the difference quotient

$$\frac{f(x+u)-f(x)}{u}$$

approaches a definite limit as u approaches zero through positive values, and approaches the same or a different limit as u approaches zero through negative values. *Considered as a function of u, the quotient has at most a finite jump for* $u=0$. The same is true of the function $\phi(u)$ defined by the formula

$$\phi(u) = \frac{f(x+u)-f(x)}{2\sin\frac{1}{2}u} = \frac{f(x+u)-f(x)}{u}\cdot\frac{\frac{1}{2}u}{\sin\frac{1}{2}u},$$

since $(\frac{1}{2}u)/(\sin\frac{1}{2}u)$ has the limit 1. For any other value of u between $-\pi$ and π this $\phi(u)$, considered as a function of u for a fixed value of x, is continuous if $f(x+u)$ is continuous, and has a finite jump if $f(x+u)$ has a finite jump. *The function $\phi(u)$ is continuous from $-\pi$ to π except for a finite number of finite jumps.* For the value of x in question, by (9)

$$S_n(x) - f(x) = \frac{1}{\pi}\int_{-\pi}^{\pi}\phi(u)\sin(n+\tfrac{1}{2})u\,du.$$

Hence, by the Corollary of Theorem II, $S_n(x)-f(x)$ approaches zero as n becomes infinite.[1] This proves

THEOREM III. *If $f(x)$ is continuous except for a finite number of finite jumps in a period, its Fourier series converges to the value $f(x)$ at every point where $f(x)$ is continuous and has a finite right-hand derivative and a finite left-hand derivative.*

7. *Uniform convergence.* Theorem III applies in particular to any function satisfying the hypotheses of Theorem I, and it is now established without question that under the conditions of Theorem I the series converges *to the value $f(x)$* for all values of x. To that extent Theorem I is superseded by Theorem III.

[1] For graphs illustrating the convergence in special cases see, e.g., Byerly, *Fourier's Series and Spherical Harmonics*, Boston, 1895, pp. 62–64.

The earlier theorem, however, gives important additional information with regard to the manner of convergence, under its own more restrictive hypotheses.

Let $f(x)$ be a function for which the conditions of Theorem I are satisfied By the conclusion of that theorem, together with the fact that the sum of the series is $f(x)$,

$$|f(x) - S_n(x)| = \left| \sum_{k=n+1}^{\infty} (a_k \cos kx + b_k \sin kx) \right| \leqq \sum_{k=n+1}^{\infty} \frac{2C}{k^2}.$$

It is clear that

$$\frac{1}{k^2} = \int_{k-1}^{k} \frac{du}{k^2} < \int_{k-1}^{k} \frac{du}{u^2},$$

since $u<k$ throughout the interior of the interval of integration, and hence

$$\sum_{k=n+1}^{\infty} \frac{1}{k^2} < \sum_{k=n+1}^{\infty} \int_{k-1}^{k} \frac{du}{u^2} = \int_{n}^{\infty} \frac{du}{u^2} = \frac{1}{n}.$$

So

$$|f(x) - S_n(x)| \leqq \frac{2C}{n},$$

for all values of x. The fact that the remainder does not exceed a quantity which is *independent of* x and which approaches zero as n becomes infinite is expressed by saying that the series is *uniformly convergent.* The conclusion thus noted may be recorded as

THEOREM IV. *If $f(x)$ is a function which has a continuous second derivative except for a finite number of corners in a period, its Fourier series converges uniformly to the value $f(x)$ for all values of x.*

8. *Weierstrass's Theorem for trigonometric approximation.* The preceding, like Theorem I, holds in particular for a broken-line function. Its application to such a function is not merely of interest in itself, but can be used to prove an important general theorem, known as Weierstrass's theorem, which says that any continuous function of period 2π can be approximately represented with any assigned degree of accuracy by a suitably constructed trigonometric sum. By a trigonometric sum is meant an expression of the form

$$\begin{aligned} &\alpha_0/2 + \alpha_1 \cos x + \alpha_2 \cos 2x + \cdots + \alpha_n \cos nx \\ &\qquad + \beta_1 \sin x + \beta_2 \sin 2x + \cdots + \beta_n \sin nx, \end{aligned}$$

with any constant coefficients α_k, β_k.

Let $f(x)$ be any function which has the period 2π and is continuous for all values of x. It is clear that it can be approximated by a broken-line function as closely as may be desired. In terms of geometric representation this can be accomplished simply by marking points close enough together on the graph of

$f(x)$ and joining them in succession by line segments. Let ϵ be any positive number, arbitrarily small, and let $g(x)$ be a broken-line function constructed so that the difference $|f(x)-g(x)|$ is not merely less than ϵ, but less than $\epsilon/2$, for all values of x. Let $T_n(x)$ be the partial sum of the Fourier series *for the function* $g(x)$, through the terms involving $\cos nx$ and $\sin nx$. Since the series converges uniformly to the value $g(x)$, it will be possible to take n so large that $|g(x)-T_n(x)|<\epsilon/2$ for all values of x; if C_0 is the constant given by Theorem I, and entering into the proof of Theorem IV, as applied to $g(x)$, it is sufficient to take $n>4C_0/\epsilon$, so that $2C_0/n<\epsilon/2$. Then

$$|f(x) - T_n(x)| < \epsilon$$

for all values of x, and this is the essence of the conclusion to be proved. It may be stated as

THEOREM V (*Weierstrass's Theorem*). *If $f(x)$ is any continuous function of period 2π, and if ϵ is any positive number, arbitrarily small, it is possible to construct a trigonometric sum $T_n(x)$ so that*

$$|f(x) - T_n(x)| < \epsilon$$

for all values of x.

If the Fourier series for $f(x)$ itself were known to converge uniformly to the right value there would of course be no need of bringing in the auxiliary function $g(x)$; but the Fourier series for a given function is *not* necessarily convergent if the function is merely assumed to be continuous. While continuous functions having divergent Fourier series are of complicated structure, and not likely to be encountered except when cited expressly for purposes of illustration, the fact that such functions exist gives significance to the general theorem of Weierstrass, which holds for all continuous functions without exception.

9. *Completeness of the series.* An important consequence for the theory of Fourier series, which will be stated first and then proved, is

THEOREM VI. *If $f(x)$ is a continuous function of period 2π whose Fourier coefficients are all zero, then $f(x)=0$ identically.*

The hypothesis means that

$$\int_{-\pi}^{\pi} f(x) \cos kx\, dx = \int_{-\pi}^{\pi} f(x) \sin kx\, dx = 0$$

for all integral values of k. It follows that

$$\int_{-\pi}^{\pi} f(x) T_n(x) dx = 0$$

if $T_n(x)$ is any trigonometric sum whatever. Let M be the maximum of $|f(x)|$; the conclusion to be proved means of course that $M=0$, but that is not assumed

for the time being. Let ϵ be any positive quantity. Corresponding to the positive quantity $\epsilon/[2\pi(M+1)]$ let a trigonometric sum $T_n(x)$ be constructed according to Weierstrass's theorem so that

$$|f(x) - T_n(x)| \leqq \frac{\epsilon}{2\pi(M+1)}$$

for all values of x. Then

$$\int_{-\pi}^{\pi} [f(x)]^2 dx = \int_{-\pi}^{\pi} f(x)[f(x) - T_n(x)]dx$$

$$\leqq \int_{-\pi}^{\pi} M \cdot \frac{\epsilon}{2\pi(M+1)} dx = \frac{M\epsilon}{M+1} < \epsilon.$$

Since this is true no matter how small ϵ is taken, it must be that

$$\int_{-\pi}^{\pi} [f(x)]^2 dx = 0,$$

from which it follows that $f(x) \equiv 0$, as the theorem asserts.

When the Fourier coefficients for a function are all zero, the corresponding series is of course convergent, and has zero for its sum. The theorem may be regarded as a statement that under these conditions, if the function is continuous, it is identical with the sum of the series. It is a special case of the general proposition, not yet proved in this paper, that any continuous function is equal to the sum of its Fourier series if the series is uniformly convergent. From the point of view of demonstration the special case is not trivial; on the contrary, it contains the essence of the general theorem, which can be deduced from it almost immediately with the aid of certain standard theorems on uniformly convergent series. These theorems, which will not be proved here, are to the effect that a uniformly convergent series of continuous functions represents a continuous function and can be integrated term by term. If they are assumed as known the reasoning proceeds as follows. Let $f(x)$ be the given continuous function, let a_k and b_k be its Fourier coefficients, and let $h(x)$ be the sum of the series, supposed uniformly convergent:

$$h(x) \equiv \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx).$$

By one of the theorems cited $h(x)$ also is continuous. The series will still be uniformly convergent, and so integrable term by term, if multiplied through by $\cos kx$ or $\sin kx$. When the integration is performed it is seen by means of (3) that

$$\int_{-\pi}^{\pi} h(x) \cos kx\, dx = \pi a_k, \qquad \int_{-\pi}^{\pi} h(x) \sin kx\, dx = \pi b_k.$$

Comparison of these formulas with (2), by means of which a_k and b_k are defined, shows that

$$\int_{-\pi}^{\pi} [f(x) - h(x)] \cos kx\, dx = \int_{-\pi}^{\pi} [f(x) - h(x)] \sin kx\, dx = 0$$

for all values of k, and application of Theorem VI to the difference $f(x)-h(x)$ gives $f(x)-h(x)\equiv 0$.

10. *Convergence at a point of discontinuity.* Throughout the discussion of convergence so far it has been assumed that the function is continuous at least at the point where convergence is to be proved, though in Theorem III it may have discontinuities elsewhere. By way of introduction to a treatment of convergence at a point of discontinuity let $F_0(x)$ be the particular function obtained by the following construction: it is defined by the formula $(\pi-x)/2$ for $0<x<2\pi$, is made periodic by repetition of these values in successive intervals of length 2π, and at the points of discontinuity $x=2k\pi$, $k=0, \pm 1, \pm 2, \cdots$, it is expressly given the value zero. Its graph, except for isolated points, is thus made up of an infinite succession of straight line segments, each with slope $-\frac{1}{2}$, the right-hand end of each segment being π units below the left-hand end of the next, while for the values of x corresponding to breaks in the graph the value of the function is not the end value belonging to either of the segments concerned, but is half-way between them. Let a_k, b_k be the Fourier coefficients of this function $F_0(x)$. Since the function satisfies the identity $F_0(-x)\equiv -F_0(x)$, the integral of $F_0(t)\cos kt$ from $-\pi$ to 0 cancels the integral from 0 to π, and $a_k=0$. It is readily found by explicit calculation that $b_k=1/k$. So $F_0(x)$ has the Fourier series

$$\sin x + \tfrac{1}{2}\sin 2x + \tfrac{1}{3}\sin 3x + \cdots . \tag{10}$$

It is known from Theorem III, without further inquiry, that the series must converge to $F_0(x)$ at all points where $F_0(x)$ is continuous.[1] And when x is zero or any integral multiple of 2π the convergence of the series to the prescribed value is immediately apparent, since each term then reduces to zero separately. So $F_0(x)$ is represented by the series for *all* values of x. The value assigned to the function at the points of discontinuity of course has no influence on the determination of the coefficients; the essential observation is that the series (10) does converge at these points, and the value to which it converges is the one designated.

Now let $f(x)$ be any function of period 2π which is continuous except for a finite number of finite jumps in a period, which actually does have a finite jump for $x=0$, and which has a derivative from the right and a derivative from the

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[1] In particular, setting $x=\pi/2$ gives

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots,$$

in agreement with the result obtained by setting $x=1$ in the power series for arc tan x. So the present reasoning incidentally gives a proof of the validity of the power series for $x=1$.

left there; the last requirement is naturally interpreted to mean that the function has a derivative from the right if defined for $x=0$ by the limit approached from the right, and a left-hand derivative if defined by the limit from the left, or in other words that each segment of the graph has a determinate finite slope where the break occurs. Let the limits approached by $f(x)$ from the right and from the left be denoted by $f(0+)$ and $f(0-)$, and let $D=f(0+)-f(0-)$. Let $f_0(x)=f(x)-(D/\pi)F_0(x)$. This function approaches the limit $[f(0+)+f(0-)]/2$ as x approaches zero from the right, and has the same limit for approach from the left; moreover it has a right-hand and a left-hand derivative for $x=0$, by the corresponding hypothesis on $f(x)$. So its Fourier series converges for $x=0$ to the value $[f(0+)+f(0-)]/2$, by direct application of Theorem III. As the Fourier series for $f(x)$ can be obtained by adding the series for $f_0(x)$ and the series for $(D/\pi)F_0(x)$, and as the latter converges to the value zero, it is seen that the series for $f(x)$ converges to the mean value $[f(0+)+f(0-)]/2$.

Similar reasoning is applicable in the case of a finite jump for any other value of x. The question is merely that of convergence at the point where the discontinuity occurs; convergence at points of continuity is already taken care of by Theorem III. If carried through in detail, the proof would involve something amounting to explicit verification of the fact that the Fourier series for $F_0(x-c)$ as a function of x is

$$\sum_{k=1}^{\infty}\left(\frac{\cos kc}{k}\sin kx-\frac{\sin kc}{k}\cos kx\right)=\sum_{k=1}^{\infty}\frac{\sin k(x-c)}{k},$$

if c is any constant. One way of accomplishing this is as follows: As the Fourier coefficients for $F_0(x)$ are known to be 0 and $1/k$,

$$\frac{\sin kx}{k}=\frac{1}{\pi}\int_{-\pi}^{\pi}F_0(t)\cos k(t-x)dt.$$

Let $F_0(x-c)$ be denoted by $F(x)$, and let A_k, B_k be its Fourier coefficients. Then

$$\begin{aligned}A_k\cos kx+B_k\sin kx&=\frac{1}{\pi}\int_{-\pi}^{\pi}F(u)\cos k(u-x)du\\&=\frac{1}{\pi}\int_{-\pi}^{\pi}F_0(u-c)\cos k(u-x)du.\end{aligned}$$

By the substitution $u-c=t$ the last expression becomes

$$\frac{1}{\pi}\int_{-\pi-c}^{\pi-c}F_0(t)\cos k[t-(x-c)]dt,$$

and as integration over any period interval gives the same result this reduces to

$$\frac{1}{\pi}\int_{-\pi}^{\pi}F_0(t)\cos k[t-(x-c)]dt=\frac{\sin k(x-c)}{k}.$$

The conclusion with regard to convergence at a point of discontinuity may be summarized in

THEOREM VII. *If $f(x)$ is continuous except for a finite number of finite jumps in a period, if it has a finite jump for $x=x_0$, the limits approached from the right and from the left being $f(x_0+)$ and $f(x_0-)$, and if it has a derivative from the right and a derivative from the left at this point, its Fourier series converges for $x=x_0$ to the value $[f(x_0+)+f(x_0-)]/2$.*

11. *Least-square property.* An important characteristic of the Fourier series is the least-square property, according to which the integral

$$\int_{-\pi}^{\pi} [f(x) - S_n(x)]^2 dx,$$

where $S_n(x)$ is the partial sum of the series, has a smaller value than that which is obtained if $S_n(x)$ is replaced by any other trigonometric sum of the nth order. This property however is not needed for the present discussion of convergence, and it is not necessary to repeat a proof of it here.

12. *Summation by the first arithmetic mean.* A matter which does have an intimate connection with the preceding work is the *summation* of Fourier series by the method of the arithmetic mean. The mean in question is the quantity

$$\sigma_n(x) \equiv \frac{S_0(x) + S_1(x) + \cdots + S_{n-1}(x)}{n}.$$

Although, as has been stated, there exist continuous functions for which the sums $S_n(x)$ do not give a convergent approximation, Fejér[1] proved the striking theorem that *$\sigma_n(x)$ always converges uniformly to the value $f(x)$ if $f(x)$ is continuous.* A proof[2] will be given here which is in part different in arrangement from that of Fejér.

A preliminary observation is that the arithmetic mean converges uniformly in the case of any broken-line function of the sort previously considered. Let $g(x)$ be any such function, let $T_n(x)$ be the partial sum of its Fourier series, and let

$$\tau_n(x) \equiv (1/n)[T_0(x) + T_1(x) + \cdots + T_{n-1}(x)].$$

Let ϵ be any positive quantity, and in accordance with the uniform convergence of the series let p be a number so large that $|g(x) - T_k(x)| < \epsilon/2$ everywhere for $k=p$ and for all larger values of k. For $n>p$,

[1] L. Fejér, *Untersuchungen über Fouriersche Reihen*, Mathematische Annalen, vol. 58 (1904), pp. 51–69.

[2] For the method cf. A. Haar, *Zur Theorie der orthogonalen Funktionensysteme*, Dissertation, Göttingen, 1909, reprinted in Mathematische Annalen, vol. 69 (1910), pp. 331–371.

$$g(x) - \tau_n(x) = \frac{1}{n} \sum_{k=0}^{n-1} [g(x) - T_k(x)]$$
$$= \frac{1}{n} \sum_{k=0}^{p-1} [g(x) - T_k(x)] + \frac{1}{n} \sum_{k=p}^{n-1} [g(x) - T_k(x)].$$

Let the sums from 0 to $p-1$ and from p to $n-1$ be denoted respectively by Σ_1 and Σ_2, so that $g(x) - \tau_n(x) = (1/n)(\Sigma_1 + \Sigma_2)$. Then

$$|g(x) - \tau_n(x)| \leqq \frac{1}{n}|\Sigma_1| + \frac{1}{n}|\Sigma_2|.$$

In Σ_2 each difference $g(x) - T_k(x)$ is less than $\epsilon/2$ in absolute value, and as the number of terms in the summation is not greater than n it is certain that $|\Sigma_2| < n\epsilon/2$ and $(1/n)|\Sigma_2| < \epsilon/2$. The sum Σ_1 does not depend on n; if G is the maximum of its absolute value, $(1/n)|\Sigma_1| < \epsilon/2$ as soon as $n > 2G/\epsilon$. When the last condition is satisfied,

$$|g(x) - \tau_n(x)| < \epsilon$$

for all values of x. When any positive ϵ is chosen, no matter how small, the inequality is satisfied for all values of n from a certain point on, and $\tau_n(x)$ thus converges uniformly toward $g(x)$.

It is readily seen (though this is not necessary for present purposes) that the method of proof is of more general applicability, and that if any series whatever is convergent the corresponding means will converge to the same value. The significance of the process of "summation" lies in the fact that the means will sometimes converge when the original series does not.

From the identity (6), written with k in place of n, it is seen that

$$\sigma_n(x) = \frac{1}{n} \sum_{k=0}^{n-1} S_k(x) = \frac{1}{n\pi} \int_{-\pi}^{\pi} f(t) \left[\sum_{k=0}^{n-1} \frac{\sin (k + \frac{1}{2})(t - x)}{2 \sin \frac{1}{2}(t - x)} \right] dt.$$

The relations

$$\sin \tfrac{1}{2}v \sum_{k=0}^{n-1} \sin (k + \tfrac{1}{2})v = \tfrac{1}{2} \sum_{k=0}^{n-1} [\cos kv - \cos (k + 1)v]$$
$$= \tfrac{1}{2}(1 - \cos nv) = \sin^2 \tfrac{1}{2}nv$$

give

$$\frac{1}{\sin \frac{1}{2}v} \sum_{k=0}^{n-1} \sin (k + \tfrac{1}{2})v = \frac{\sin^2 \frac{1}{2}nv}{\sin^2 \frac{1}{2}v}$$

and hence

$$\sigma_n(x) = \frac{1}{n\pi} \int_{-\pi}^{\pi} f(t) \frac{\sin^2 \frac{1}{2}n(t - x)}{2 \sin^2 \frac{1}{2}(t - x)} dt.$$

For the study of convergence an important difference between $\sigma_n(x)$ and $S_n(x)$ is that the trigonometric factor in the last integral is always positive or zero, while the corresponding factor in (6) is of variable sign. This is the underlying reason for the greater simplicity of some of the properties of $\sigma_n(x)$. (It is not implied that the present formulas are in any sense to be regarded as superseding the earlier ones; the original sums $S_n(x)$ continue to be of more fundamental significance and have important advantages of their own, and in particular are likely to converge more rapidly when they do converge.)

In the case of a function $f(x)$ which is identically 1 each $S_k(x)$ reduce to 1, and $\sigma_n(x)$ consequently is identically 1 also, for any value of n, so that

$$1 \equiv \frac{1}{n\pi}\int_{-\pi}^{\pi} \frac{\sin^2 \frac{1}{2}n(t-x)}{2\sin^2 \frac{1}{2}(t-x)}\,dt. \tag{11}$$

An inference from (11) is that for any $f(x)$, if M is a number such that $|f(x)| \leqq M$ for all values of x,

$$|\sigma_n(x)| \leqq \frac{1}{n\pi}\int_{-\pi}^{\pi} M\,\frac{\sin^2 \frac{1}{2}n(t-x)}{2\sin^2 \frac{1}{2}(t-x)}\,dt = M. \tag{12}$$

It is possible now to proceed to the proof of the main convergence theorem for $\sigma_n(x)$. Let $f(x)$ be an arbitrary continuous function of period 2π, and $\sigma_n(x)$ the arithmetic mean of the first n partial sums of its Fourier series. Let ϵ be any positive quantity. Let $g(x)$ be a broken-line function constructed so that

$$|f(x) - g(x)| < \epsilon/3$$

for all values of x, and let $r(x) = f(x) - g(x)$. Let $\tau_n(x)$ and $\rho_n(x)$ be the arithmetic means pertaining to $g(x)$ and $r(x)$ respectively. Then

$$\sigma_n(x) = \tau_n(x) + \rho_n(x), \qquad \sigma_n(x) - f(x) = [\tau_n(x) - g(x)] + [\rho_n(x) - r(x)],$$

and

$$|\sigma_n(x) - f(x)| \leqq |\tau_n(x) - g(x)| + |r(x)| + |\rho_n(x)|.$$

The definition of $r(x)$ makes

$$|r(x)| < \epsilon/3.$$

By application of (12) to the function $r(x)$, whose maximum absolute value is less than $\epsilon/3$, it is seen that

$$|\rho_n(x)| < \epsilon/3$$

for all x and all n. As it has been shown that $\tau_n(x)$ converges uniformly toward $g(x)$,

$$|\tau_n(x) - g(x)| < \epsilon/3$$

for all values of n from a certain point on. For such values of n it follows that

$$|\sigma_n(x) - f(x)| < \epsilon,$$

and this relation, satisfied for all values of x, expresses the property of uniform convergence. This completes the proof of

THEOREM VIII. *If $f(x)$ is any continuous function of period 2π, and $\sigma_n(x)$ the arithmetic mean of the partial sums $S_0(x), S_1(x), \cdots, S_{n-1}(x)$ of its Fourier series, $\sigma_n(x)$ converges uniformly toward $f(x)$ for all values of x.*

13. *Weierstrass's Theorem for polynomial approximation.* Inasmuch as $\sigma_n(x)$ is a trigonometric sum, Theorem V is incidentally obtained again as a corollary of Theorem VIII, though with the order of presentation followed here the first proof is simpler.

In conclusion it may be noted that the better known theorem of Weierstrass which relates to polynomial approximation is also immediately obtainable from the work that has been done. The earlier form of proof will be preferred again.

Let $f(x)$ be a function of x which is continuous for $-1 \leqq x \leqq 1$. Let $x = \cos\theta$, and let $f(x) = f(\cos\theta) = \phi(\theta)$. This is a function of period 2π which is defined and continuous for all values of θ, and which furthermore satisfies the relation $\phi(-\theta) \equiv \phi(\theta)$. Let ϵ be any positive quantity, and let $g(\theta)$ be defined for $0 \leqq \theta \leqq \pi$ as a broken-line function such that $|\phi(\theta) - g(\theta)| < \epsilon/2$ throughout the interval. Then if $g(\theta)$ is defined for $-\pi \leqq \theta \leqq 0$ by the relation $g(-\theta) = g(\theta)$, and for values of θ outside the interval $(-\pi, \pi)$ by the requirement that it shall have the period 2π, it is a function to which Theorem IV is applicable, and $|\phi(\theta) - g(\theta)| < \epsilon/2$ for all values of θ. Let $T_n(\theta)$ be the partial sum of the Fourier series for $g(\theta)$, and let n be taken so large that by virtue of the uniform convergence $|g(\theta) - T_n(\theta)| < \epsilon/2$ everywhere, and hence $|\phi(\theta) - T_n(\theta)| < \epsilon$. Since $g(-\theta) \equiv g(\theta)$ each sine coefficient b_k in $T_n(\theta)$ is zero; the cosine of $k\theta$ is expressible as a polynomial of the kth degree in $\cos\theta$ for each value of k; and hence $T_n(\theta)$ is a polynomial in $\cos\theta$, which may be denoted by $P_n(\cos\theta)$ or $P_n(x)$. As $\phi(\theta) \equiv f(x)$, an approximating polynomial has been constructed for $f(x)$ so that

$$|f(x) - P_n(x)| < \epsilon$$

for $-1 \leqq x \leqq 1$.

This result can be extended to any interval (a, b) by the change of variable $y = (2x - a - b)/(b - a)$. Any continuous function of x for $a \leqq x \leqq b$ is a continuous function of y for $-1 \leqq y \leqq 1$; an approximating polynomial in terms of y can be found for this function; and any polynomial in y is a polynomial of the same degree in x. The general conclusion can be stated as

THEOREM IX (*Weierstrass's Theorem for polynomial approximation*). *If $f(x)$ is any continuous function for $a \leqq x \leqq b$, and if ϵ is any positive number, arbitrarily small, it is possible to construct a polynomial $P_n(x)$ so that*

$$|f(x) - P_n(x)| < \epsilon$$

for $a \leqq x \leqq b$.

The discussion of convergence of Fourier series given in the earlier sections

can be carried over in part to the case of Legendre series and to more general developments in series of polynomials. This is done, together with an extension to less elementary parts of the theory, in a paper in the Annals of Mathematics.[1]

THE POSTULATIONAL METHOD IN MATHEMATICS[2]

By E. V. HUNTINGTON, Harvard University

It is an interesting coincidence that the beginning of the Century of Progress which is now being celebrated in Chicago coincides almost exactly with the beginning of the postulational method in mathematics.

The postulational method in mathematics may be said to have had its origin in the non-Euclidean geometries of Lobachevsky and Bolyai which were published almost exactly one hundred years ago.

A systematic development of postulational technique began with Peano and his school in Italy in 1889, and Hilbert's "Foundation of Geometry" attracted wide attention to the method ten years later. In the United States important papers were published early in the twentieth century by O. Veblen, E. H. Moore, L. E. Dickson, B. A. Bernstein, H. M. Sheffer, and many others, in this field. After a short period of semi-quiescence, renewed interest in the postulational method has been recently shown by numerous books and papers, both on the mathematical side and on the philosophical side. The semi-popular accounts of the method in E. T. Bell's "The Queen of the Sciences," in the Century of Progress Series, and in C. J. Keyser's little book on "Thinking about Thinking," indicate the widespread popular interest in the subject. A rapidly increasing number of technical papers have been based on E. H. Moore's notion of "complete independence;" and the debate on the foundations of pure logic, which has centered for over twenty years around the monumental "Principia Mathematica" of Whitehead and Russell, is being carried on with renewed vigor in very recent years. Moreover in many widely separated fields of science the methods of attack on fundamental scientific problems are coming more and more to resemble the postulational method used in mathematics. P. W. Bridgman's "operational" theory in physics, although built up (as V. Lenzen has shown) from an entirely different point of view, may be mentioned as of particular interest in this connection.

Progress is so rapid that it is doubtless too early to attempt any assessment of the permanent value of the postulational method. All that I shall undertake to do in this brief paper is to examine one or two typical examples of simple sets

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[1] D. Jackson, *Series of orthogonal polynomials*, Annals of Mathematics, vol. 34 (1933), pp. 527–545.

[2] A paper read before the Mathematical Association of America at the joint meeting with Section A of the A.A.A.S. held in Chicago, June 20, 1933.

of postulates and to point out certain characteristics of the postulational method which this examination will suggest.

I propose to examine, for illustration, two systems (K, R) in which K is a class of elements $A, B, C, \cdots$, and R is a triadic relation.

The System of Betweenness

Let us consider first the system (K, R) in which K is the class of points $A, B, C, \cdots$ on a straight line, and the relation $R(ABC)$ is the familiar triadic relation of "betweenness."

If any three distinct points A, B, C of the line are presented (A first, B second, C third) for observation, we are supposed to be able to judge whether or not B lies between A and C. Thus, if B lies between A and C, we write "ABC in R," which means that A, B, C are in the relation R; and if B does not lie between A and C, we write "ABC not in R."

Any judgment of the form "ABC is in R," or "ABC is not in R," may be called a *primary observation.* A primary observation gives a direct answer to a "yes-or-no" question in regard to a particular situation. These primary observations constitute what we may call the "*first level*" of thought in this discussion.

In contrast with these primary observations another type of observation is to be noted. For example, in the given system of points on a line under the relation of betweenness, it is clear that no matter what set of points we observe, the combination "ABC-in-R and CBA-not-in-R" will *never occur.* A judgment to the effect that such and such a combination of primary observations never occurs in the given system may be called a *secondary observation.* We list for reference a selection of such secondary observations on the particular system under discussion (the numbering conforming with earlier publications).[1]

In the given system (K, R) we observe:

A. The combination "ABC-in-R and CBA-not-in-R" never occurs.

C. The combination "AXY-in-R and AYX-in-R" never occurs.

2. The combination "XAB-in-R and AYB-in-R and XAY-not-in-R" never occurs.

4. The combination "AXB-in-R and AYB-in-R and AXY-not-in-R and AYX-not-in-R" never occurs.

9. The combination "ABC-in-R and ABX-not-in-R and XBC-not-in-R" never occurs.

Secondary observations of this sort constitute what we may call the *second level* in this discussion.

It has always seemed to me a very remarkable fact that the human mind has the capacity for making "secondary observations" of this kind. A secondary observation is a sweeping judgment to the effect that no matter what particular set of points we consider in the given system certain combinations of primary

[1] Transactions of the American Mathematical Society, vol. 26 (1924), pp. 257–282.

observations will never occur. At first sight one might suppose that in order to justify such a sweeping judgment it would be necessary to examine all possible sets of points,—which appears to be impossible. But as a matter of fact we do not need to make an exhaustive list of all possible cases. We somehow know on the basis of a single observation that no combination of the sort mentioned can ever occur.

The distinction between a primary observation and a secondary observation is worth dwelling upon.

In a *primary observation* what we observe is a given triad of points ABC, and the question we answer is whether this given triad "is or is not in R."

To *deny* the primary observation "ABC is in R" means to assert that "ABC is not in R;" and to deny the primary observation "ABC is not in R" means to assert that "ABC is in R." The two observations "ABC-in-R" and "ABC-not-in-R" form a pair of contradictories; each is the denial of the other.

In a *secondary observation*, on the other hand, what we observe is the given system (K, R) as a whole, and the question we answer is whether certain combinations of primary observations "do or do not occur in the given system."

To *deny* the secondary observation "the combination ABC-in-R and CBA-not-in-R never occurs in the system," means to assert that "the combination ABC-in-R and CBA-not-in-R does occur at least once in the system." The discovery of a single instance of the combination in question would be sufficient to contradict the secondary observation that this combination never occurs in the system.—So much for the system of betweenness.

The System of Cyclic Order

Let us now consider another equally familiar system (K, R), namely the system in which K is the class of points on the circumference of a circle and R is the relation of cyclic order.[1]

In the new system, as before, if any three distinct points A, B, C on the circumference are presented for observation (A first, B second, C third), we are supposed to be able to judge whether or not these points satisfy the given relation (in this case the relation of cyclic order).

Among the secondary observations listed above some will be found to hold and some to fail in the new system.

For example, the secondary observation marked A will fail in the new system, but the following secondary observation will hold:

In the new system (K, R), we observe:

E. The combination "ABC-in-R and CAB-not-in-R" never occurs.

Hypothetical Laws

By thus considering a variety of systems (K, R) we are led to regard such statements as those numbered A, C, E, 2, 4, 9 above no longer as *observations*

[1] Proceedings of the National Academy of Sciences, vol. 10 (1924), pp. 74–78.

made upon a given system (K, R) but as *hypothetical laws* which may be observed to "hold" in some systems (K, R) and observed to "fail" in others. The study of the interrelations between these hypothetical laws constitutes the "*third level*" in this discussion,—and it is only on this third level that mathematical theorems appear. Before giving an example of a mathematical theorem, it will be well to recapitulate the three "levels" to which our discussion has led us.

THE FIRST LEVEL (CLASSIFICATION OF INDIVIDUAL TRIADS)

On the first level we consider a given triad and inquire whether it is or is not "in R." The result is a "primary observation" which concerns only the given triad, and no "variable" is involved.

What the relation R does is to provide a means of classification of the individual triads in a given system. If a given triad $A_1B_1C_1$ "passes the test," we write, for brevity, $A_1B_1C_1\epsilon R$ and if another given triad $A_2B_2C_2$ "fails to pass the test" we write $A_2B_2C_2\epsilon R'$. (The epsilon notation ϵ, taken from Peano, may be read "belongs to"). Whatever triad $A_0B_0C_0$ is presented for observation, we know that either $A_0B_0C_0\epsilon R$ or else $A_0B_0C_0\epsilon R'$ will hold; but the combination $ABC\epsilon R$ and $ABC\epsilon R'$ will never occur.

THE SECOND LEVEL (CLASSIFICATION OF SYSTEMS K, R)

On the second level, we consider a given system (K, R) as a whole and inquire whether it does or does not satisfy a given "hypothetical law." The result is a "secondary observation," in which the system (K, R) as a whole, and not an individual triad, is the subject of the observation.

What a "hypothetical law" does is to provide a classification of the systems (K, R) themselves. Each "law" listed above provides a separate test. For example, Law 9 is a test on which every system (K, R) will either pass or fail.

If a given system $S_1=(K_1, R_1)$ "passes the test 9," we write $S_1\epsilon 9$, so that, in convenient symbolism, $S_1\epsilon 9$ means:

$$A\epsilon K_1\cdot B\epsilon K_1\cdot C\epsilon K_1\cdot X\epsilon K_1\cdot ABC\epsilon R_1\cdot ABX\epsilon R_1'\cdot XBC\epsilon R_1' : \rightarrow : 0.$$

Here the notation "$\rightarrow 0$" means that the combination of observations preceding this symbol never occurs; the dots serve merely to separate the several observations.

If given system S_2 "fails to pass the test," we write $S_2\epsilon 9'$, where $S_2\epsilon 9'$ means:

$$\exists(A_0, B_0, C_0, X_0 \text{ in } K_2) \text{ such that } [A_0B_0C_0\epsilon R_2\cdot\&\cdot A_0B_0X_0\epsilon R_2'\cdot\&\cdot X_0B_0C_0\epsilon R_2'].$$

Here the notation "$\exists\cdots$" means "there exists at least one set of elements $\cdots$" such that each one of the observations indicated will hold.

The two statements $S\epsilon 9$ and $S\epsilon 9'$ are contradictories, in the sense that whatever system $S=(K, R)$ is presented for observation, we know that either $S\epsilon 9$ or else $S\epsilon 9'$ will hold; and that the combination $S\epsilon 9$ and $S\epsilon 9'$ will never occur.

Similarly, each of the other laws mentioned above, like Law A, Law C, etc., will provide its own basis of classification of the systems (K, R).

THE THIRD LEVEL (INTERRELATIONS AMONG HYPOTHETICAL LAWS)

When now we rise to the third level, we consider combinations of several hypothetical laws taken simultaneously.

As we have seen, any single law, regarded as a means of classifying the systems (K, R), will effect a dichotomy. Law 4, for example, divides the class of systems (K, R) into two mutually exclusive subclasses, which may be called 4 and $4'$, and if any system S is given, it will certainly belong to one or other of these subclasses, but not to both; that is, one and only one of the two possibilities $S\epsilon 4$ and $S\epsilon 4'$ will occur.

But the combination of any two laws taken together, as for example, Law 4 and Law 9, will divide the class of systems (K, R) into four subclasses, or compartments. Thus, if S is any given system, one and only one of the following four possibilities will occur:

$$(S\epsilon 4 \cdot S\epsilon 9); \quad (S\epsilon 4 \cdot S\epsilon 9'); \quad (S\epsilon 4' \cdot S\epsilon 9); \quad (S\epsilon 4' \cdot S\epsilon 9').$$

Similarly, the combination of any three laws, taken together, will divide the class of systems into 2^3, or 8, compartments; and so in general, any n laws, taken together, will divide the class into 2^n compartments.

The important question now is, whether any of these compartments may be "empty." If no compartment is "empty"—that is, if each of the 2^n compartments contains at least one actual system—then the n laws are said to be "completely independent" in the sense of E. H. Moore. If on the other hand any one of the compartments is "empty" we have a case in which some of the laws depend upon others; and the assertion (or denial) that such-and-such a compartment is "empty" is precisely what we mean by a "mathematical proposition."

An example of such a mathematical proposition concerning the laws 2, $2'$; 4, $4'$; and 9, $9'$, is the following:

$$P: \qquad S\epsilon 2 \cdot S\epsilon 9 \cdot S\epsilon 4' : \rightarrow : 0,$$

which says that the compartment $(2.9.4')$ is empty. This means that there is no system S which obeys the combination of three laws 2, 9, and $4'$.

To *deny* this proposition P means to assert that there exists at least one system S_0 which belongs in this compartment, that is, at least one system S_0 which has the three properties mentioned. Hence if we denote the denial of P by P', then P' will mean

$$\exists(S_0) \text{ such that } [S_0\epsilon 2 \cdot \& \cdot S_0\epsilon 9 \cdot \& \cdot S_0\epsilon 4'].$$

The two propositions P and P' are logical contradictories; one and only one is valid. To indicate that P is valid, we write $(1 \rightarrow P)$, which may be read: "starting from the first principles, we are led to P"; or simply, "truth leads to P."

The same fact may be expressed also by writing $P' \rightarrow 0$, which may be read: "P' leads to contradiction," or "P' leads to nothing." In other words, to prove the theorem that $1 \rightarrow P$, it is sufficient to prove the theorem that $P' \rightarrow 0$.

A Sample of a Mathematical Proof

Let us now briefly examine what appears to be involved in the proof of a mathematical theorem, as exemplified by the proof of the theorem just mentioned, namely: If S is any system (K, R), then

$$P: \quad (S\epsilon 2)\cdot(S\epsilon 9)\cdot(S\epsilon 4'): \to :0.$$

The proof runs as follows: Suppose we consider the expression P' where, by definition, P' means

(1) $$P' \to \exists(S_0) \text{ such that } [S_0\epsilon 2\cdot\&\cdot S_0\epsilon 9\cdot\&\cdot S_0\epsilon 4'].$$

From (1), $P'\to(S_0\epsilon 4')$. Here, by definition, $S_0\epsilon 4'$ means

$$\exists(A_0, B_0, X_0, Y_0 \text{ in } K_0)$$

such that

$$[A_0X_0B_0\epsilon R_0\cdot\&\cdot A_0Y_0B_0\epsilon R_0\cdot\&\cdot A_0X_0Y_0\epsilon R_0'\cdot\&\cdot A_0Y_0X_0\epsilon R_0'];$$

so that

(2) $$P' \to (A_0X_0B_0\epsilon R_0),$$

and

(3) $$P' \to (A_0Y_0B_0\epsilon R_0),$$

and

(4) $$P' \to (A_0X_0Y_0\epsilon R_0'),$$

and

(5) $$P' \to (A_0Y_0X_0\epsilon R_0').$$

Again from (1), $P'\to(S_0\epsilon 9)$; whence, by definition of $S_0\epsilon 9$,

(6) $$P' \to [A_0X_0B_0\epsilon R_0\cdot A_0X_0Y_0\epsilon R_0'\cdot Y_0X_0B_0\epsilon R_0': \to :0].$$

By (6) and (2),

(7) $$P' \to [A_0X_0Y_0\epsilon R_0'\cdot Y_0X_0B_0\epsilon R_0': \to :0].$$

By (7) and (4),

(8) $$P' \to [Y_0X_0B_0\epsilon R_0': \to :0].$$

Hence by (8),

(9) $$P' \to (Y_0X_0B_0\epsilon R_0).$$

Again from (1), $P'\to(S_0\epsilon 2)$; whence, by definition of $S_0\epsilon 2$,

(10) $$P' \to [A_0Y_0B_0\epsilon R_0\cdot Y_0X_0B_0\epsilon R_0\cdot A_0Y_0X_0\epsilon R_0': \to :0].$$

By (10) and (3),

(11) $$P' \rightarrow [Y_0X_0B_0\epsilon R_0 \cdot A_0Y_0X_0\epsilon R_0' : \rightarrow :0].$$

By (11) and (9),

(12) $$P' \rightarrow [A_0Y_0X_0\epsilon R_0' : \rightarrow :0].$$

Hence by (12),

(13) $$P' \rightarrow (A_0Y_0X_0\epsilon R_0).$$

But from (5) and (13), $P' \rightarrow 0$. Therefore $1 \rightarrow P$, as was to be proved.

A Few of the Properties of the Implicative Relation

Let us examine briefly a few of the properties which characterize the deductive processes, as exemplified in the proof just presented.

Let $a_0, b_0, c_0, \cdots$, represent actual primary observations on given triads.

Let $a, b, c, \cdots$, represent hypothetical primary observations, which may be true for some triads and false for others; and let a' represent the contradictory of a.

Let $i_0, j_0, k_0, \cdots$, represent actual secondary observations on given systems.

Let $i, j, k, \cdots$, represent hypothetical secondary observations, which may be true for some systems and false for others; and let i' represent the contradictory of i.

(For brevity, let $x, y, z, \cdots$, represent any hypothetical observations, primary or secondary.)

Let $p, q, r, \cdots$, represent mathematical propositions, that is, statements which assert that such-and-such combinations of hypothetical observations never occur in any system. For example, suppose p asserts that the combination ijk does not occur in any system, then the contradictory proposition p' will assert that there exists at least one system S_0 in which this combination does occur; and we write $p \rightarrow (ijk \rightarrow 0)$ and $p' \rightarrow (i_0j_0k_0)$. It is understood that in any combination the order in which the letters are written is immaterial; and the arrow, $\rightarrow$, may be read "leads to," or "implies."

Then a few of the principles which the implicative relation ($\rightarrow$) is observed to obey are the following.

I. $p \rightarrow (x_0y_0z_0) : \rightarrow : p \rightarrow (x_0).$

That is, if the proposition p leads to $(x_0y_0z_0)$, then this fact itself will lead to the further fact that p leads to (x_0).

II. $p \rightarrow (x_0y_0z_0 \rightarrow 0) \cdot p \rightarrow x_0 : \rightarrow : p \rightarrow (y_0z_0 \rightarrow 0).$

That is, if it is found that p leads to $(x_0y_0z_0 \rightarrow 0)$ and also that $p \rightarrow x_0$, then it follows that p leads to $(y_0z_0 \rightarrow 0)$.

III. $p \rightarrow x_0 \cdot p \rightarrow x_0' : \rightarrow : p \rightarrow 0.$

That is, if it is found that p leads to x_0 and also that p leads to x_0', then p leads to a contradiction and is to be rejected.

IV. (a) $p \to x_0 : \to : p \to (x_0' \to 0)$; and (b) $p \to (x_0 \to 0) : \to : p \to x_0'$.

That is, if p implies that a certain observation occurs, then p implies that the contradictory observation does not occur; and conversely.

V. $p \to i \cdot i \to (a_0 b_0 c_0) : \to : p \to (a_0 b_0 c_0)$.

That is, if p leads to i, where i is a secondary observation, and i leads to $(a_0 b_0 c_0)$, where a_0, b_0, c_0 are primary observations, then p leads to $(a_0 b_0 c_0)$. Similarly,

VI. $p \to j \cdot j \to (abc \to 0) : \to : p \to (abc \to 0)$.

These statements I–VI, and others like them, are written down first as actual observations on the particular system—let us say, tentatively, the system $\Sigma = (K_1, K_2, x, ', \to)$—which we call the system of deduction; but by treating the symbols K_1, K_2, x, $'$, $\to$ (or whatever they ought to be) as variables instead of constants, those statements become hypothetical laws which may be valid in some systems Σ and not valid in others.

Just as the earlier statements A, C, 2, 4, 9 were obtained originally as actual observations on a particular system $S = (K, R)$ and were afterwards regarded as hypothetical laws providing a means of classification of all systems S, so the statements I–VI, obtained in the first place as actual observations on a particular system $\Sigma = (K_1, K_2, x, ', \to)$, may properly be regarded as hypothetical laws providing a means of classification of all systems Σ.

In this sense, it is just as legitimate to talk about a "set of postulates for the theory of deduction" as it is to talk about a "set of postulates for the theory of betweenness." In the one case we are classifying systems $S = (K, R)$; in the other case we are classifying systems $\Sigma = (K_1, K_2, x, ', \to)$.

Many attempts have been made to establish such a set of postulates for the theory of deduction, but no one of them has as yet secured universal acceptance.[1]

The above statements I–IV are not proposed as a definite set of postulates; it is not even clear exactly how the "base" $(K_1, K_2, x, ', \to)$ ought to be analyzed. The point that I am attempting to make is that the problem of finding a set of postulates for the theory of deduction is mathematically just as legitimate a problem as the problem of finding a set of postulates for the theory of betweenness or the theory of Boolean algebra or the theory of geometry. In each case the postulational problem is a problem of classification; and all the postulational technique—consistency, independence, complete independence, etc.—appears to be just as applicable in the one case which is not yet satisfactorily solved as it is in the other cases where solutions are already familiar.

[1] For a new point of view in regard to the "formal" and "informal" theories of deduction in Russell and Whitehead's Principia Mathematica, see a forthcoming paper by the present writer which will appear in the Bulletin of the American Mathematical Society early in 1934.

If this conclusion is correct, I hope that this paper may help in some degree to encourage further work in the direction of a definitive set of postulates for the theory of deduction.

QUESTIONS, DISCUSSIONS, AND NOTES

EDITED BY R. E. GILMAN, Brown University, Providence, Rhode Island

The department of Questions and Discussions in the Monthly is open to all forms of activity in collegiate mathematics, including the teaching of mathematics, except for specific problems, especially new problems which are reserved for the department of Problems and Solutions.

A PRACTICAL INSURANCE PROBLEM FOR COURSES IN THE MATHEMATICS OF INVESTMENT[1]

By C. N. REYNOLDS, West Virginia University

Many courses in the mathematics of investment are brought to a climax with the computation of terminal reserves for net level premium life insurance policies. Some go further and introduce the computation of a policy's contribution to the company's surplus by means of an important formula developed by Sheppard Homans then actuary of the Mutual Life Insurance Company of New York, in a letter dated March 13, 1868. This formula however is no longer used for the computation of dividends.[2] In either case, the course closes with material of no great practical value to the average student. Most of us accept without question our insurance dividends and cash surrender values.

This note is a report on the successful introduction, during the past four years, of the more immediately practical problem of determining the rate of interest secured by the policy-holder on his policy considered as an investment. This problem arises as one chooses a policy and also when one considers the advisability of surrendering a policy.

Homan's value of the policy's contribution to the surplus during the (n+1)th policy year may be expressed as follows:[3]

$$({}_nT_x + G_x - e)(1 + i') - {}_{n+1}T_x - q'_{x+n}(1 - {}_{n+1}T_x), \tag{1}$$

where x is the age at which the policy is issued,

${}_nT_x$ is the terminal reserve at the end of the nth policy year,

G_x is the gross premium, of which

e is the portion used for the company's expenses,

i' is the average rate of interest secured on the company's investments,

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[1] Publication No. 94 of the Division of Industrial Sciences, W. V. U.

[2] See *A Review of . . . Methods of Surplus Distribution,* J. C. Rietz, Record of the American Institute of Actuaries, vol. 11, no. 23, June, 1922, p. 106. and *A Process for Calculating Annual Dividends by the Karup Method of Attained-Age Valuation,* Harold A. Grout, Transactions of the Actuarial Society of America, vol. 27, part 1, no. 75, May, 1926, p. 59.

[3] See Kuhn and Morris, *Mathematics of Finance,* p. 249.

q'_{x+n} is the probability of the policy-holder's death during the $(n+1)$th policy year according to the company's actual experience.

This expression is written from the company's point of view.

From the policy-holder's point of view, the portion of the contribution to the surplus which is of significance is a known quantity, δ_{x+n}, the dividend credited to him at the end of the $(n+1)$th policy year. For him, the terminal reserves are to be replaced by the cash surrender values, ${}_nV_x$, as given in his policy, which are usually equal to zero during the first three years.[1] Just as brokerage fees are a portion of the sum invested in buying bonds upon which an average interest return might be computed, so the policy-holder, having invested the gross premium, G_x, will omit from his consideration the term "$-e$" of (1). The actual mortality factor, q'_{x+n}, experienced by the larger companies at the present time is slightly less than that indicated in the American Men's Ultimate Mortality Table[2] ($AM^{(5)}$) which is, in turn, much less than that indicated in the American Experience Table of Mortality (published in 1868). It if is not convenient to use this table, we may replace q'_{x+n} by kq_{x+n}, a constant multiple of the mortality factor obtained from the American Experience Table of Mortality. For the twenty-year period just closing, k might conservatively be taken as 0.60, but it has been rising since 1929 and in 1931, it was about 0.65 for the larger companies. On the other hand, our interest factor $(1+i')$ becomes an unknown quantity $(1+i_{n+1})$ which varies from year to year during the history of a policy. Our new equation is:

$$(2) \qquad [{}_nV_x + G_x](1 + i_{n+1}) = \delta_{x+n} + {}_{n+1}V_x + q'_{x+n}(1 - {}_{n+1}V_x).$$

By omitting "$G_x - e$" or "G_x" from (1) or (2) respectively, they become applicable to paid-up policies. By omitting the cost of insurance term $q'_{x+n}(1 - {}_{n+1}T_x)$ or $q'_{x+n}(1 - {}_{n+1}V_x)$ from (1) or (2) respectively, they become applicable to participating deferred life annuity contracts.

If we consider a particular policy in a given company and its dividend history for a period of twenty years, we have for $n = 0, 1, 2, \cdots, 19$, twenty equations (2), the solutions of which $i_1, i_2, \cdots, i_{20}$ give us the history of the investment value of our policy. Whenever, as is usually the case, the cash surrender value of a policy is zero until three annual premiums have been paid, the earlier values of i_{n+1} in our sequence are negative. For a given n, the values of i_{n+1} are monotone increasing functions of G_x which, in turn, depends upon the type of policy involved. In other words, the insurance agent is correct in urging

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[1] Complete information concerning cash surrender values for the first twenty years of the life of a policy is given in the current rate books of most of the larger insurance companies. The rate books of the Metropolitan and John Hancock companies are exceptions. Dividend histories over periods of ten or twenty years may be secured for the usual types of policies when issued at quinquennial age intervals from such handbooks as the Unique Manual Digest and the Flitcraft publications.

[2] See *American-Canadian Mortality Investigation*, 1900–1915, Actuarial Society of America.

the value of limited payment policies and endowment policies from the investment point of view.

In problems of this type, mean values for the interest rate taken over a period of years are not particularly significant since the actual situation is more adequately described by our sequence of values of i_{n+1}.

If, however, we replace i_{n+1} by r_k in (2), then sum our equations from $n=0$ to $n=k-1$, and then solve for r_k, we obtain a weighted mean of the i_{n+1}'s, each annual rate being weighted with the sum which accumulates interest at that rate for one year. Thus r_k is the average rate of simple interest for the first k years of the policy.

If we replace i_{n+1} by r_k in (2), multiply the equation by $(1+r_k')^{k-n-1}$ and sum from $n=0$ to $n=k-1$, we obtain the equation

$$
\begin{aligned}
(3)\qquad G_x(1+r_k')^k + \sum_{n=1}^{k-1}(G_x - \delta_{x+n-1} - q'_{x+n-1}(1 - {}_nV_x))(1+r_k')^{k-n} \\
- (\delta_{x+k-1} + {}_kV_x + q'_{x+k-1}(1 - {}_kV_x)) = 0.
\end{aligned}
$$

Solving this equation by a contracted form of Horner's method, we obtain a rate of compound interest equivalent to our varying rates of simple interest for the first k policy years. Here each $(1+i_{n+1})$ has been weighted with the value of the money invested in the $(n+1)$th policy year accumulated by compound interest at rate r_k' to the end of our period of k years. Since this mean value is in no sense a "true" value and since Horner's method will, from the point of view of an average class in the mathematics of investment, be a vague recollection of a misery which is past, I prefer to present to such classes the problem of evaluating r_k rather than r_k'.

Using the methods indicated in this note[1] with q'_{x+n} taken from the AM$^{(5)}$ Mortality Table, we find for policies issued in 1909 by the Mutual Life Insurance Company of New York to persons twenty-five years of age, the following results:

Ordinary Life	$i'_{20}=.039$	$r_{20}=.005$	$r'_{20}=.004$
20 Payment Life	$i'_{20}=.043$	$r_{20}=.026$	$r'_{20}=.022$
20 Year Endowment	$i'_{20}=.044$	$r_{20}=.041$	$r'_{20}=.033$

For a John Hancock Retirement Annuity Contract with \$100 annual premium, (premiums, dividends, and cash surrender values being independent of age, this is merely a savings plan) we find that $i'_{20}=.044$, $r_{20}=.049$ and $r'_{20}=.038$.

An Operational Formula

By H. E. Dow, University of Vermont

In the discussion of linear differential equations with real or imaginary constant coefficients, rules are given for obtaining in special cases the particular integral of

[1] The tabular arrangement of the computation involved in these problems in suggested by that of any investment schedule, and is, accordingly, omitted from this note.

$$F(D)y = f(x),$$

where $F(D)$ is a polynomial in $D \equiv d/dx$, and $f(x) = \sin px$ or $f(x) = \cos px$. These special cases are that $F(D)$ contains only even powers of D or else contains only odd powers of D. A rule yielding a solution when both odd and even powers of D are present is seldom given. A general rule for obtaining a solution in this case may be found in Fry's *Differential Equations*,[1] but, to the student, this method appears artificial. C. A. Hutchinson presented a method[2] to be applied in this case, but we may develop a method which uses a different approach and which carries his results a step farther.

We shall attack the problem by using the exponential expression for cos px and sin px,

$$\cos px = (e^{ipx} + e^{-ipx})/2, \quad \sin px = (e^{ipx} - e^{-ipx})/2i,$$

where p is a real or complex constant.

We shall also need to know the particular solution of

$$F(D)y = Ae^{ax}.$$

Substituting $y = Be^{ax}$, we have $BF(a) = A$, whence $B = A/F(a)$ if $F(a) \neq 0$, and

$$\frac{1}{F(D)} Ae^{ax} = \frac{Ae^{ax}}{F(a)}.$$

Now to return to our initial problem, dealing with the case that $f(x) = \sin px$. We may then write

$$\sin px = (e^{ipx} - e^{-ipx})/2i$$

$$\frac{1}{F(D)} \sin px = \frac{1}{F(D)} \frac{e^{ipx}}{2i} - \frac{1}{F(D)} \frac{e^{-ipx}}{2i}.$$

But we know how to find solutions of the expressions on the right. We obtain

$$\frac{1}{F(D)} \sin px = \frac{e^{ipx}}{2iF(ip)} - \frac{e^{-ipx}}{2iF(-ip)}$$

provided $F(ip)F(-ip) \neq 0$.

Let us then find constants a and b such that[3]

$$(1) \qquad \begin{aligned} a + bi &= F(ip) \\ a - bi &= F(-ip) \end{aligned}$$

---

[1] Cf. T. C. Fry, *Differential Equations*, page 162.

[2] Cf. *An Operational Formula*, this MONTHLY, vol. 40 (1933), page 482.

[3] This is possible. By Cramer's Rule, we may find unique solutions of $x+yi=F(ip)$, $x-yi = F(-ip)$, since

$$\begin{vmatrix} 1 & i \\ 1 & -i \end{vmatrix} = -2i \neq 0.$$

and the condition that $F(ip)F(-ip)\neq 0$, is replaced by the condition that $a^2+b^2\neq 0$. In this form we have,

$$\frac{1}{F(D)}\sin px = \frac{1}{2i}\left(\frac{e^{ipx}}{a+bi} - \frac{e^{-ipx}}{a-bi}\right)$$

$$= \frac{1}{a^2+b^2}\left[\frac{a(e^{ipx}-e^{-ipx})}{2i} - \frac{b(e^{ipx}+e^{-ipx})}{2}\right].$$

$$\frac{1}{F(D)}\sin px = \frac{1}{a^2+b^2}(a\sin px - b\cos px), \tag{2}$$

where a and b are constants satisfying equations (1).

In a similar manner we show that,

$$\frac{1}{F(D)}\cos px = \frac{1}{a^2+b^2}(a\cos px + b\sin px), \tag{3}$$

where a and b are constants satisfying equations (1).

The case most often met in practice is that p is real and the coefficients of $F(D)$ are real. In this special case a is the real part of $F(ip)$ while b is the real coefficient of the imaginary part of $F(ip)$. Therefore we have found a rule covering all cases and which admits of easy application to the ordinary cases.

RECENT PUBLICATIONS

EDITED BY R. A. JOHNSON, Brooklyn College of the City of New York

All books for review should be sent directly to the editor of this department, at Brooklyn College, 66 Court Street, Brooklyn, N. Y., and not to any of the other editors or officers of the Association.

REVIEWS

The Relativity Theory Simplified. By Max Talmey, M.D. New York, Falcon Press, 1932 xi+186 pages. \$1.85.

Max Talmey, a doctor of medicine and a personal friend of Einstein, has succeeded as probably no other man has in producing a book, explaining the relativity theory which any person of ordinary intelligence can read with understanding. His powers of logical but simple exposition are remarkable. With no use of imperfect analogies, from which incorrect impressions too often come, with a scrupulous care to avoid all mysticism and ideas merely fanciful, and leaving out all philosophical discussion, Dr. Talmey has explained the facts so clearly that any person with an interest in this subject can grasp them. A non-scientist reader, not accustomed to the slowness with which he must read scientific work in order to assimilate it, should be cautioned not to read more than one or two of the short chapters of this book at a time, and to think them over carefully before reading further. The mind will then more easily become

accustomed to this unusual line of thought, and the reader will become aware that he as well as the fictitious "only twelve men in the world beside Einstein" can understand the relativity theory.

The concluding chapter on the formative period of Einstein's life by this personal friend is an interesting light on the life of the great man whose theory bears his name.

C. S. ATCHISON

Elementary Mechanics of Solids. By H. A. Baxter. London, Blackie and Son Limited, 1933. 281 pages.

The present text, prepared to cover courses for the Higher School Certificate in England, is well suited for use in American colleges and technical institutes. An idea of the material treated can be gained from a glance at the chapter headings: I—Motion of a Particle; II—The Interaction of Two Particles; III—Some Important Cases (including the value of g, harmonic motion, and projectiles); IV—Systems of Particles; V—Motion of a Rigid Body in One Plane; VI—Energy of Rigid Bodies Moving in One Plane; VII—Centres of Inertia, Moments of Inertia, Dimensions, Motions of a Solid (Momentum, Axis, Moment of Momentum).

In his preface the author states that he has "aimed at the development of the mechanics of solids from the axioms laid down by Newton." In accomplishing this aim the author has treated the subject matter rigorously throughout, making quite apparent the direct dependence of the principles developed upon the axioms set forth at the outset. A major virtue of the book is the simplicity and clarity of the proofs and of the illustrative examples. In the complicated developments reference is made to the place in the text where the principles and formulas used are first given.

It is intended that students using this text will have already had some training in calculus, or will be taking a course in calculus concurrent with the course in mechanics. In order to meet the needs of the second group of students, the author develops the notion of the differential and the derivative before he makes use of it. For the subject matter of Chapter II, however, it is expected that the student has acquired a knowledge of simple integration. Thereafter free use is made of the calculus.

A wealth of problems has been given at the close of each chapter. The author has taken great pains in grading the problems in order of difficulty and complexity. Each set is divided into groups such that problems within a group involve a principle not yet introduced in preceding groups. In addition, the author furnishes a complete set of answers. To the present reviewer it seems that "Mechanics of Solids" should prove quite valuable to both teachers and students in an introductory course in Mechanics.

In closing a word might be said about the physical makeup of the book. The type and setup are such as to make for easy reading. The diagrams are clearly

and well drawn. The text is free of misprints. In short the publishers are to be thanked for giving this book an excellent and pleasing appearance.

S. B. LITTAUER

Brief Course in Plane and Spherical Trigonometry. By H. A. Davis and L. A. Chambers. New York, American Book Company, 1933. 136 pages. $2.00 with tables; $1.50 without tables.

In this text, the authors have made omissions from the material usually given, and by so doing have succeeded in making the essentials stand out clearly, and have saved time for the more unusual matter to which they devote some later chapters.

In presentation of topics the authors follow the custom of devoting a chapter to the acute angle before taking up the general angle, a preference not shared by the reviewer who objects to definitions of trigonometric functions that will not apply to obtuse angles. This early chapter is followed by one on logarithms and their application to the solution of right triangles, with a rich supply of problems.

At this point the general angle is introduced, and reduction formulas are given in very brief, pointed style, with applications. Further use of the reduction formulas is made in connection with the graphs. For most of the graphs, line values of the functions are plotted, at intervals of $22°30'$, with clear exposition. The explanation of the graph for the cotangent would, however, probably offer some students the opportunity for errors and misconceptions. In connection with graphs, the reviewer is delighted to find addition of ordinates.

Geometric proof is given for $\sin (x-y)$, $\cos (x-y)$ as well as for $\sin (x+y)$ and $\cos (x+y)$, where x, y, and $x+y$ are acute, while analytical proof is shown for $\sin (x+y)$ where $x+y$ is obtuse. For addition formulas a very good, though somewhat meager list of identities is given, while a larger emphasis is laid on identities of inverse functions, in a later chapter, following the solution of triangles. Besides a good list of trigonometric equations of the usual type, the authors present simultaneous trigonometric equations.

In the solution of triangles, the order of laws is chosen which gives the student, for each method of solution, an appropriate check for accuracy. While the law of cosines is proved, no encouragement is given the student for its use in solving triangles even in the case where the sides are represented by one digit numbers. Besides a large supply of examples for formal solutions, the authors list a number of problems to be solved.

Areas are given a separate chapter, with more emphasis than most teachers will use. An excellent point in this chapter is the exposition of the relation of the triangle to the inscribed and circumscribed circles, the often omitted demonstration of the value of the radius of the inscribed circle in terms of the sides of the triangle, and the uncommon proof of the law of sines by means of the circumscribed circle. The medians, also, are given in terms of the sides.

In the "Imaginary in Trigonometry," DeMoivre's Theorem is used both for computation and development of theory. Especially unusual is it to find in a brief trigonometry the series expansion for sin x, cos x, and e^x, with the consequent Euler equations of relation between the trigonometric and exponential functions. Hyperbolic functions and their inverses also appear.

Thirty pages are devoted to a brief but adequate practical treatment of spherical trigonometry. The examples include applications to astronomy and navigation. The excellent figures will aid the student here.

An appendix contains some elementary information on Kepler's laws of planetary motion, equinoxes, the moon, stars, eclipses, tides, time, and observations, with mention of refraction, parallax, and aberration.

The reviewer found few misprints. Some small details came to attention: arrows are sometimes omitted from angles where they would aid, ∞ is used without benefit of plus or minus sign, there is lack of explanation of the error of five place tables in computing angles supposedly correct to the second, and directed line segments are designated by one letter. These matters are minor, however, and one sees, instead of them, the success of the authors in presenting a concise, well arranged text, with an unusual amount of thought-provoking material.

M. E. WELLS

The Geometry of Repeating Design and Geometry of Design for High Schools. By A. D. Bradley. New York, Teachers College, Columbia University, 1933. 127 pages. $1.50.

As the title of the book indicates, the material is presented in two parts, the first of which deals with theory, the second with theorems and applications. The exhibit of repeating design and their constructions assembles here a large amount of material from scattered sources, and the author classifies and makes additions with continual reference to the sources, particularly on pattern design, tile work, and mosaics, giving the principal theorems various authors have written on repeating design. However, the technical work on groups and Diophantine equations is avoided, as the aim of this study is to make a simple presentation of repeating design in a "manner to make clear its value as professionalized subject matter in geometry." The author presents motions which generate patterns as a problem in the theory of groups but simply enough for understanding by plane geometry students.

The chief configurations covered may be classified as regular polygons, at rest and also under translation and rotation, "regular point systems" of Sohncke, "circle packing" of Niggli, the "par-hexagon," triangles and star polygon designs.

The chapters of greatest interest to the teacher of plane geometry comprise Part II. For the class accustomed to thinking independently and originating theorems and applications, this part will serve as a storehouse of suggestive material. Exercises are arranged in careful sequence with theorems, giving the

student the necessary feeling of collaboration in building up the whole structure of his geometry, instead of the too frequent feeling of working on some extra and unessential ornamentation of a structure already presented in completed form. As the author follows the report of the National Committee on Mathematical Requirements the book may easily be used as a substantial part of a course in Plane Geometry.

While the book is "intended to function as a source book of problems for teachers of plane geometry" it is hoped by the reviewer that teachers who use the book will see that students have easy access to copies for frequent and prolonged study, with time allowed for working out problems, as it is only by actually constructing configurations in design that the masterful sense of proportions, as well as deep consciousness of significance of line relations and angle magnitude, can be developed.

The book is of convenient size, and is exceedingly well printed. A large bibliography is appended.

M. E. WELLS

MATHEMATICS CLUBS

EDITED BY F. M. WEIDA, The George Washington University, Washington, D. C.

All reports of club activities, suggestions and topics for club programs, and material of interest to clubs should be sent to F. M. Weida, The George Washington University, Washington, D. C. All manuscripts should be typewritten, with double spacing, and with margins at least one inch wide.

CLUB ACTIVITIES
1932–33

THE PI MU EPSILON MATHEMATICAL FRATERNITY

Pi Mu Epsilon is an academic fraternity in institutions of university grade. Its primary aim is the advancement of mathematics and scholarship. It is a living, active, working fraternity of scholars in which the members are engaged in study and research and the exchange of ideas in the field of mathematical science.

Pi Mu Epsilon of the Ohio State University

The officers for the past year were: Jack T. Kent, Director; David Dawson, Vice Director; Brandon G. Rightmire, Treasurer; Marjorie Leffler, Secretary.

We had forty-six active members. Eleven were initiated on May 27, 1932. Nineteen students were invited to membership at a meeting on May 8, 1933 and these students were initiated on May 24, 1933.

During the past year, the chapter sponsored two visiting lecturers: Professor J. R. Musselman of Western Reserve University and Professor I. A. Barnett of the University of Cincinnati. A calendar of the club's activities other than regular business meetings follows:

October 31, 1932: "The area and boundary of minimal surfaces" by Dr. E. F. Beckenbach.

November 7, 1932: "Subharmonic functions and minimal surfaces" by Dr. E. F. Beckenbach.

December 9, 1932: "Collineation groups and their configurations" by Dr. J. R. Musselman of Western Reserve University. This lecture was followed by an informal dinner.

December 19, 1932: "On subharmonic functions" by Professor Tibor Radó.

January 28, 1933: We had a party which was attended by fifty-nine members and guests.
February 6, 1933: "Riemannian geometry in function space" by Dr. Henry P. Thielman.
February 20, 1933: "On uniform convergence" by J. W. T. Suckau.
March 6, 1933: "The use of complex numbers in the geometry of the triangle" by Dr. J. H. Weaver.
March 10, 1933: "Functional calculus" by Professor I. A. Barnett of the University of Cincinnati. This lecture was followed by an informal dinner.
May 8, 1933: "On the solution of a certain partial differential equation in the neighborhood of a singular point" by H. H. Alden.
May 22, 1933: "Hensel's *p-adic* numbers" by W. J. Robinson.
May 24, 1933: Initiation banquet.

JACK T. KENT, *Director*

Pi Mu Epsilon of the University of Nebraska

The officers for the past year were: Hubert A. Arnold, Director; Mrs. Alice P. Thompson, Vice Director; Cedric W. Richards, Secretary; Paul Bartunek, Treasurer.

We had sixty-two active members. Five were initiated on May 25, 1932 and fifteen were initiated on January 31, 1933.

The primary aim of the society is to promote interest in scholarship and especially in mathematics. An average of at least 85% in mathematics for those who have completed the calculus and an average of 95% for those just finishing it are the requirements for eligibility.

The meetings and programs were as follows:

November 1, 1932: "Sailors, cocoanuts and monkeys" by Mr. Fred B. Daniels.
November 29, 1932: "The nature of mathematics" by Mr. Dearborn; "The oscillograph and electric currents" by Mr. Bollman.
December 13, 1932: "Tides" by Mr. Lowry; "Roots of equations" by Miss Vita Oberlander.
January 31, 1933: The annual initiation banquet was held at the Hotel Nebraskan. Professor M. G. Gaba was the toastmaster. Dr. Candy gave a short talk, followed by Professor Brenke who described methods used in European Colleges. Mr. Daniels talked about his experiences in German Universities. Fifteen new members were elected.
March 7, 1933: "Mathematical fields" by Mr. Roberts.
April 6, 1933: "Magic squares" by Dr. Candy.
May 10, 1933: The annual Pi Mu Epsilon examinations were held. The prizes of ten dollars each were won by Mr. James Marvin and Mr. M. Nuernberger.
May 16, 1933: "History of mathematics" by Miss Dolores S. Bernhardt; "A ladder problem leading to an equation of the fourth degree" by Mr. Richards. The names of seven new members were voted upon and accepted.

We had two committees, the program committee and the examinations committee. The members who served on the program committee were: Mr. Richards, Miss M. M. Wicker, and Mr. R. B. Thompson. The members who served on the examinations committee were: Mr. Lowry and Mr. Roberts.

CEDRIC W. RICHARDS, *Secretary*

Pi Mu Epsilon of Ohio Wesleyan University

The officers for the past year were: Charles Hiller, Director; Kenneth Ulm, Vice Director and Treasurer; Gretchen JoHantgen, Recording Secretary; Professor Rufus Crane, Permanent Secretary. These officers were elected at the May 1932 meeting by the unanimous vote of all the members present. The fraternity had twenty active members and an average attendance of thirteen.

The primary aim of the fraternity is an increased interest in mathematics. In order to be elected to membership one must be a major in the department of mathematics, and must have shown a real interest in the subject. One must have finished one semester of calculus with a point average of 2.2 in mathematics and a general point average of 1.8.

The meetings and programs were as follows:

November 30, 1932: "Modern geometry of the triangle" by Professor Rufus Crane.

February 8, 1933: "Groups" by Raymond Felts.

March 15, 1933: "Numbers" by Kenneth Cummins.

March 29, 1933: "Complex numbers as related to geometry" by Professor J. H. Weaver of the Ohio State University.

April 19, 1933: "The quantum theorem" by Edwin Smith.

May 10, 1933: "Algebraic symbols" by Hiawatha Louder.

Several business meetings were also held by the group, to discuss eligibility of new members and election of officers. On January 11, 1933, initiation services were held for four new members and a banquet given in their honor. Following the banquet, short speeches were given by both faculty and student members.

GRETCHEN JOHANTGEN, *Secretary*

LOCAL MATHEMATICS CLUBS

The Euclidean Circle of Indiana University

The Euclidean Circle is the oldest departmental club at Indiana University. It was established in 1886 as a mathematics and physics club and was reorganized in 1907 under the present name as a mathematics club.

The officers for the past year were: Louis H. Chaney, President; Mary E. Stranburg, Vice President and Treasurer; Albert Cecil Windell, Secretary.

All officers are elected at the next to the last meeting of each school year. No faculty member is eligible to hold office. The officers are nominated by a nominating committee composed of all the outgoing seniors of the club and are elected by popular vote of the active members of the club. Active members are those who have paid their dues.

Membership includes the faculty members of the department of mathematics, graduate and undergraduate students of Indiana University who have had no less than fifteen hours of mathematics in a college or university. Initiation into the club for new members is held at the beginning of each semester. The membership for this year was sixty-three.

Regular meetings of the club were held once a month. Special meetings, programs and business meetings may be called by the President at any time.

The purpose of the Euclidean Circle is two-fold—social and educational. The club concerns itself with subjects of educational interest in mathematics and strives to promote better acquaintance among the students and faculty of the mathematics department.

The meetings and programs were as follows:

October 3, 1932: "A report of the International Mathematics Congress held in Switzerland during the summer of 1932" by Assistant Professor T. W. Moore, delegate from Indiana University.

November 7, 1932: At this meeting we initiated twelve new members.

December 12, 1932: Party at the home of Dean Wells, member of the mathematics department and Dean of Women at Indiana University. At this meeting, we had the following program: "College students of Germany" by Eckhard Stegman, exchange student from Koenigsberg, Germany; "Morley's theorem" by Muriel Adams; "A problem of division" by Elza Scotten; "Mathematical games" by Professor Hennel.

February 9, 1933: "Mathematics of the Universe" by Professor Cogshall, head of the Astronomy Department of Indiana University. At this meeting we initiated fourteen new members.

March 13, 1933: "Some faulty geometrical theorems" by Elmer Hagerty and Ralph McClain; "Mathematical puzzles and jig-saw puzzles" by Entertainment Committee.

April 4, 1933: "The State High School mathematics contest" by Miss Bittner of Indiana University Extension Division; "Mathematics wise and otherwise" by Professor C. G. F. Franzen, Professor of Secondary Education at Indiana University.

May 1, 1933: Election of officers for 1933–1934. "Trick problems" by John Vendes.

May 19, 1933: Annual Euclidean Circle picnic held this year at McCormick's Creek State Park.

The Euclidean Circle reports a very successful year and has done very much to inspire and stimulate interest in the knowledge and growth of mathematics at Indiana University. The club co-operated with the Extension Division of the University in holding the State High School Mathematics Contest here and several contests for awards were held within the club itself.

The Euclidean Circle extends to all mathematics clubs in the United States and elsewhere best wishes for success.

ALBERT CECIL WINDELL, *Secretary*

The Mathematics Club of Kansas State College of Agriculture and Applied Science

We held monthly meetings at which papers were read by members of the department staff. The meetings were followed by brief social gatherings at which tea and wafers were served. Dr. W. T. Stratton of the department acted as chairman at the various sessions. Our students in mathematics are cordially invited to attend. The primary aim of the club is to stimulate interest in mathematics on the part of the students.

The meetings and programs were as follows:

November 17, 1932: "The five noted travelers in the W. A. G. plane" by Professor R. D. Dougherty.
December 15, 1932: "Point sets" by Dean R. W. Babcock.
January 19, 1933: "College geometry" by Dr. W. T. Stratton.
February 17, 1933: "The normal curve" by Professor A. E. White.
March 17, 1933: "Invariants" by Emma Hyde; "Conformal mapping" by Thirza A. Mossman.
April 21, 1933: "Glimpses into the history of mathematics" by Dr. U. G. Mitchell of the University of Kansas.

The last meeting was sponsored by the mathematical club and the science club of the college. It consisted of an illustrated lecture which was highly interesting and largely attended.

B. L. REMICK, *Head of Department of Mathematics*

PROBLEMS AND SOLUTIONS

EDITED BY OTTO DUNKEL, H. L. OLSON, AND W. F. CHENEY, JR.

ELEMENTARY PROBLEMS

Send all communications about Elementary Problems and Solutions to W. F. Cheney, Jr. Dept. Box 35, Storrs, Connecticut.

The Department of Elementary Problems welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics.

Problems may be submitted unaccompanied by their solutions.

PROBLEMS FOR SOLUTION

E 75. *Proposed by C. W. Munshower, Colgate University.*

Show that in any plane triangle the product of the sum of the ratios of the sides to the radii of the corresponding escribed circles, and the ratio of the sum of the sides to the sum of the radii of the escribed circles, is equal to four.

E 76. *Proposed by Raphael Robinson, University of California at Berkeley.*

Determine a so that the curve $y=a^x$ is tangent to the line $y=x$.

E 77. *Proposed by W. R. Ransom, Tufts College.*

Prove that in the factorial of each positive integer, the last non-zero digit is even.

E 78. *Proposed by Ruth G. Mason, Berkeley, California.*

In this diagram of a long division problem, the x's denote unknown digits not all alike. One certain digit is represented by the letter A wherever that digit occurs. Determine all the digits and prove the solution unique.

```
x A) x x x x x x x x x(x x x x x A x
     x A x
     -----
         x x x
         x x x
         -----
           x x x
           x A x
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             x x x
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               x x
               x A
               ---
                 x x
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                 ---
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E 79. *Proposed by W. C. Janes, Kansas State College.*

Prove that the centroid of the plane section of a parallelopiped through the extremities of three concurrent edges, is a point of trisection of the diagonal concurrent with those edges.

E 80. *Proposed by H. E. Stelsôn, Kent State College, Kent, Ohio.*

A rectangular board has the length l (with the grain) and the width w (across the grain). From it a square table top is to be made with the grain running all one way. The board may be sawed both parallel and perpendicular to its length; but the table top may have joints only with the grain, and *not* across the grain. What is the largest table top that can be so made?

SOLUTIONS

E 34. [1933, 241]. *Proposed by V. F. Ivanoff, San Francisco, California.*

Solve $X^{-X}=(-X)^X$ and justify the number of solutions.

Solution by R. P. Agnew, Cornell University.

We change the proposer's notation by writing z for X. In the first place, we may ask whether there are any integer values of z for which $z^{-z}=(-z)^z$. The

value $z=0$ cannot be a solution since z^{-z} and $(-z)^z$ are not defined for $z=0$. If $z=1$ then $z^{-z}\neq(-z)^z$; and if $z>1$ then $0<z^{-z}<1$ while $(-z)^z=(-1)^z z^z$ is in absolute value greater than 1 and again $z^{-z}\neq(-z)^z$. We see as easily that there are no negative integer solutions. Therefore $z^{-z}=(-z)^z$ has no integer solutions.

In the second place we may ask whether there are any values of z not integers for which $z^{-z}=(-z)^z$. Here we meet the difficulty that neither of the functions z^{-z} or $(-z)^z$ is single-valued and we are forced to call upon the theory of complex variables for definitions. If $w=\rho(\cos\phi+i\sin\phi)=\rho e^{i\phi}$, where $\rho>0$ and $-\pi<\phi\leqq\pi$, then by definition $\log w$ is the multiple valued function

$$\log w = \log\rho + i\phi + 2n\pi i,$$

where $\log\rho$ is the real logarithm of ρ and n is an integer, positive, zero, or negative; and for each complex number $t=u+iv$, w^t is the multiple valued function

$$w^t = e^{t\log w} = e^{u\log\rho-\phi v-2n\pi v+i(v\log\rho+\phi u+2n\pi u)}.$$

We note that w^t never vanishes, and that if neither u nor v vanishes, then both the modulus and amplitude of w^t depend on n.

Henceforth we regard z as a complex variable represented in rectangular form by $z=x+iy$ and in polar form by $z=\rho e^{i\phi}$, ρ and ϕ being functions of x and y, where $\rho>0$ and $-\pi<\phi\leqq\pi$. Solving the equation $z^{-z}=(-z)^z$ is equivalent to determining the values of z for which the multiple valued function

$$F(z) = z^{-z} - (-z)^z \tag{1}$$

vanishes. Using the fact that $-z=-x-iy=\rho e^{i(\phi\pm\pi)}$ where we take the $+$ sign if $\phi\leqq 0$ and the $-$ sign if $\phi>0$, we find from definitions given above that the values of the multiple valued functions z^{-z} and $(-z)^z$ are

$$\begin{aligned} z^{-z} &= e^{-x\log\rho+\phi y+2p\pi y+i[-y\log\rho-\phi x-2p\pi x]} = \phi_p(z)\\ (-z)^z &= e^{x\log\rho-(\phi\pm\pi)y-2q\pi y+i[y\log\rho+(\phi\pm\pi)x+2q\pi x]} = \psi_q(z), \end{aligned} \tag{2}$$

where p and q assume all integer values. Corresponding to each pair of integers m and n, let $F_{m,n}(z)$ denote the particular single-valued function obtained from (1) by taking $p=m$ and $q=n$ in (2); then

$$F_{m,n}(z) = \phi_m(z)\left[1 - e^{\alpha_{m+n}(z)+i\beta_{m+n}(z)}\right]$$

where $\alpha_{m+n}(z)=2x\log\rho-(2\phi\pm\pi)y-2(m+n)\pi y$ and $\beta_{m+n}(z)=2y\log\rho+(2\phi\pm\pi)x+2(m+n)\pi x$. Since $\phi_m(z)$ never vanishes and $e^{\alpha+i\beta}=1$ if and only if $\alpha=0$ and β is an integer multiple of 2π, we see that $F_{m,n}(z)=0$ if and only if the coordinates x, y, ρ, ϕ which determine z satisfy both of the equations

$$2x\log\rho - (2\phi\pm\pi)y - 2(m+n)\pi y = 0 \tag{3}$$

$$2y\log\rho + (2\phi\pm\pi)x + 2(m+n)\pi x = 2k \tag{4}$$

for some integer value of k.

Using (3), we can show that there is no value of z such that $F_{m,n}(z)=0$ for

two different values of the sum $m+n$. This means that a number z_0 can be a solution of $z^{-z}=(-z)^z$ only in the sense that it is a solution if one chooses definitely related branches (i.e., determinations) of the multiple valued functions z^{-z} and $(-z)^z$, and hence that from one point of view the given equation has no solutions whatever. On the other hand, (3) and (4) show that if z_0 is a solution of $F_{\mu,\nu}(z)=0$, then it is also a solution of $F_{m,n}(z)=0$ provided $m+n=\mu+\nu$.

It remains for us to determine whether, when m and n are fixed, the equation $F_{m,n}(z)=0$ has solutions. In other words, we ask whether one can specify a branch of each of the functions z^{-z} and $(-z)^z$ and then find values of z such that on these branches $z^{-z}=(-z)^z$.

Using (3), we can show that there is no pair of indices m and n such that $F_{m,n}(z)=0$ has real solutions. We next investigate pure imaginary solutions. If $F_{m,n}(z)=0$ has a solution $z=iy$, then $x=0$, $\phi=\pm\pi/2$, $2\phi\pm\pi=0$, $\rho=|y|$, and equations (3) and (4) become $(m+n)y=0$ and $y\log|y|=k\pi$. Since $y=0$ does not furnish a solution, we see that if $F_{m,n}(z)=0$ has pure imaginary solutions, then $m+n=0$. If $m+n=0$, then $F_{m,n}(z)=0$ has for pure imaginary solutions the infinite set of numbers iy for which y does not vanish and satisfies one of the equations $y\log|y|=0, \pm 1, \pm 2, \cdots$. In particular, $y=\pm 1$ gives the solutions $\pm i$.

We come finally to the question of solutions of $F_{m,n}(z)=0$ which are neither real nor pure imaginary. For such values of z we have $x\neq 0$, $y\neq 0$, $\rho>0$, and $\phi\neq -\pi/2, 0, \pi/2, \pi$. Using the relations $x^2+y^2=\rho^2$ and $x/y=\tan\phi$, we see that the system of equations (3) and (4) is for these values of z equivalent to the system

$$\rho = e^{(\phi\pm\pi/2+N\pi)\tan\phi} \tag{5}$$

$$\rho\log\rho = k\pi\sin\phi \tag{6}$$

where we have set $N=m+n$. Therefore the solutions of $F_{m,n}(z)=0$ which are neither real nor pure imaginary are given by $z=\rho e^{i\phi}$ where $\rho>0$, $-\pi<\phi<\pi$, $\phi\neq 0, \phi\neq\pm\pi/2$, ρ and ϕ satisfy (5), and ρ and ϕ satisfy (6) for some integer k.

By an analysis of the functions defined by (5) and (6), we can show that in case $N=m+n=0$, $F_{m,n}(z)=0$ has no solutions other than the pure imaginary ones already found; and in case $N=m+n\neq 0$, $F_{m,n}(z)=0$ has an infinite set of solutions which are neither real nor pure imaginary. In the latter case, approximations to the solutions may, for each fixed N, be found by drawing, in the regions $-\pi<\phi<0$, $0<\phi<\pi$ of a plane with rectangular coordinates (ρ, ϕ), the single curve defined by (5) and the family of curves defined by (6) for $k=0, \pm 1, \pm 2, \cdots$ and estimating coordinates of points of intersection.

E 47. [1933, 361]. *Proposed by D. C. Duncan, University of California at Berkeley.*

By methods of elementary plane geometry construct an equilateral triangle having a vertex upon each of three general lines in a plane, given the position of

one vertex. Consider the case when the lines are parallel, and also the case in which the three lines are replaced by three concentric circumferences. What determines the number of solutions in the last case?

Solution by L. S. Johnston, University of Detroit.

This problem may be considered corollary to the following much more general problem: Given any three circumferences in a plane and any triangle, construct a triangle similar to the given triangle having a vertex upon each of the three circumferences, the position of one vertex being given.

Let C_1, C_2 and C_3 be the three circumferences, of radii r_1, r_2 and r_3 respectively. Let the point L be given on C_1, and let PQR be the given triangle. We seek to construct a triangle similar to PQR with the vertex corresponding to P at L and the vertices corresponding to Q and R on C_2 and C_3 respectively.

Construction. (1) Let O be the center of C_2, and let LO intersect C_2 at M. Construct the triangle LOG similar to PQR, taking the vertices in the same order. Let the parallel to OG through M intersect LG at N.

(2) With G as center and GN as radius, strike an arc cutting C_3 at A and B.

(3) Construct m such that $m/LA = LO/LG = LM/LN$. With L as center and m as radius, strike an arc cutting C_2 at D such that the angles OLD and GLA have the same sense of rotation.

(4) Construct n such that $n/LB = LO/LG = LM/LN$. With L as center and n as radius, strike an arc cutting C_2 at E such that the angles OLD and GLB have the same sense of rotation.

(5) Construct the triangles LAD and LBE, which satisfy the given conditions.

Proof. (1) By construction $LD/LA = LO/LG = OM/GN = OD/GA = PQ/PR$.

(2) Therefore the triangles LOD and LGA are similar, and angle OLE equals angle GLA.

(3) Since angle GLO equals angle ALD, triangle LAD is similar to triangle PQR.

(4) By analogous reasoning triangle LBE is similar to triangle PQR.

Discussion. Since G may be on either side of LO, there are in general four solutions. If the rôles of C_2 and C_3 had been interchanged in the construction, no new solutions would have arisen, as may be readily verified.

Since the construction and proof given above are independent of the shape of the triangle PQR and of the relative positions of the circumferences, it is evident that the problem proposed for concentric circumferences has been solved. To determine the number of solutions, we assume for convenience that $r_1 > r_2 > r_3$. It is immediately apparent from a figure that there are two, one or no solutions according as r_1 is less than, equal to, or greater than $r_2 + r_3$.

To apply the construction to the case of three general lines in the plane, we merely have to substitute in our hypothesis, three lines l_1, l_2 and l_3 for the three circumferences. Then LM becomes the perpendicular from L to l_2, and OM and GN become infinite. The arc whose center is G and whose radius is GN

becomes the perpendicular to LN at N. Since this line intersects l_3 at one point only, there are two solutions, one for each of the possible positions of N. Again, since this construction is independent of the relative positions of the three lines and of the form of the triangle PQR, we have solved the problem for the case of three parallel lines.

It is of course obvious that no use is ever made of the assumption that the point L is on a circumference or on a line, so that it is sufficient to give the location of the point L, without stating that it is on any locus.

The first part of this problem is essentially the same as problem 3441 [1930, 380], the solution of which was published in April, 1931, a less simple solution than the one given here, however. The second part of the proposed problem is set as an exercise in some elementary texts, e.g., Wentworth's Plane and Solid Geometry, 1899 edition, page 249, exercise 583.

Solved also by Roy MacKay, E. P. Starke and Simon Vatriquant.

E 48. [1933, 422]. *Proposed by Norman Anning, University of Michigan.*

If the squares of the sines of a set of angles are in harmonic progression, show that the squares of the tangents of the same angles are also in harmonic progression.

Solution by H. E. H. Greenleaf, De Pauw University, Greencastle, Ind.

(1) When the squares of the sines are in harmonic progression, then the squares of the cosecants must be in arithmetic progression.

(2) Since $\cot^2 x = \csc^2 x - 1$, the squares of the cotangents must also be in arithmetic progression.

(3) Therefore the squares of the tangents, which are the reciprocals of the squares of the cotangents, will be in harmonic progression.

Solved also by H. T. R. Aude, L. M. Bauer, Everett Haynes, G. A. Lyle, Roy MacKay, Charles Molloy, W. R. Ransom, A. W. Richeson, T. L. Smith, E. P. Starke, C. W. Trigg, Simon Vatriquant, R. C. Yates and the proposer.

E 49. [1933, 422]. *Proposed by Arnold Dresden, Swarthmore College.*

In the following problem in long division it is required to determine the non-zero digits, which are designated by x's. The zeros are properly shown.

$$\begin{array}{r@{\,}ccccccccccc@{\,}l}
x\,0\,x\,0\,x) & x & x & x & 0 & x & 0 & x & 0 & 0 & x & 0 & (x\,0\,x\,0\,0\,x \\
 & \underline{x} & \underline{x} & \underline{0} & \underline{x} & \underline{0} & \underline{x} & & & & & & \\
 & & & & x & 0 & x & x & 0 & & & & \\
 & & & & \underline{x} & \underline{0} & \underline{x} & \underline{0} & \underline{x} & & & & \\
 & & & & & & x & 0 & x & 0 & x & 0 & \\
 & & & & & & \underline{x} & \underline{0} & \underline{x} & \underline{0} & \underline{x} & \underline{0} & \\
 & & & & & & & & x & 0 & 0 & 0 &
\end{array}$$

Solution by B. C. Zimmerman, Orange Walk, British Honduras.

Call the x's of the divisor and quotient a, b, c, a' b' and c' in order.

1. From the first partial product, $b<a$ and $c<a$.

2. From the third partial product and the results of (1), $c'=5$, and a, b and c are each even.

3. A comparison of the three partial products shows that $b'<a'$ and $b'<c'$.

4. The fourth-column addition tells us that $ab'+ba'=10$.

5. Considering (1), (2), and (3), the only solutions of (4) are

(5A) $6b' + 2\cdot 2 = 10$, whence $a = 6$, $b = 2$, $a' = 2$ and $b' = 1$, or else

(5B) $4b' + 2\cdot 3 = 10$, whence $a = 4$, $b = 2$, $a' = 3$ and $b' = 1$.

6. The sixth-column addition tells us that $(ac'/10)+bb'+ca'=10$.

7. Substituting from (3) and (5A) in (6) leads to a fractional value of c, which is impossible, and eliminates the possibility of (5A).

8. Substituting from (3) and (5B) in (6) gives $c=2$.

9. Thus the unique solution depends on the divisor, 40,202, and the quotient, 301,005, which lead to the dividend, 12,101,010,010 and the remainder 7000.

Solved also by W. E. Buker, M. L. Constable, Daniel Finkel, E. L. Harp, Jr., W. R. Ransom, Rev. M. A. Scheier, Claude Shannon, E. P. Starke, Simon Vatriquant and the proposer.

E 50. [1933, 423]. *Proposed by H. T. R. Aude, Colgate College.*

Two fractions, F_1 and F_2, when written in a certain scale of notation, are $0.3737\cdots$ and $0.7373\cdots$ respectively. When written in a second scale of notation, these same fractions are $0.2525\cdots$ and $0.5252\cdots$ respectively. Find the fractions and the bases used for the two scales.

Solution by T. L. Smith, Carnegie Institute of Technology, Pittsburgh, Pa.

1. Letting the bases of the two scales of notation be a and b, we have

$$F_1 = 3/a + 7/a^2 + 3/a^3 + 7/a^4 + \cdots = 2/b + 5/b^2 + 2/b^3 + 5/b^4 + \cdots$$

$$F_2 = 7/a + 3/a^2 + 7/a^3 + 3/a^4 + \cdots = 5/b + 2/b^2 + 5/b^3 + 2/b^4 + \cdots.$$

2. Since $1+1/x^2+1/x^4+\cdots=x^2/(x^2-1)$, these equations become

$$F_1 = (3a + 7)/(a^2 - 1) = (2b + 5)/(b^2 - 1) \quad \text{and}$$

$$F_2 = (7a + 3)/(a^2 - 1) = (5b + 2)/(b^2 - 1).$$

3. Thence by division, $(3a+7)/(7a+3)=(2b+5)/(5b+2)$, whence $a=(1-29b)/(b-29)$.

4. If the last equations of (2) and (3) are now solved simultaneously, the only values of a and b greater respectively than 7 and 5 (which condition is implied in the statement of the problem) are $a=11$ and $b=8$.

5. Therefore $F_1=1/3$ and $F_2=2/3$.

Solved also by Roy MacKay, W. R. Ransom, E. P. Starke, R. E. Starr, Simon Vatriquant and the proposer.

E 51. [1933, 423]. *Proposed by J. Rosenbaum, The Milford School, Milford, Conn.*

Prove that there are just three pairs of non-negative integers (x, y), which satisfy the equation, $3\cdot 2^x+1=y^2$, and determine them.

Solution by Raymond Garver, University of California at Los Angeles.

By inspection, $x=0$ gives $y=2$, but $x=1$ and $x=2$ make y irrational. If $2<x$, y must be odd, and y^2-1 must be divisible by 3. That is, y must be of the form $6k\pm 1$, with k a positive integer. Then $3\cdot 2^x+1=36k^2\pm 12k+1$, or $2^{x-2}=k(3k\pm 1)$. When $k=1$, $2^{x-2}=2$ or 4, $x=3$ or 4, and the corresponding values of y are 5 and 7 respectively. But if k is greater than 1, the right side of the equation $2^{x-2}=k(3k\pm 1)$ contains an odd factor >1, and can lead to no integral solution. Therefore the only integral solutions are (0, 2), (3, 5) and (4, 7).

Solved also by M. A. Heaslet, W. R. Ransom, T. L. Smith, E. P. Starke, C. W. Trigg, Simon Vatriquant and the proposer.

E 52. [1933, 423]. *Proposed by Moshe Abrahami, Case School of Applied Science.*

Find the area of a triangle in terms of the altitude, interior angle bisector, and median, all from the same vertex of the triangle.

Solution by L. M. Bauer, Menaul School, Albuquerque, N.M.

Let the triangle be ABC, and so label it that $c<b$. Let the altitude be $AD=h$, the interior angle bisector $AU=t$, and the median $AM=m$.

1. In the right triangle, ABD, $c^2=h^2+(a/2-MD)^2$.
2. Since U lies in the segment MB, $(BU)^2=(a/2-MU)^2$.
3. Since the interior angle bisector of a triangle divides its opposite side in the ratio of the other two sides,

$$(BU)^2[h^2+(a/2+MD)^2] = c^2(a/2+MU)^2.$$

4. Dividing (1) by (2), cross-multiplying, subtracting from (3) and simplifying, gives

$$c^2/(BU)^2 = MD/MU.$$

5. From (3) and (4) comes

$$\frac{h^2+(a/2+MD)^2}{(a/2+MU)^2} = \frac{MD}{MU}.$$

6. If (5) be solved for a, and simplified by using the fact that $MD-MU=UD$, there results

$$a = 2[(h^2+MD\cdot UD)(MD-UD)/(UD)]^{1/2}.$$

7. The area of the triangle $=ah/2$, and if the segments MD and UD be replaced by their equivalents $(m^2-h^2)^{1/2}$ and $(t^2-h^2)^{1/2}$, this area becomes

$$h\{[h^2+(m^2-h^2)^{1/2}(t^2-h^2)^{1/2}][(m^2-h^2)^{1/2}-(t^2-h^2)^{1/2}]/(t^2-h^2)^{1/2}\}^{1/2},$$

or, if we set $m^2-h^2=p^2$ and $t^2-h^2=q^2$, this same area appears as

$$h[(p-q)(h^2/q+p)]^{1/2}.$$

Discussion. It is interesting to note that when $h<t<m$, the construction is always possible and the area formula always leads to a real area. If $h=t=m$, there are an infinite number of isosceles triangles satisfying the given conditions, the construction is impossible and the formula is indeterminate. Finally, if $m<t<h$, the construction is impossible and the formula gives an imaginary area.

Solved also by Roy MacKay, Simon Vatriquant and the proposer.

E 53. [1933, 423]. *Proposed by Morgan Ward, California Institute of Technology.*

If s is a positive integer, prove that

$$D_x^s(x^s \ln x) = s!(\ln x + 1 + \tfrac{1}{2} + \tfrac{1}{3} + \cdots + 1/s).$$

Solution by G. A. Lyle, U. S. Naval Academy, Annapolis, Md.

1. The theorem is obviously true when $s=1$ and when $s=2$.
2. Assume that the theorem is true when $s=n$, so that

$$D_x^n(x^n \ln x) = n!(\ln x + 1 + \tfrac{1}{2} + \tfrac{1}{3} + \cdots + 1/n).$$

3. Then

$$\begin{aligned} D_x^{n+1}(x^{n+1}\ln x) &= D_x^n D_x(x^{n+1}\ln x) = D_x^n[(n+1)x^n \ln x + x^n] \\ &= (n+1)D_x^n(x^n \ln x) + D_x^n x^n \\ &= (n+1)n!(\ln x + 1 + \tfrac{1}{2} + \tfrac{1}{3} + \cdots + 1/n) + n!(n+1)/(n+1) \\ &= (n+1)![\ln x + 1 + \tfrac{1}{2} + \tfrac{1}{3} + \cdots + 1/(n+1)]. \end{aligned}$$

4. Since the theorem is true when $s=1$, and since if it is true for any positive integer value of s it is also true for the next higher positive integral value of s, it follows by mathematical induction that the general theorem is true.

Solved also by Raymond Garver, M. A. Heaslet, Roy MacKay, W. R. Moffitt, F. C. Smith, T. L. Smith, E. P. Starke, Simon Vatriquant and the proposer.

Advanced Problems

Send all communications about Advanced Problems and Solutions to Otto Dunkel, Washington University, St. Louis, Mo. All manuscript should be typewritten, with double spacing, and with margins at least one inch wide.

Problems containing results believed to be new, or extensions of old results are especially sought. The editorial work would be greatly facilitated if, on sending in problems, proposers would

also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known textbooks or results found in readily accessible sources will not be proposed as problems for solution in this department. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

PROBLEMS FOR SOLUTION

3662. *Proposed by W. B. Campbell, Rangoon, Burma.*

Within what region must the point P lie, in order that it be possible to draw four real normals from it to the ellipse $x=a\cos\phi$, $y=b\sin\phi$, $a>b$.

3663. *Proposed by Morgan Ward, California Institute.*

If r is greater than zero, define the positive integers $\sigma_{0r}, \sigma_{1r}, \cdots, \sigma_{rr}$ by the identity

$$(x+1)(x+3)(x+5)\cdots(x+2r-1) \equiv \sigma_{0r}x^r + \sigma_{1r}x^{r-1} + \cdots + \sigma_{rr};$$

if r equals zero, let $\sigma_{00}=1$. Show that for any integral t, $k \geqq 0$

$$\sum_{s=0}^{2k+1} \frac{(-1)^s}{s!}\ {}_{2k+1+t}C_{s+t}\sigma_{s,s+t} = 0.$$

3664. *Proposed by Otto Dunkel, Washington University.*

The three independent variables u, v, w are the parameters of three systems of surfaces such that through each point P there passes one and only one surface of each system, and the three surfaces cut orthogonally at P. Let $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ be the unit vector tangents at P to the three curves u, v, w which are determined at that point, and such that these three vectors form a right handed rectangular system. Also let $ds=h_1du$ be the differential of arc at P along the curve u, and let h_2 and h_3 have similar definitions. Prove the relations

$$\frac{\partial \mathbf{a}}{\partial u} = -\frac{1}{h_2}\frac{\partial h_1}{\partial v}\mathbf{b} - \frac{1}{h_3}\frac{\partial h_1}{\partial w}\mathbf{c},$$

$$\frac{\partial \mathbf{a}}{\partial v} = \frac{1}{h_1}\frac{\partial h_2}{\partial u}\mathbf{b},$$

$$\frac{\partial \mathbf{a}}{\partial w} = \frac{1}{h_1}\frac{\partial h_3}{\partial u}\mathbf{c},$$

with two other sets obtained by cyclic interchanges; and show also that on each surface, say $w=$constant, the parametric curves $u=$constant and $v=$constant are lines of curvature.

See Weatherburn, *Advanced Vector Analysis*, p. 12, where equations are given which easily result from the above, but which are not so convenient for obtaining the curl of a vector in curvilinear coordinates, p. 16, formula (28).

3665. *Proposed by Vincent Kelley, London, England.*

Let $f(u)$ be an odd or even integral function of u commonly written as $\Theta_{g,h}^{(n)}(u)$, which satisfies the relations

$$f(u+\pi)=(-1)^g f(u), \qquad f(u+\pi T)=(-1)^h q^{-n}e^{-2nui}f(u),$$

where g and h can be 0 or 1, and n is a positive integer. As usual, $q=e^{\pi T i}$, where T is any complex quantity $a+ib$, with b positive.

(i) Prove that if $n-g$ and $n-h$ are both odd, and $u=\frac{1}{2}\pi$ is a zero of odd order of $f(u)$, then so also is $u=\frac{1}{2}\pi T$.

(ii) Prove that if $f(u)$ has no zero of the form $\lambda\pi$ or $\mu\pi T$, where λ and μ are real, then must $g=0$ and $h=0$.

3666. *Proposed by Martin Rosenman, Brooklyn, N. Y.*

Set up a one-to-one correspondence between the points in the open interval $0<x<1$ and the points in the closed interval $0\leqq x\leqq 1$.

SOLUTIONS

3563. [1932, 429]. *Proposed by Otto Dunkel, Washington University.*

To construct approximately an angle which is $1/n$th of a given angle, let the given angle be BAC, with C as any convenient point on the side AC. Take D on BA produced so that $AD=AC$, and draw DC. Lay off from A along BA produced lengths equal to $(n-1)DC/2$ and $(n-1)AC$ terminating, respectively, in M and N. Determine the point P which divides MN in the ratio $(8-n)$: $4(n-2)$; then the angle BPC is approximately $1/n$th of the angle BAC.

This construction is exact for $n=2$, 4 and for the trivial case $n=1$. For $n=3$ it is essentially the method given by Wedderburn in the second part of his problem 2972 [1922, 224]; and the error in this case was discussed in the solution [1925, 96–97]. For $n=3$ a more convenient but less accurate method may be obtained by taking P as the mid-point of MN. Determine the approximate error for given angles not greater than 45° for the cases $n=5$, 7, and for the modified construction when $n=3$.

Solution by the Proposer.

The above approximate constructions result as particular solutions when we endeavor to determine the constants in

$$(1)\qquad \tan\delta=\frac{a\sin\theta}{a\cos\theta+2b\cos\frac{1}{2}\theta+c}, \qquad \delta=\frac{\theta}{n}+\epsilon,$$

so that δ will be very nearly θ/n for very small angles θ. The error ϵ is easily shown by trigonometric transformations to be given by

$$\tan\epsilon=N/D,$$

$$(2)\qquad \begin{aligned} N&=a\sin\frac{n-1}{n}\theta-b\left(\sin\frac{n+2}{2n}\theta-\sin\frac{n-2}{2n}\theta\right)-c\sin\frac{\theta}{n},\\ D&=a\cos\frac{n-1}{n}\theta+b\left(\cos\frac{n+2}{2n}\theta+\cos\frac{n-2}{2n}\theta\right)+c\cos\frac{\theta}{n}\cdot\end{aligned}$$

Developing N in powers of θ, we obtain for the first three terms

$$\begin{aligned} & [(n-1)a - 2b - c]\frac{\theta}{n}, \\ (3) \qquad & -[2(n-1)^3 a - (3n^2+4)b - 2c]\frac{\theta^3}{2\cdot 3!n^3}, \\ & +[8(n-1)^5 a - (5n^4+40n^2+16)b - 8c]\frac{\theta^5}{8\cdot 5!n^5}. \end{aligned}$$

In order to obtain an approximation of the fifth order in θ, we set the coefficients of θ and θ^3 equal to zero. On eliminating c from the resulting two linear equations in a, b, c, we easily find a solution

$$(4) \qquad a = 3n, \qquad b = 2(n-1)(n-2), \qquad c = (n-1)(8-n).$$

After inserting these values the equation (1) becomes the analytical expression for the geometric construction of the angle BPC of the problem; and hence $\delta = \angle BPC$.

After inserting the values (4) in the third term of (3) and dividing by $3n^2$, the value of D for $\theta = 0$, we obtain the first term of the development of $\tan\epsilon$, which is the same as that for ϵ. Hence for very small values of θ in radians we have for ϵ in radians the approximate equations

$$\begin{aligned} & \epsilon_n = \frac{(n-1)(n-2)(n-4)(7n+4)}{2\cdot 6!n^5}\theta^5, \\ (5) \qquad & \epsilon_3 = -\frac{\theta^5}{6998.4}, \qquad \epsilon_5 = \frac{\theta^5}{9615.4-}, \qquad \epsilon_7 = \frac{\theta^5}{5073.8+}. \end{aligned}$$

We shall now find an upper bound for ϵ for $\theta \leqq \pi/4$ and for $n = 5$. After factoring N we have

$$\tan\epsilon_5 = \frac{2^5\left[5\cos\frac{2\theta}{5} + 12\cos\frac{3\theta}{10} + 18\cos\frac{\theta}{5} + 20\cos\frac{\theta}{10} + 10\right]\sin\frac{\theta}{5}\sin^4\frac{\theta}{20}}{5\cos\frac{4\theta}{5} + 8\cos\frac{7\theta}{10} + 8\cos\frac{3\theta}{10} + 4\cos\frac{\theta}{5}}.$$

Replacing the bracket in the numerator by its value for $\theta = 0$, and the denominator by its value for $\theta = \pi/4$, and noting that $\epsilon_5 < \tan\epsilon_5$, we have

$$\begin{aligned} & \epsilon_5 < 92.052\sin\frac{\theta}{5}\sin^4\frac{\theta}{20}, \\ (6) \qquad & < \frac{\theta^5}{8690}, \qquad 0 < \theta \leqq \frac{\pi}{4}. \end{aligned}$$

For $n = 7$ we may write N as the product of the two factors

and

$$48 \sin \frac{\theta}{7} \sin^4 \frac{\theta}{28}$$

$$\left[14 \cos \frac{4\theta}{7} + 56 \cos \frac{\theta}{2} + 140 \cos \frac{3\theta}{7} + 240 \cos \frac{5\theta}{14} + 344 \cos \frac{2\theta}{7} + 440 \cos \frac{3\theta}{14} + 516 \cos \frac{\theta}{7} + 560 \cos \frac{\theta}{14} + 287\right].$$

The denominator D is

$$3\left[7 \cos \frac{6\theta}{7} + 20 \cos \frac{9\theta}{14} + 20 \cos \frac{5\theta}{14} + 2 \cos \frac{\theta}{7}\right],$$

and we may obtain an upper bound for ϵ_7 in the same manner.

For the trisection approximation of the problem it will be found that $a=1$, $b=\frac{1}{2}$, $c=1$, and

$$\tan \delta = \frac{\sin \theta}{\cos \theta + \cos \frac{1}{2}\theta + 1} \cdot$$

These values satisfy the first equation of (3) but not the second. For very small angles,

$$\epsilon = \frac{\theta^3}{648}, \qquad \delta = \frac{\theta}{3} + \epsilon.$$

We also have

$$\tan \epsilon = \frac{4\left[2 \cos \frac{\theta}{3} - 1\right]}{2 \cos \frac{2\theta}{3} + \cos \frac{5\theta}{6} + \cos \frac{\theta}{6} + 2 \cos \frac{\theta}{3}} \sin \frac{\theta}{3} \sin^2 \frac{\theta}{12} \cdot$$

The fraction on the right has a maximum value for $\theta=\pi/4$ in the interval $0\leqq\theta\leqq\pi/4$, and we obtain an upper bound for ϵ by setting $\theta=\pi/4$ in this fractional part. We find in this way

$$\epsilon < .6841 \sin \frac{\theta}{3} \sin^2 \frac{\theta}{12}, \quad \theta \leqq \frac{\pi}{4},$$
$$< \frac{\theta^3}{631} \cdot$$

The construction given by Kennedy in his paper *Angle Division*, in this MONTHLY [1932, 478] for improving any good trisection approximation may be generalized. Let δ_1 be a good approximation to $\theta/5$, where $\theta = \angle BAC$, $\delta_1 =$

$\angle BP_1C$ and P_1 lies on the extension of BA. Let Q_1 be the other point on CP_1 so that $AQ_1=AC$; M_1, the other point on AP_1 so that $Q_1M_1=Q_1A$; Q_2, the other point on CP_1 so that $M_1Q_2=M_1Q_1$; and finally, P_2 on AP_1 so that $Q_2P_2=Q_2M_1$. Then $\delta_2=\angle BP_2C$ is a better approximation. A formula for the final error ϵ_2, which is sufficiently accurate when ϵ_1 is as small relative to θ as in the given construction, is

$$\epsilon_2 = -\frac{4\left(3+2\cos\frac{2\theta}{5}\right)}{2\left(\cos\frac{4\theta}{5}+\cos\frac{2\theta}{5}\right)+1}\sin^2\frac{\theta}{5}\,\epsilon_1,$$

$$\delta_1 = \frac{\theta}{5}+\epsilon_1, \qquad \delta_2 = \frac{\theta}{5}+\epsilon_2.$$

In the interval $0\leqq\theta\leqq\pi/4$ the fraction in the above formula does not vary much and has its maximum in this interval for $\theta=\pi/4$. Inserting this value in the fraction we obtain the following less exact formula, but which is accurate enough for judging the results

$$\epsilon_2 = -0.17350^2\epsilon_1, \qquad \theta\leqq\pi/4.$$

For $\theta=45°$, $\epsilon_1=6.96''$ by the construction of the problem for $\theta/5$, and the construction just described reduces this error to $\epsilon_2=-0.74''$. By repeated applications of such successive approximations the error can be made as small as is desired.

3599. [1933, 116]. *Proposed by Roy MacKay, Albuquerque, New Mexico.*
If $f(0)=1$ and

$$f(x) = \left(\frac{x}{e^x-1}\right)^k \text{ for } x\neq 0,$$

where k is a positive integer $\geqq 2$; prove that

$$\left[\frac{d^r}{dx^r}f(x)\right]_{x=0} = \frac{(-1)^r\sigma_r}{\binom{k-1}{r}},$$

where r is any positive integer $\leqq k-1$; and σ_r is the elementary symmetric function of the numbers $1, 2, 3, \cdots, (k-1)$, that is σ_r is the sum of the products of these numbers taken r at a time.

Solution by Morgan Ward, California Institute.

Let $w=e^x-1$, and let $F(x)$ be any function of x analytic for $|x|$ sufficiently small. Then by Lagrange's formula, (Goursat-Hedrick, *Mathematical Analysis*, vol. I, p. 405, formula 52)

(1) $$F(x) = F(0) + c_1w + c_2w^2 + \cdots + c_kw^k + \cdots$$

where

$$c_k = \frac{1}{k!}\left[\frac{d^{k-1}}{dx^{k-1}} F'(x)\left(\frac{x}{e^x - 1}\right)^k\right]_{x=0}.$$

Now let $F(x) = e^{vx}$. Then by Leibniz' theorem

$$\frac{d^{k-1}}{dx^{k-1}} F'(x)\left(\frac{x}{e^x - 1}\right)^k = \sum_{r=0}^{k-1}\binom{k-1}{r} v^{k-r}e^{vx}\frac{d^r}{dx^r}\left(\frac{x}{e^x - 1}\right)^k.$$

For convenience of printing, let

$$\left[\frac{d^r}{dx^r}\left(\frac{x}{e^x - 1}\right)^k\right]_{x=0} = \left[\frac{d^r}{dx^r} f(x)\right]_{x=0} = u_r, \quad r = 0, 1, \cdots, k - 1.$$

Then on setting $x=0$ in the identity just written, we obtain

(2) $$c_k = \frac{1}{k!} \sum_{r=0}^{k-1}\binom{k-1}{r} v^{k-r}u_r.$$

But since $w = e^x - 1$, $F(x) = (1+w)^v$. Hence if x is such that $|w| < 1$, we have

(3) $$F(x) = \sum_{k=0}^{\infty} \frac{v(v-1)\cdots(v-k+1)}{k!} w^k.$$

Comparing (1), (2) and (3), we have

(4) $$v(v-1)\cdots(v-k+1) = \sum_{r=0}^{k-1}\binom{k-1}{r} v^{k-r}u_r.$$

But with the proposer's notation,

$$v(v-1)\cdots(v-k+1) = v^k - \sigma_1v^{k-1} + \cdots + (-1)^r\sigma_rv^{k-r} + \cdots + (-1)^{k-1}\sigma_{k-1}v.$$

Therefore comparing the coefficient of v^{k-r} on both sides of (4), we obtain

$$u_r = \frac{(-1)^r\sigma_r}{\binom{k-1}{r}}$$

which is the result stated.

Solved also by the proposer.

Note by the Editors. A proof is given by O. Schlömilch in his *Compendium der Höheren Analysis*, zweiter Band, dritte Auflage, 1879, p. 25, formula 45.

3601. [1933, 179]. *Proposed by M. Markowitz, Brooklyn, New York.*

Prove or disprove that

$$H = [(n-1)^2 F_x F_y - n(n-1) F F_{xy}]/xy,$$

where F is a homogeneous function of x and y of order n, and H is its Hessian, i.e., $H = F_{xx}F_{yy} - (F_{xy})^2$.

Solution by Max Shiffman, College of the City of New York.

Differentiating Euler's equation for homogeneous functions,

$$xF_x + yF_y = nF,$$

partially with respect to x and y, we obtain

$$xF_{xx} + yF_{xy} = (n-1)F_x,$$
$$xF_{xy} + yF_{yy} = (n-1)F_y,$$

respectively.

Consider these last equations as two simultaneous equations in x and y, and solve for x:

$$xH = \begin{vmatrix} (n-1)F_x & F_{xy} \\ (n-1)F_y & F_{yy} \end{vmatrix}.$$

Substituting for F_{yy} its value as given in the last equation above, and expanding the determinant, we obtain

$$H = (n-1)/xy[(n-1)F_x F_y - F_{xy}(xF_x + yF_y)].$$

But $xF_x + yF_y = nF$, and therefore

$$H = [(n-1)^2 F_x F_y - n(n-1) F F_{xy}]/xy,$$

which was to be proved.

Solved also by Frank Ayres, Jr., H. W. Bailey, H. J. Hamilton, J. D. Hill, W. V. Parker, H. D. Ruderman, S. Vatriquant, F. Underwood, and a contributor who failed to give his name.

3602. [1933, 179]. *Proposed by Sigmond Moroh, Brooklyn, New York.*

In a circle with the center O let CD be the diameter perpendicular at M to the chord AB. Through M two chords EF and ST are drawn arbitrarily; and let the lines ET and SF cut AB in Q and P, respectively. Prove that QM is equal to MP.

I. Solution by Margaret M. Young, Brooklyn College of the City of New York.

Angles TSF and TEF are equal since they intercept the same arc: denote each by $\angle 1$. For the same reason $\angle SFE$ and $\angle STE$ are equal: denote each by $\angle 2$. Also $\angle SMP = \angle QMT = \angle 3$; $\angle PMF = \angle EMQ = \angle 4$. Set $AB = 2a$.

Then applying the law of sines to the triangles SMP, PMF, EMQ, and QMT, we have

$$(1)\quad SP = \frac{PM \sin \angle 3}{\sin \angle 1}, \qquad (2)\quad PF = \frac{PM \sin \angle 4}{\sin \angle 2},$$

$$(3)\quad EQ = \frac{QM \sin \angle 4}{\sin \angle 1}, \qquad (4)\quad QT = \frac{QM \sin \angle 3}{\sin \angle 2}.$$

From the theorem that, if two chords of a circle intersect, the product of the segments on one equals the corresponding product on the other, we have $a^2-\overline{PM}^2=SP\cdot PF$, $a^2-\overline{QM}^2=EQ\cdot QT$. Hence

$$a^2 - \overline{PM}^2 = \frac{\overline{PM}^2 \sin \angle 3 \sin \angle 4}{\sin \angle 1 \sin \angle 2}, \qquad a^2 - \overline{QM}^2 = \frac{\overline{QM}^2 \sin \angle 3 \sin \angle 4}{\sin \angle 1 \sin \angle 2}.$$

By solving for PM and QM it will be seen that they are equal.

II. Solution by J. W. Blincoe, University of Virginia.

The problem as stated is a metric specialization of a projective theorem which may be stated as follows:

Suppose lines AB and CD intersecting at M are conjugate lines with respect to a conic, point R on AB being the pole of line CD with respect to the conic. Furthermore, suppose that an arbitrary line through M intersects the conic at points E and F, and that a second arbitrary line through M intersects the conic at points S and T. Now if lines SF and ET intersect line AB in points P and Q, and, if lines ES and FT intersect AB in points P' and Q' respectively, then M is the harmonic conjugate of R with respect to P and Q and also with respect to P' and Q'.

Draw the complete quadrangle on points $EFST$, letting lines ET and SF intersect at point U and lines ES and FT intersect at point V. Now U, V, and M are diagonal points with respect to quadrangle $EFST$. Therefore point M is the pole of line UV with respect to the conic. Since the pole of UV lies on CD, the pole of CD must lie on UV. Hence R is collinear with U and V. Now draw line VM intersecting SF and ET in points W and N respectively. By virtue of the harmonic properties of quadrangle $EFST$ the points $VWMN$ form a harmonic range. But range $RPMQ$ is perspectively related to range $VWMN$ with point U as the center of perspectivity. Hence range $RPMQ$ is likewise harmonic. By a similar argument we may prove that range $RP'MQ'$ is harmonic.

This completes the proof of the projective theorem. The metric theorem now follows immediately. If line CD is a diameter, then R is the point at infinity on line AB and M is the harmonic conjugate of the point at infinity with respect to P and Q, also with respect to P' and Q'. Therefore M bisects segments PQ and $P'Q'$. If, in particular, the conic is a circle, then line AB is perpendicular to line CD as stated in the original problem.

Solved also by Frank Ayres, Jr., M. Charosh, J. W. Clawson, A. Gelbart, R. Goormaghtigh, Alice A. Grant, B. Hoffman, A. S. Householder, Dorothy Huff, L. S. Johnston, B. C. Kieler, J. E. La Fon, R. MacKay, A. Pelletier, W. O. Pennell, O. J. Ramler, H. D. Ruderman, F. Underwood, S. Vatriquant, and G. A. Williams.

3605. [1933, 179]. *Proposed by H. S. Thurston, University of Alabama.*

Prove that every square matrix whose elements are given by the relation,

$$a_{ij} = \begin{cases} (-1)^{j-1}\binom{j-1}{i-1}, & i<j \\ (-1)^{i-1}, & i=j \\ 0, & i>j, \end{cases}$$

is a square root of the unit matrix I.

Solution by E. E. Strock, Yale University.

Let

$$(b_{ij}) = (a_{ij})(a_{ij}) = \left(\sum_{k=1}^{n} a_{ik}a_{kj}\right).$$

If $i=k=j$, then $a_{ik}a_{kj}=(-1)^{2i-2}=1$. If $i>k$ or $k>j$, then $a_{ik}a_{kj}=0$. Hence if $i=j$, $b_{ii}=1$; and if $i>j$, $b_{ij}=0$. If $i<j$,

$$\begin{aligned} b_{ij} &= \sum_{k=i}^{j}(-1)^{j+k}\binom{k-1}{i-1}\binom{j-1}{k-1} \\ &= \frac{(j-1)!}{(i-1)!}\sum_{k=i}^{j}(-1)^{j+k}\frac{1}{(k-i)!(j-k)!} \\ &= \frac{(j-1)!}{(i-1)!}\sum_{k=i}^{j}(-1)^{j+k}\binom{j-i}{k-i}\Big/(j-i)! \\ &= \frac{(j-1)!(-1)^{j+i}}{(i-1)!(j-i)!}(1-1)^{j-i} = 0. \end{aligned}$$

Hence (b_{ij}) is the unit matrix I.

Solved also by A. E. Andersen, Frank Ayres, Jr., J. A. Bullard, W. V. Parker, E. S. Quade, and Raphael Robinson.

NEWS AND NOTICES

Readers are invited to contribute to the general interest of this department by sending items to Professor J. H. Weaver, Ohio State University, Columbus, Ohio.

At the meeting of the Trustees of the Association in Cambridge on December 29, 1933, the following resolution was unanimously adopted:

Whereas the Secretary-Treasurer of the Association, W. D. Cairns, has served the Association faithfully and with notable efficiency since the foundation of the Association in 1915, and

Whereas the conduct of the affairs of the Association, more particularly the astute management of its financial affairs during the present depression, has been extraordinary in that the funds remained intact to perhaps a greater extent than is true of any other corporate body known to any members of the Board of Trustees and in that there is a favorable balance on the business of the past year;

Now, therefore, be it resolved by the Board of Trustees of the Mathematical Association of America that the thanks of the Association be extended to the secretary-treasurer and that the president be authorized and requested to transmit to him an expression of the deep appreciation of his service to this Association.

The fifth Carus Mathematical Monograph will be issued from the press early in March, 1934. It is entitled "History of Mathematics in America before 1900" by David Eugene Smith and Jekuthiel Ginsburg. It is to be uniform in make-up with the four preceding Monographs and to be supplied to members of the Association, individual and institutional, at the cost price $1.25 when ordered directly through the Secretary of the Association. It will be distributed to the general public at the list price of $2.00 through the Open Court Publishing Company of Chicago, Illinois, now located at 149 East Huron Street. Copies of the preceding Monographs may still be obtained through the Secretary at the cost price by members, one copy to each, who have previously neglected to order.

An American Decimal-Metric Code has been drafted and urged for adoption to assist the administration in effecting economic and educational improvement under the National Recovery Act. The purpose of the Decimal-Metric Code as stated by the committee is "To establish a single standard for the expression of all values, measures, weights, angles, temperatures, etc., which is best adapted for American national and international service." If this code is adopted all civilized countries of any economic importance except Great Britain will use decimal-metric units.

The founding of an international mathematical journal, "Compositio Mathematica" is announced by its publisher, P. Noordhoff, of Groningen. Its editorial staff consists of representatives from all the principal mathematical countries. The American representatives are Professors L. P. Eisenhart, Einar Hille, Solomon Lefschetz, and Oswald Veblen.

Dr. Richard Brauer, formerly of the University of Koenigsberg, is spending the current year at the University of Kentucky as visiting professor of mathematics.

Professor G. D. Birkhoff of Harvard University received an honorary degree from the University of Poitiers, France, at the recent Quin-centenary celebration at that university.

The Comstock Prize of $2500 was presented to Percy W. Bridgeman, Hollis Professor of Mathematics and Philosophy at Harvard University, at a dinner of the National Academy of Sciences held at Lowell House, Harvard University, on the evening of November 21. The report of the chairman of the Comstock Fund Committee, Max Mason, reads in part as follows: "Most of Bridgeman's work falls into three categories, the first, so peculiarly his own, the behavior of materials under high pressure; the second, the properties of single crystals at normal pressure; and the third, the application of thermodynamics to electrical phenomena."

At the ceremonies commemorating the fiftieth anniversary of the graduate school of the University of Pennsylvania, Professor L. P. Eisenhart, Dean of the Graduate School of Princeton University, was awarded the degree, Doctor of Science.

Dr. Henry H. Pixley, instructor in the College of the City of Detroit, has been granted leave of absence to permit him to accept a position as mathematical economist in the Division of Economic Research and Planning of the National Recovery Administration.

Professor G. C. Evans of Rice Institute at Houston, Texas, has been appointed to succeed M. W. Haskell, Professor Emeritus at the University of California at Berkeley, who retired at the end of last year.

At Pennsylvania State College, Associate Professors Orrin Frink, C. A. Rupp, and C. C. Wagner have been promoted to full professorships, Assistant Professors H. B. Curry and I. M. Sheffer have been promoted to associate professorships, and Dr. T. C. Benton has been promoted to an assistant professorship.

Doctor R. S. Burington has been promoted to an assistant professorship of mathematics at the Case School of Applied Sciences.

W. H. Ingram has been appointed lecturer at the College of the City of New York.

Professor John von Neumann, Princeton University, has been appointed professor of mathematics at the Institute for Advanced Study.

Dr. J. L. Dorroh has been appointed to an instructorship at Johns Hopkins University.

J. K. Long, who for the past four years was instructor in mathematics at Purdue University, died December 30, 1933, at the age of 28. He was a member of the Association for several years.

Professor A. B. Morton of the Georgia School of Technology died October 13, 1933. He was a charter member of the Association.

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DIRECTORY

EDITORIAL CORRESPONDENCE should be addressed to the EDITOR-IN-CHIEF, W. B. CARVER, White Hall, Cornell University, Ithaca, N.Y.

BOOKS FOR REVIEW should be addressed to R. A. JOHNSON, Brooklyn College, 66 Court Street, Brooklyn, N.Y.

BUSINESS CORRESPONDENCE should be addressed to the SECRETARY-TREASURER of the Association, W. D. CAIRNS, 33 Peters Hall, Oberlin, Ohio.

CHANGE OF ADDRESS: Members should send notice of any change of address to the SECRETARY-TREASURER, W. D. CAIRNS, Oberlin, Ohio, before the 10th of each month.

MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Eighteenth Summer Meeting of the Association, Williamstown, Mass., Sept. 3-4, 1934.

The following is a list of the sections of the Association, with dates of those section meetings which have been scheduled for 1934 and reported to the Secretary.

ALLEGHENY MOUNTAIN, Pittsburgh, Pa., Feb. 10; Washington, Pa., May 5.
ILLINOIS, Jacksonville, May 4-5.
INDIANA, La Fayette, May 11-12.
IOWA, Des Moines, April 20-21.
KANSAS, Topeka, Mar. 17.
KENTUCKY, May.
LOUISIANA-MISSISSIPPI, Jackson, Miss., Mar. 23-24.
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Williamsburg, Va., May.
MICHIGAN, Ann Arbor, Mar. 17.
MINNESOTA, Northfield, May 12.
MISSOURI.
NEBRASKA.
OHIO, Columbus, Apr. 5.
OKLAHOMA, Oklahoma City, Feb. 9.
PHILADELPHIA, Philadelphia, Dec. 1.
ROCKY MOUNTAIN.
SOUTHEASTERN, University, Ala., Mar. 30-31.
SOUTHERN CALIFORNIA, Riverside, Mar. 3.
TEXAS.
WISCONSIN, Oshkosh, May 5.

AFFILIATED ORGANIZATIONS: THE NEW ENGLAND ASSOCIATION OF TEACHERS OF MATHEMATICS. THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS.

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In view of the unique and significant relation of Professor Herbert Ellsworth Slaught to the Mathematical Association of America and in recognition of his long-continued and important contributions to the cause of American mathematics and to the activities of this Association, the members of the Association, in session at Cambridge, Massachusetts, December 29, 1933, voted unanimously to suspend the By-Laws and to elect Professor Slaught Honorary President for life.

THE EIGHTEENTH ANNUAL MEETING OF THE ASSOCIATION

The eighteenth annual meeting of the Mathematical Association of America was held at Cambridge, Massachusetts, on Friday and Saturday, December 29 and 30, 1933, in affiliation with the American Association for the Advancement of Science and the American Mathematical Society. Three hundred eighty were in attendance at the meetings, including the following two hundred one members of the Association:

C. R. Adams, Brown University
R. P. Agnew, Cornell University
R. C. Archibald, Brown University
T. B. Ashcraft, Colby College
C. S. Atchison, Washington and Jefferson College

R. W. Babcock, Kansas State Agricultural College
F. H. Bailey, Massachusetts Institute of Technology
N. H. Ball, Princeton University
H. C. Barber, English High School, Boston
R. W. Barnard, University of Chicago
Ralph Beatley, Harvard Graduate School of Education
A. A. Bennett, Brown University
William Betz, Rochester Public Schools
G. D. Birkhoff, Harvard University
L. M. Blumenthal, Institute for Advanced Study
Julia W. Bower, Connecticut College
H. W. Brinkmann, Swarthmore College
A. B. Brown, Columbia University
B. H. Brown, Dartmouth College
E. C. Brown, Worcester Polytechnic Institute
R. E. Bruce, Boston University

S. S. Cairns, Lehigh University
W. D. Cairns, Oberlin College
R. H. Cameron, Brown University
B. H. Camp, Wesleyan University
A. D. Campbell, Syracuse University
G. A. Campbell, American Tel. and Tel. Co.
Mildred E. Carlen, Brown University
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A. B. Coble, University of Illinois
L. W. Cohen, University of Kentucky
J. L. Coolidge, Harvard University
Lennie P. Copeland, Wellesley College
H. B. Curry, Pennsylvania State College
D. R. Curtiss, Northwestern University
E. H. Cutler, Lehigh University

H. M. Dadourian, Trinity College
D. R. Davis, New Jersey State Teachers College
J. W. Davis, Dorchester High School
C. E. Dimick, U. S. Coast Guard Academy
L. L. Dines, Carnegie Institute of Technology
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Tomlinson Fort, Lehigh University
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W. L. Hart, University of Minnesota
L. A. Hazeltine, Stevens Institute of Technology

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E. V. Huntington, Harvard University
W. A. Hurwitz, Cornell University

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Dunham Jackson, University of Minnesota
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C. F. Roos, N.R.A., Washington, D. C.
M. F. Rosskopf, Brown University
S. G. Roth, New York University
George Rutledge, Massachusetts Institute of Technology

M. A. Scheier, Catholic University
Mabel F. Schmeiser, Institute for Advanced Study
I. J. Schoenberg, Institute for Advanced Study
Hazel E. Schoonmaker, Hartwick College
Wladimir Seidel, Harvard University

JOSEPH SEIDLIN, Alfred University
R. S. SHAW, College of the City of New York
I. M. SHEFFER, Pennsylvania State College
L. L. SILVERMAN, Dartmouth College
H. L. SLOBIN, University of New Hampshire
CLARA E. SMITH, Wellesley College
I. W. SMITH, North Dakota Agricultural College
VIRGIL SNYDER, Cornell University
JOSEPH SPEAR, Northeastern University
VIVIAN E. SPENCER, University of Pennsylvania
E. R. STABLER, Harvard University
ANNA A. STAFFORD, Institute for Advanced Study
MARION E. STARK, Wellesley College
A. L. STARRETT, Harvard University
M. H. STONE, Harvard University
R. E. STREET, Harvard University
M. HELEN SULLIVAN, Mt. St. Scholastica College
J. L. SYNGE, University of Toronto
J. D. TAMARKIN, Brown University
J. H. TAYLOR, George Washington University
J. S. TAYLOR, University of Pittsburgh
J. M. THOMAS, Duke University
ARTHUR TILLEY, New York University
C. C. TORRANCE, Institute for Advanced Study
MARIAN M. TORREY, Goucher College
J. I. TRACEY, Yale University
F. E. ULRICH, Union College
H. E. WAHLERT, New York University
J. L. WALSH, Harvard University
WARREN WEAVER, Rockefeller Foundation
F. M. WEIDA, George Washington University
MARIE J. WEISS, Sophie Newcomb College
MARY EVELYN WELLS, Vassar College
V. H. WELLS, Williams College
ANNA PELL WHEELER, Bryn Mawr College
J. K. WHITTEMORE, Yale University
D. V. WIDDER, Harvard University
EVELYN P. WIGGIN, Randolph-Macon Woman's College
R. L. WILDER, Institute for Advanced Study
A. H. WILSON, Haverford College
ELIZABETH W. WILSON, Radcliffe College
W. A. WILSON, Yale University
F. S. WOODS, Massachusetts Institute of Technology
W. D. WRAY, Cornell University
FRANCES M. WRIGHT, Elmira College
FRANCES W. WRIGHT, Harvard University
MABEL M. YOUNG, Wellesley College
S. D. ZELDIN, Massachusetts Institute of Technology

The sessions of the American Association for the Advancement of Science began on Wednesday evening with an address by the retiring president, Doctor J. J. Abel, and the reception which followed this at Hotel Statler in Boston. The president of the American Association, Professor H. N. Russell, presided, and addresses of welcome were given by Doctor K. T. Compton and Doctor Harlow Shapley. The second annual Hector Maiben lecture on "The faith of reverent science" was given on Thursday afternoon by Doctor W. M. Davis, professor emeritus of geology, Harvard University. The Rumford medal of the American Academy of Arts and Sciences was presented to Doctor Harlow Shapley Saturday evening at a joint meeting of the Academy and the American Association; Doctor Shapley gave a very entertaining address on "The anatomy of a disordered universe."

The Council of the American Association met Wednesday afternoon and each morning thereafter, the Mathematical Association being represented by Professor J. I. Tracey and Secretary Cairns. The Council elected Professor R. D. Carmichael vice-president and chairman of Section A for the year 1934, and Professor R. E. Langer member of the Section committee.

One very delightful feature of the week was the program given Wednesday afternoon in honor of the visiting scientists by the Boston Symphony Orchestra through the cooperation of the conductor, Doctor Serge Koussevitzky, and the

directors of the orchestra. A large number of the visiting mathematicians attended the very pleasant reception given by Professor and Mrs. J. L. Coolidge at Lowell House Wednesday afternoon. A group of mathematicians was among the one hundred forty who were present at the testimonial dinner given to Doctor J. McKeen Cattell Wednesday evening in recognition of his long and valuable service to the cause of American science. The formal tribute to Doctor Cattell was given in an address by Professor John Dewey on "The obligation of science." On Friday afternoon the visitors were entertained at tea by President Ada L. Comstock and associates of Radcliffe College. Among the other attractions of the week was the opportunity afforded on Thursday of visiting the Isabella Stewart Gardner Museum, complimentary tickets being furnished to the mathematicians. After the session on Thursday afternoon there was an exhibit of the important machines, charts and the electrical integraph owned by Massachusetts Institute of Technology.

The headquarters for the mathematicians was the Commander Hotel, a very convenient location inasmuch as the meetings were held at the neighboring Alice Mary Longfellow Hall of Radcliffe College.

The annual dinner of the mathematicians was held at Walker Memorial Building of Massachusetts Institute of Technology Thursday evening, with a record attendance of two hundred ninety-five, Professor J. L. Coolidge presiding. President K. T. Compton of the Institute made the guests welcome to all the week's features of interest. With further witty introductions the toastmaster introduced President Coble, who spoke on a "new deal" to interest mathematicians afresh in the active work of the Society and of the very great desirability of having these interests cared for and guided by those who are themselves mathematicians; and President Dresden, who said that the two mathematical organizations have pulled together in every good sense of the word in promoting a true collaboration of mathematicians, that we must make a definite effort to improve the conditions of mathematics in the schools, that to this end we must place in the schools as teachers persons who are keenly interested in the subject and who may develop the teaching of secondary school mathematics so as to make its value more evident. Professor Helen Merrill told how in the earlier high school days boys and girls sent there for the sole purpose of studying, without outside diversions, took a sound course in elementary algebra in one year and plane and solid geometry in a second year; she contrasted with this the profound changes which recent years have brought and deplored the replacing of serious subjects by those of lesser value and the concessions which we are making on the score of usefulness. Professor Archibald Henderson spoke of the wide separation of science and art and of the rôle of mathematics in relation to these. He happily quoted Daniel Gregory Mason as replacing the word "mathematics" in Bertrand Russell's definition of mathematics by the word "music." He discussed further the question of reality in science and described the revolution in mathematical physics whereby the physicist instead of talking in terms of mechanical models trusts to his imagination and deals in a world of symbols. "The

Pegasus of today is not merely a scientist but an artist as well." He closed by referring to some striking analogies between art and science.

The American Mathematical Society held sessions on Tuesday afternoon, Wednesday forenoon and afternoon for the reading of shorter papers. On Thursday morning the Bôcher Prize was awarded to Professors Marston Morse and Norbert Wiener, the former for his paper on "Calculus of variations in the large" in the Transactions, and the latter for his paper on "Tauberian theorems" in the Annals for January 1932. Each gave a very able description of the work for which the award was made. This was followed by an address by Georges Valiron, exchange professor at Harvard University, on "Schwarz's Lemma; its extensions and applications." On Thursday afternoon a session was held at the Massachusetts Institute of Technology on "General analysis" with papers by Professors T. H. Hildebrandt, R. W. Barnard, and M. H. Stone.

The program of the Mathematical Association consisted of a joint session with Section A and the Mathematical Society on Friday morning, and two sessions on Friday afternoon and Saturday morning. Professor C. N. Moore presided at the joint session, and Vice-president A. A. Bennett and President Arnold Dresden at the sessions on Friday afternoon and Saturday morning respectively. Recognition was given at the Saturday morning session of the able work done by the program committee under the chairmanship of Professor W. C. Graustein. At the joint session a motion offered by Professor W. L. Hart was adopted unanimously expressing the gratefulness of the visitors to Professor J. L. Walsh and the other members of the committee on local arrangements, including the Junior Mathematical Club, to President Ada L. Comstock and her staff, to Professor and Mrs. J. L. Coolidge, and to the presidents and divisions of mathematics of Harvard University and Massachusetts Institute of Technology and to the ladies of the two divisions, for the many favors extended. The program follows, together with abstracts of some of the papers, numbered in accordance with their place on the program.

Joint Session of the Association with Section A of the American Association and the American Mathematical Society

1. "Linear groups and finite geometries" by Professor H. H. Mitchell, University of Pennsylvania, retiring vice-president of Section A.

2. "The rise and fall of projective geometry" by Professor J. L. Coolidge, Harvard University, representing the Mathematical Association.

1. The paper by Professor Mitchell, or an abstract of it, will appear in an early number of this Monthly.

2. The address by Professor Coolidge will appear in full in an early number of the Monthly.

First Session of the Association

1. "The study of the history of mathematics" by Doctor George Sarton, Harvard University.

2. "Conformal mapping with applications to aerofoil theory" by Doctor J. G. Estes, Massachusetts Institute of Technology.

3. "The Euclidean process in quadratic fields" by Professor H. W. BRINKMANN, Swarthmore College.

1. Doctor George Sarton spoke in a very informal vein of the studies to which he has devoted a good part of his life. He insisted on the need of organizing the study and teaching of the history of mathematics more seriously than has been done heretofore except in a very few places. For most mathematicians, history is simply a recreation—and that is all right—but such recreation cannot be sound unless it becomes the principal concern of a few scholars. Much has already been done in the United States by G. B. Halsted, Florian Cajori, D. E. Smith, R. C. Archibald, L. C. Karpinski, L. E. Dickson, *et al.*, and there is now a special journal *Scripta Mathematica*, not to mention *Isis*. Yet all this is only a beginning, for the task remaining to be done is immense. Most problems relative to ancient, mediaeval, and oriental mathematics should be left to specialists who will be more generally found among historians and archaeologists interested in mathematics rather than among professional mathematicians. On the other hand, the latter—if they be historically minded—have abundant opportunities for research in the field of nineteenth century mathematics which historians have as yet exploited only superficially. The progress in mathematics and mathematical physics made during the last twenty years has been so deep and revolutionary that it has now become possible by way of contrast to see the nineteenth century in a better perspective and with greater objectivity. To realize how interesting such studies could be it will suffice to refer, e.g., to a recent paper by D. J. Struik: Outline of a history of differential geometry (Isis 19, 92–120, 161–91, 1933).

In the second part of his speech Doctor Sarton dealt with the teaching of the history of mathematics, giving some practical advice suggested to him by his own experience. The purpose of courses on the subject cannot be to create historians (these must find themselves) but to educate mathematicians. A gentle mathematician should know the main facts of the mathematical past and have some familiarity with the main heroes, even as a good citizen should have some knowledge of the past events of his own country: we should not expect either of them to be a specialized historian, but we have every right to expect them to have some mathematical or civic education, not simply technical instruction. The course should aim at explaining very simply the development of the main mathematical ideas, the gradual subdivision of mathematics into branches, the growth of those branches, their interactions. It is well also to explain exchanges of influence between mathematical and other scientific ideas, even if this entails a few excursions into physics, astronomy, chemistry, etc.; and the constant interrelations between mathematics and every other aspect of life (e.g., music, religion). In his own course Doctor Sarton tries to give characterizations of the great mathematicians, and thinks that if his students remember nothing else but keep in their mind living pictures of these great men a worthwhile result has already been obtained. It is expedient to devote a couple of lessons at the end of the course to a brief explanation of historical methods, bibliography, the

ways of ascertaining what is already known on a definite subject and of increasing that knowledge. These extra lessons should be facultative and not submitted to examination; they would afford means of awakening historical vocations and of directing the teacher's attention and solicitude to his most promising pupils. Finally no Master's degree should be granted except to Bachelors who have proved that they are sufficiently acquainted with the historical background of their studies and that they have learned to appreciate the human elements involved. The study of the history of mathematics will not help them to be better mathematicians, not at all, but they will be raised up to a higher spiritual level, they will be gentler men and finer teachers.

2. Joukovsky aerofoils are obtained from the circle $(x-m\cos\delta)^2+(y-m\sin\delta)^2=a^2$, where $z=me^{i\delta}$ is in the first quadrant, by applying to it Joukovsky's conformal transformation $\zeta=z+c^2/z$. The point of intersection of the circle with the negative axis of reals is $z=-c$.

If we can find the flow about the circle, we will know the flow about the figure into which the circle is mapped; i.e., the aerofoil. We get the flow about the circle by combining in order several flows. To a source-sink combination we add a uniform flow, and then let the source and sink approach each other, their strengths becoming infinite as the distance separating them approaches zero. The resulting flow is the flow about a circle (or cylinder). To get an angle of attack in the aerofoil plane, we combine the above flow with a circulatory motion. Mapping this flow by use of Joukovsky's transformation, we get a flow about the aerofoil making an arbitrary angle of attack with the aerofoil. By use of Blasius's formulae we then are able to calculate the lift and the moment of the aerofoil for any angle of attack.

3. The paper by Professor Brinkmann will appear in the MONTHLY.

SECOND SESSION OF THE ASSOCIATION

1. "Logical definitions of extension, class, and number" by Professor A. N. WHITEHEAD, Harvard University.

2. "The Richard paradox" by Professor ALONZO CHURCH, Princeton University.

3. "The present situation in the foundations of mathematics" by Doctor KURT GÖDEL, The Institute for Advanced Study.

1. The Association was honored in having the address by Professor Whitehead. An abstract of the paper has been kindly furnished by Doctor W. V. Quine.

In identifying a cardinal number n with the class of all n-element classes of given type, the theory of the foundations of arithmetic in *Principia Mathematica* entails (1) the existence of an infinity of isomorphic arithmetics corresponding to the infinity of logical types, and (2) the dependence of number upon accidents of factual existence (as seen in particular in the need for the axiom of infinity). Whitehead now sketches an alternative theory which is free from these

anomalies. The new theory is accompanied by a new theory of classes and of relations.

Extension is rested upon a partial calculus of propositions, developed in terms of alternation ($p \cup q$) and *identity* (as against mere equivalence). Postulates are adopted according to which alternation has relatively to the connective of identity the properties of commutativity, associativity, idempotence, non-invertibility *et al.*, and possesses a unique identity element Λ, identified arbitrarily with the falsehood "$(p)\cdot p\cdot \sim p$." On the other hand alternation is allowed no absorptive (=universal) element relatively to identity.

Inclusion of propositions, $p \subset q$, is defined to mean: $q = p \cup q$.

The unary operation of *indication* upon a term x yields a proposition Ec!x, "Behold x!" Any trivially true proposition involving x suffices as definiens for "Ec!x"; the definiens chosen, suggested by Quine, is: $(\exists\phi)\cdot\phi x$.

This constitutes adequate machinery for a theory of classes, provided that identity as applied to propositions is taken in a sufficiently strict sense to make for diversity of Ec!x and Ec!y where $x \neq y$. Granted this condition, the unit class of x can be construed as the proposition Ec!x; in general, the class of terms $a, b, c, \cdots$ can be construed as the proposition Ec!$a \cup$Ec!$b \cup$Ec!$c \cup \cdots$. The following definitions define respectively "x is a member of p," "p is a class" and "the class of terms x satisfying the condition ϕx."

$$x \epsilon p \cdot \;=\mathrm{Df}\cdot \mathrm{Ec}!x \subset p$$

$$\mathrm{Cls}!p \cdot \;=\mathrm{Df}:\cdot x\epsilon p \cdot \supset_x \cdot x \epsilon q : \supset_q \cdot p \subset q$$

$$\hat{x}(\phi x)\cdot \;=\mathrm{Df}:\cdot (\imath p) : \mathrm{Cls}!p : \phi x \cdot \equiv_x \cdot x\epsilon p.$$

Where the propositions p and q are classes, $p \cup q$ and $p \subset q$ become the ordinary class sum and class inclusion. Λ is the null class. The absence of an absorptive element for alternation results in the absence of a universal class V and thus extrudes various antimonies for the avoidance of which, in *Principia Mathematica*, the theory of class types was required.

The cardinal numbers 0, 1, 2, $\cdots$ are defined respectively as Λ, Ec!Λ, Ec!$\Lambda \cup$ Ec!Ec!Λ, $\cdots$. The definition of inductive cardinal in general depends upon the notion Ord'a, which is the infinite class Ec!$a \cup$ Ec!Ec!$a \cup$ Ec!Ec!Ec!a $\cup \cdots$. Ord'a is formally defined as follows:

$$\mathrm{Ord}`a \cdot = \mathrm{Df}\cdot \hat{x}[\phi a : \phi y \cdot \supset_y \cdot \phi(\mathrm{Ec}!y) : \supset_\phi \cdot \phi x].$$

The class of all inductive cardinals is then defined, in effect, thus:

$$\mathrm{NCind}\cdot = \mathrm{Df}\cdot \hat{p}[(\exists r)\cdot p = \hat{q}(\mathrm{Ord}`r \subset \mathrm{Ord}`q \cdot \mathrm{Ord}`q \subset \mathrm{Ord}`\Lambda)]$$

Preparatory to the theory of relations, a scheme for the generation of order is introduced by a definition explaining the use of subscripts:

$$x_n \cdot \;=\mathrm{Df}\cdot \mathrm{Ec}!x = n.$$

Thus x_1, x_2 etc. are the propositions "Ec!$x=1$," "Ec!$x=2$" etc. The ordered

couple x, y is defined as $x_1 \cup y_2$; analogously for ordered triples, etc. Then dyadic relations are introduced as classes of ordered couples, triadic relations as classes of ordered triples, and so on.

2. The paper by Professor Church will appear in an early number of the MONTHLY.

3. The paper by Doctor Gödel will appear in full in the MONTHLY.

MEETINGS OF THE BOARD OF TRUSTEES

Ten members of the out-going and of the incoming Board were present at the Cambridge meetings.

The following twenty-six persons and two institutions were elected to membership on applications duly certified:

To Individual Membership

S. K. BANERJI, D.Sc. (Calcutta) Offg. Director General of Observatories, India; Hon. Prof., Applied Physics, Royal Inst. of Science, Bombay, India

C. S. CARLSON, M.S. (Iowa) Asso. Prof., St. Olaf Coll., Northfield, Minn.

Sister ROSE MARGARET COOK, M.S. (Notre Dame) Instr., Loretto Heights Coll., Loretto, Colo.

W. H. GAVER, A.B. (Randolph-Macon) Prof., Newberry Coll., Newberry, S. C.

H. H. GOLDSTINE, B.S. (Chicago) Grad. student, Univ. of Chicago, Chicago, Ill.

HYACINTH GRABBE. Grad. student, Catholic Univ. of America, Washington, D. C.

L. A. HAZELTINE, M.E. (Stevens) Prof., Physical Math., Stevens Inst. of Tech., Hoboken, N. J.

Sister ESTHER M. KENNA, Prof., Coll. of St. Elizabeth, Convent Station, N. J.

L. A. MACCOLL, A.M. (Columbia) Engr., Bell Telephone Labs., New York, N. Y.

CLIFFORD MARBURGER. Instr., Franklin and Marshall Acad., Lancaster, Pa.

G. E. MESTLER. Computation Dept., Genesee State Park Commission, Rochester, N. Y.

L. T. MOSTON. Ph.D. (Harvard) Prof., Waynesburg, Pa.

H. J. O'CONNOR, B.S. in Ch. (Niagara Univ.) Teacher, Trott Vocational School, Niagara Falls, N. Y.

Rev. A. J. REILLY, S.T.B. (Catholic Univ.) Missionary, Catholic Mission, Shasi, Hupeh, China

S. G. ROTH, A.M. (North Carolina) Instr., New York Univ., New York, N. Y.

MAX SASULY, M.S. (Chicago) Senior Statistician, N.R.A., Division of Research and Planning, Washington, D. C.

F. W. SOHON, Ph.D. (Georgetown) Director of Seismological Observatory; head of dept. of math., Georgetown Univ., Washington, D. C.

E. P. STARKE, Ph.D. (Columbia) Prof., Rutgers Univ., New Brunswick, N. J.

M. H. STONE, Ph.D. (Harvard) Asso. Prof., Yale Univ., New Haven, Conn

EARL THOMAS, B.S. (Trinity Univ.) Grad. Fellow, Louisiana State Univ., Baton Rouge, La.

J. M. THOMAS, Ph.D. (Johns Hopkins) Assoc. Prof., Duke Univ., Durham, N. C.

H. E. VAUGHAN, A.M. (Michigan) Univ. fellow, Univ. of Michigan, Ann Arbor, Mich.

J. A. WARD, A.B. (Davidson Coll.) Fellow, Louisiana State Univ., Baton Rouge, La.

J. E. WOOLHISER, B.S. in E.E. (George Washington) Scientific Aide, U. S. Coast and Geodetic Survey, Washington, D. C.

W. D. WRAY, A.B. (Haverford) Grad. student, Cornell Univ., Ithaca, N. Y.

E. N. YEAGER, M.S. (Notre Dame) Prof., Toledo Teachers Coll., Toledo, Ohio

To Institutional Membership

OKLAHOMA A. AND M. COLLEGE, Stillwater, Oklahoma

CARNEGIE INSTITUTE OF TECHNOLOGY, Pittsburgh, Pa.

The financial report of the Secretary-Treasurer for the year 1933 was presented, approved by Professor Slaught for the Finance Committee; and this was accepted, subject to inspection by a sub-committee. Professors Bennett and Hart later examined the report and the evidences of assets and found them to be satisfactory.

The Trustees approved the formation of the Allegheny Mountain Section, and the draft of the By-Laws of the section as presented.

It was voted that the Cleveland Trust Company be designated as one of the depositaries of the Association and that said Company be authorized to pay out funds on deposit with it upon checks and vouchers signed by the Secretary-Treasurer and the Manager of the Association.

It was voted unanimously to recommend to the Mathematical Association at its business meeting Friday afternoon that the By-Laws of the Association be suspended by unanimous vote and that Professor H. E. Slaught be elected Honorary President for life, including the privileges of honorary life membership.

In view of a letter from Professor B. F. Finkel to the editor-in-chief, and on motion of Professor Carver, it was voted to ask Professor Slaught on the part of the Editorial Committee to write to Professor Finkel expressing our appreciation of his services for the Monthly, regretting that he has decided to retire from active responsibility for the Department of Problems and Solutions, and requesting that he allow his name to be continued as an associate editor.

The following were appointed associate editors of the Monthly for the year 1934, as nominated by Professor Carver:

W. F. Cheney	R. E. Gilman	R. E. Sanger
N. A. Court	R. A. Johnson	D. E. Smith
Otto Dunkel	B. W. Jones	J. H. Weaver
B. F. Finkel	J. R. Musselman	F. M. Weida
	H. L. Olson	

Through Professor W. B. Ford, chairman of the committee on the conditions of award of the Chauvenet Prize, the following report was presented:

It is recommended (1) that the phrase "best expository paper" be changed to "a noteworthy expository paper." (2) that the statement "The purpose of the prize is to stimulate expository contributions in mathematical journals" be changed to "The purpose of the prize is to stimulate expository contributions in mathematical journals on the part of the younger American scholars. The award is made on the basis of the findings of a committee of three appointed each triennium by the President." (3) that the prize continue to be given only to members of the Association.

It was voted to accept the recommendations and to thank the committee for their services.

The Trustees voted to re-nominate Professors Anna Pell Wheeler and H. S. Vandiver as associate editors of the Annals for a term of three years beginning January 1934.

It was voted to accept with thanks the invitation of the department of

mathematics of the University of Michigan to hold the 1935 summer meeting of the Association at the University of Michigan.

The Trustees adopted a plan for reprinting the "Outline of the History of Mathematics" by Professor R. C. Archibald. Details as to the purchase of this important publication will be announced somewhat later.

Other items of business were considered which had to do with (1) the commission to study the training and utilization of advanced students in mathematics; (2) business details connected with the Fifth Carus Monograph; (3) possible cooperation with other educational organizations in a plan of tests in college mathematics.

Annual Business Meeting of the Association

The Secretary announced the names of those who had been elected to membership at the meeting of the Trustees. He reported also the deaths of the following members:

W. H. Andrews, Professor of mathematics, Kansas State Agricultural College, (May 26, 1933)
Vevia Blair, Teacher, Horace Mann School, New York. (July 9, 1933)
R. L. Borger, Professor of mathematics, Ohio University, Athens, Ohio. (December 26, 1932)
R. L. Cary, American Friends Service Committee, Berlin, Germany. (October 15, 1933)
E. S. Crawley, Professor of mathematics, University of Pennsylvania. (October 17, 1933)
Harry English, Retired chief examiner of public schools, Washington, D. C. (June 8, 1933)
Daniel Kreth, Surveyor and engineer, Wellman, Iowa. (1932)
M. J. McCue, Professor of civil engineering, University of Notre Dame. (October 10, 1932)
H. S. Myers, Professor of mathematics, Southwestern College, Winfield, Kans. (October 23, 1932)
E. H. Moore, Professor of mathematics, University of Chicago. (December 30, 1932)
A. B. Morton, Professor of mathematics, Georgia School of Technology. (October 13, 1933)
Mary J. Quigley, Professor of mathematics, Teachers College of the City of Boston. (August 9, 1932)
Ormond Stone, Professor of astronomy, University of Virginia. (January 17, 1933)
F. B. Williams, Professor of mathematics, Clark University. (March 7, 1933)

A very happy event occurred at the business session when the members of the Association adopted by unanimous vote the recommendation of the Trustees that the By-Laws be suspended and that Professor H. E. Slaught be elected Honorary President for life, this to include the privileges of honorary life membership.

The election of officers for the year 1934 resulted in the following, as reported by the tellers, C. S. Atchison and F. W. Perkins:

For Vice-Presidents: A. A. Bennett, 209 votes; T. C. Fry, 191 votes; E. P. Lane, 231 votes; E. B. Stouffer, 167 votes.

For additional members of the Board of Trustees, to serve until January 1937: R. W. Brink, 197 votes; D. R. Curtiss, 245 votes; H. T. Davis, 128 votes; Otto Dunkel, 186 votes; T. H. Hildebrandt, 195 votes; C. C. MacDuffee, 142 votes; H. L. Rietz, 280 votes; J. L. Walsh, 205 votes.

The following were accordingly declared elected:

Vice-Presidents: A. A. Bennett, Brown University; E. P. Lane, University of Chicago.

Additional members of the Board of Trustees: R. W. BRINK, University of Minnesota; D. R. CURTISS, Northwestern University; H. L. RIETZ, University of Iowa; J. L. WALSH, Harvard University.

REPORT OF THE SECRETARY-TREASURER AS TREASURER, DECEMBER 12, 1933

RECEIPTS

Balance Dec. 12, 1932		$11,199.97
1931 indiv. dues	11.00	
1932 indiv. dues	305.76	
1932 instit. dues	45.50	
1932 subscriptions	25.10	
1933 indiv. dues	6,089.75	
1933 instit. dues	719.20	
1933 subscriptions	810.42	
Initiation fees	202.00	
Life membership	63.44	
Advertising	321.50	
Sale copies of MONTHLY	132.49	
Sale First Carus Mon.	12.50	
Sale Second Carus Mon.	5.00	
Sale Third Carus Mon.	7.50	
Sale Fourth Carus Mon.	8.75	
Sale Rhind Papyrus	605.00	
Received *Annals* subscription	2.50	
Int. Oberlin Savgs. Bk.	99.18	
Int. Peoples Bkg. Co.	55.40	
Int. Cleveland Trust Co.	25.00	
Int. Hardy Fund	120.00	
Int. Certifs. of deposit	38.56	
Int. from Genl. Endowment Fund	675.00	
Int. Carus Fund	115.63	
Int. Chace Fund	223.13	
Int. Chauvenet Fund	35.00	10,754.31
Total 1933 receipts to date		21,954.28
Total expenditures		15,086.12
Balance to end of 1933 business		6,868.16

EXPENDITURES

Publisher's bills (Nov. '32–Oct. '33)		$ 4,722.35
Publisher's bills (paper for 1934)		764.72
President's office		34.26
Manager's office		37.07
Editor-in-chief's office		369.34
Committee on Membership		95.20
Expense *Register*		130.60
Secretary-Treasurer's office		
Postage	316.08	
Bond	11.26	
Safety deposit	4.40	
Office supplies	58.30	
Express, tel., etc.	106.60	
Clerical work	1,875.00	
Printing	156.85	
Library expense	25.00	
Paid copies MONTHLY	13.00	
Atlantic City meeting	136.33	
Chicago meeting	116.52	
Bank tax	1.92	
Paid Papyrus postage	3.18	2,824.44
Annals subvention		225.00
Paid to sections from initiation fees		96.74
Die for Association pin		15.00
Paid B. F. Finkel int. Hardy Fund		120.00
Ins. back copies of MONTHLY		18.90
Transferred to Chace Fund		500.00
Forwarded *Annals* subscriptions		2.50
Paid *Annals* subscriptions		5.00
Sustaining memb. in American Math. Society		100.00
Expense acct. Carus Mon. Fund		40.00
Award to G. H. Hardy, Chauvenet Prize		100.00
Transfer to Genl. Endowment Fund		4,885.00
Total expenditures		15,086.12
Checking account		315.66
Oberlin Savgs. Bk. acct. unrestricted		1,906.53

		Oberlin Savgs. Bk. acct. restricted	1,499.40
		Peoples Banking Co. acct.	1,636.04
		Cleveland Trust Co. savgs. acct.	2,025.00
Received on 1934 business	575.20	Certif. of deposit, unrestricted	60.73
Book balance Dec. 12, 1933	$ 7,443.36	Bank balance Dec. 12, 1933	$ 7,443.36

EXHIBIT OF THE FUNDS OF THE ASSOCIATION

CARUS MONOGRAPH FUND

Balance Dec. 12, 1932		$ 6,222.28
Receipts: Sales	$ 33.75	
Interest	245.23	
Profit in purchase of $3\frac{1}{8}\%$ U. S. Treas. Bond of 1946–49	1.19	280.17
Expenditures: Accrued interest on new bond	3.09	$ 6,502.45
On account of sections	40.00	43.09
		$ 6,459.36
Certificates of deposit, unrestricted funds	$2,650.47	
Cleveland Trust Securities Co. Gold Bond	1,000.00	
Pacific Power & Light Co. 5% gold bond, market value	770.00	
$3\frac{1}{8}\%$ U. S. Treasury Bond of 1946–49	1,000.00	
Cash in bank, restricted, certificates of participation	894.60	
Cash in bank, unrestricted	144.29	
Balance Dec. 12, 1933		$ 6,459.36

ARNOLD BUFFUM CHACE FUND

Balance Dec. 12, 1932		$ 4,582.35
Receipts: Sale Papyrus	$ 605.00	
Interest	227.12	832.12
Expenditures: Cost of $3\frac{1}{8}\%$ U. S. Treasury Bond 1946–49	1.63	$ 5,414.47
Accrued interest	2.51	4.14
		$ 5,410.33
Iowa Elec. Light & Power Co. 5% Bond	$1,000.00	
Western United Gas and Elec. Co. Bonds	2,370.00	
$3\frac{1}{8}\%$ U. S. Treasury Bond of 1946–49	1,000.00	
Certificates of deposit, unrestricted	119.77	
Certificate of deposit, Northern Trust Co., Chicago	500.00	
Cash in bank, restricted, certificates of participation	39.60	
Cash in bank, unrestricted	380.96	
Balance Dec. 12, 1933		$ 5,410.33

CHAUVENET PRIZE FUND

Balance Dec. 12, 1932	$ 619.38
Interest	35.00
	$ 654.38

Award, December 1932		100.00
		$ 554.38
Iowa Elec. Light & Power Co. 5% Bond	$ 500.00	
Cash in bank, unrestricted	54.38	
Balance December 12, 1933		$ 554.38

LIFE MEMBERSHIP FUND

Liability on life memberships Dec. 12, 1932	$ 658.13
Received on life membership payment	63.44
	721.57
To be transferred to current funds, surplus	17.88
Liability on life memberships as of Jan. 1, 1934	$ 703.69

GENERAL ENDOWMENT FUND

Balance Dec. 12, 1932		$ 7,700.00
Transfer from current funds		4,885.00
		12,585.00
Liberty Bonds	$1,000.00	
$4\frac{1}{4}$–$3\frac{1}{4}$% U. S. Treasury Bond of 1943–45	1,000.00	
Land Trust Certificate	700.00	
Cleveland Trust Investment Co. Gold Bond	1,000.00	
Idaho Power Co. 5% Bonds	2,000.00	
Northwestern Electric Co. Bonds	3,000.00	
Texas Power and Light Co. 5% Bonds, market value	885.00	
Iowa Elec. Light and Power Co., 5% Bonds	3,000.00	
Balance Dec. 12, 1933		$12,585.00

Of the funds on hand, indicated in the first division of this financial report, $380.96 belongs to the Arnold Buffum Chace Fund (This amount has since been added to the certificate of deposit with the Northern Trust Company), $54.38 belongs to the Chauvenet Prize Fund, $144.29 belongs to the Carus Monograph Fund, and $703.69 is held as a Life Membership Fund, representing the liability on life memberships already paid for, as of date Jan. 1, 1934. Aside from these amounts, the various funds of the Association are carried in the form shown in the inventories under the exhibit above.

When the accounts were closed Dec. 12, 1933, there remained on the total business for 1933 the following items:

BILLS RECEIVABLE

1933 individual dues	$200.00
Advertising	50.00
	$250.00

BILLS PAYABLE

Publisher's bills (Nov.–Dec. '33)		$1,000.00
Register, clerical work	20.00	
printing	500.00	520.00
Manager's office		25.00
Editor-in-chief's office		165.00
Secretary-Treasurer's office		
Postage	10.00	

Office supplies.........	5.00	
Express, tel., etc........	17.00	
Clerical work..........	158.00	
Printing...............	80.00	270.00
Annals subvention................		75.00
Carus Monograph Fund...........		144.29
Chauvenet Prize Fund............		54.38
Chace Fund.....................		380.96
Life Membership Fund...........		703.69
		$4,238.32

The estimated surplus, $7,435, of Jan. 1, 1933 was depleted by transfer of $4,885 to the General Endowment Fund, leaving the estimated surplus as $2,550. If to the balance on 1933 business shown in the report, $6,868.16, there be added the bills receivable, $250.00, and there be subtracted the estimated bills payable, $4,238.32, there results an estimated final balance on 1933 business of approximately $2,880, which represents the accumulated surplus in current funds. This narrow margin of profit of about $330 is properly to be augmented by the item of $760 covering the supply of paper for the MONTHLY for the year 1934, the Association having had a chance to purchase the supply at a favorable figure.

W. D. CAIRNS, *Secretary-Treasurer*

A CONFERENCE OF THE OFFICERS AND COMMITTEE MEMBERS OF THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS AND THE MATHEMATICAL ASSOCIATION OF AMERICA

As the result of some previous planning a group of officers and committee members of these two organizations met for luncheon and the early afternoon of Saturday, December 30, 1933, at the Commander Hotel, Cambridge, Massachusetts, to discuss informally matters of mutual interest. President William Betz, C. R. Atherton, H. C. Barber, Ralph Beatley, H. D. Gaylord, and W. D. Reeve represented the interests of the National Council. A number of the officers and commission members of the Mathematical Association were also in attendance.

Professor Beatley pointed out that the new commission of the College Board which is to consider possible changes in entrance examinations in mathematics can, along with other objects, determine the general ground over which the students are to be examined, whether they are to be comprehensive examinations, or be scored separately, or be a combination of these two. He also discussed the much debated question as to whether the Board examinations have any effect on the teaching of mathematics in the schools.

President Dresden described the functions of the two commissions recently appointed by the Mathematical Association and their bearing on the sphere of

work common to these two organizations.

President Betz spoke of a possible closer affiliation between the National Council and the American Association for the Advancement of Science as the Mathematical Association and the American Mathematical Society are affiliated. He pointed out that the National Council needs on the one hand to maintain active contact with the administrative groups such as the National Education Association but also desires to maintain closer contact with the "real mathematicians" and to keep informed as to what is going on in research. He called very strongly to the attention of the college and university men the fact that there is a very violent movement against the place of mathematics in the secondary schools, that the revision which has been going on for about ten years has been too much in the hands of leaders that did not themselves know how to revise intelligently, that a considerable number active in the National Council have more recently put before the secondary school teachers a better revision of the theory of mental discipline and a consequent revision in the material and methods of secondary school mathematics. He pleads for the active support of members of the Mathematical Association in the progressive improvement which it is hoped can be brought about in secondary mathematics through proper attention to such matters as the proper recognition of the element of utility in mathematics, of the necessity of teaching mathematics as a mode of thought and in avoidance of over-emphasis of technique.

Professor Reeve, who is editor-in-chief of "The Mathematics Teacher," the official journal of the National Council, said that a number of those present in this group have to live with the "general educators." In their defense he stated that they know that something is wrong in the teaching of mathematics but that they are not always accurate in their diagnosis. He stated that it is a great mistake that college teachers should not be interested in the teaching phase of mathematics, urging that there should be articles in the MONTHLY on teaching and that we must have greater cooperation so as to engage the interest of pupils of genius and conserve their interest for college mathematics courses. The social study group is now in the saddle and we shall all have to cooperate to save the situation.

Professor Bennett said that the attacks on mathematics are so unanimous that there must clearly be something in the situation sufficient to call for our careful consideration. With no feeling of hostility to mathematics, the educators whom he consulted agree that the tendency is now against mathematics and that no concerted effort can save the situation. These educators agree in the view that everyone should have the social and democratic advantages of graduating from a high school, and that, even presupposing good teaching, a subject is useless unless the pupil feels it to be vital to himself. As mathematicians we must not fail to consider such views as these and to evaluate them for whatever of value and soundness they contain.

In answer to a question by Professor W. L. Hart, Professor Reeve said that the attacks on the ninth grade seem to be heavier than on the other grades;

he added that he does not himself favor the requirement in mathematics after the ninth grade. Professor Reeve added that the tragic and important fact is that mathematics is a hated subject. The whole trouble lies in the *memoriter* method as contrasted with mathematics as a method of thinking. He adduced one procedure which we should adopt whether we affiliate formally or not, viz., to attend state and regional educational meetings, to use our influence to see that papers of a real mathematical character are put on these programs, and to take part ourselves in the programs with the air and the attitude of believing in mathematics.

Professor Hedrick said that we want very much not only to assist the men in secondary education but we want them to help us in our problems, for example, those of good methods in teaching college mathematics. He referred to the unpublished report of a commission on the teaching of mathematics in which, through a broadly distributed questionnaire, a great majority of secondary teachers made the answer that they took all the mathematics available in their college courses and that fifty per cent of them regarded general education courses as of little or no value to them, educational psychology being the most important of these and the history of education the least important.

Mr. Barber said that we should oppose the move of the College Board in one respect, that the Board wants examination questions that will test mathematical training, we want questions that will direct it.

In closing, Professor Moulton said that his commission to study the training and utilization of advanced students in mathematics desires very much the valuable aid that can be given by those who are active in secondary schools in teaching. Professor Cairns stated that we must have something more than mere affiliation; we must seek for a true community of interest and cooperation in active effort. President Dresden suggested that the officers of the two organizations communicate with one another as to developing definite plans for cooperation.

It is obvious that in a brief session such as this only preliminary contacts could be made, but there was a universal expression that the conference was very satisfactory, bringing as it did numerous members of the two organizations into touch on matters of common importance.

A somewhat fuller record of the discussion has been sent to those who took part in the conference and to other officers and committee members.

W. D. CAIRNS, *Secretary-Treasurer*

MODIFICATIONS IN THE AWARD OF THE CHAUVENET PRIZE

The Chauvenet Prize of one hundred dollars, as originally established by the Mathematical Association of America in 1926, was to be awarded every five years for "the best expository paper published in English during the five year period by a member of the Association," and the purpose of the prize as stated

was "to stimulate expository contributions in mathematical journals." In 1928 the five year period was reduced to three, but the guiding principles of the award have remained as worded above until the present. Wishing, however, to bring the prize more distinctly within the range of the younger American mathematicians and otherwise to clarify the existing rules, the Trustees of the Association at their recent meeting in Cambridge voted (1) that the phrase "best expository paper" be changed to "a noteworthy expository paper" and (2) that the statement "The purpose of the prize is to stimulate expository contributions in mathematical journals," be changed to "The purpose of the prize is to stimulate expository contributions in mathematical journals on the part of the younger American scholars." The award is made on the basis of the findings of a committee of three appointed each triennium by the President.

These changes will evidently bring about a wider latitude than heretofore in the selection of eligible papers and will leave the committee at the end of any one triennium with a larger degree of freedom for proceeding in whatever manner seems best in the broad interests of collegiate mathematics. The prize will continue to be given only to members of the Association, and will not be awarded for books.

The Chauvenet Prize was originally established by a gift from Professor Julian L. Coolidge during his presidency of the Association, and its frequency period was reduced to three years by an additional gift from Professor Walter B. Ford during his presidency.

The first award was made in December 1925 to Gilbert Ames Bliss for his paper on "Algebraic Functions and their Divisors" published in the Annals of Mathematics. The second award was made in December 1929 to Theophil Henry Hildebrandt for his paper on "The Borel Theorem and its Generalizations" published in the Bulletin of the American Mathematical Society in 1926. The third award, covering the three years preceding 1932, was made in December 1932 to Godfrey H. Hardy for his paper on "An Introduction to the Theory of Numbers" published in the Bulletin of the American Mathematical Society in 1929. The next award will be made in December 1935 and will cover the period 1932, 1933, 1934.

THE FALL MEETING OF THE MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA SECTION

The Fall meeting of the Maryland-District of Columbia-Virginia Section was held at The George Washington University, Washington, D. C., on Saturday, December 9, 1933.

The two sessions, morning and afternoon, were presided over by the Chairman, Professor B. Z. Linfield, of the University of Virginia. President Cloyd H. Marvin of The George Washington University gave a very interesting and beneficial address at the morning session and officially welcomed the members and guests. The Section had the pleasure of having as its guest President Arnold

Dresden, who, in addition to giving the last paper on the program, informed the Section of the recent very interesting work and aims of the Association.

The attendance was seventy-five including the following forty-seven members of the Association: O. S. Adams, Beatrice Aitchison, G. F. Alrich, Clara L. Bacon, G. A. Bingley, Archie Blake, J. W. Blincoe, W. E. Byrne, Paul Capron, C. N. Claire, G. R. Clements, Abraham Cohen, Tobias Dantzig, Alexander Dillingham, J. L. Dorroh, Arnold Dresden, J. A. Duerksen, J. H. Edmonston, P. J. Federico, Michael Goldberg, Harry Gwinner, W. M. Hamilton, Isabel Harris, F. E. Johnston, L. M. Kells, W. D. Lambert, A. E. Landry, Florence P. Lewis, B. Z. Linfield, Florence M. Mears, W. K. Morrill, F. D. Murnaghan, O. J. Ramler, J. B. Scarborough, M. A. Scheier, W. F. Shenton, Helen Sullivan, J. H. Taylor, Mildred E. Taylor, Marian M. Torrey, John Tyler, T. L. Wade, F. M. Weida, G. T. Whyburn, John Williamson, E. W. Woolard, Oscar Zariski.

The spring meeting will be held the first part of May at the College of William and Mary, Williamsburg, Virginia.

The following eight papers were read:

1. "A property of Steiner's deltoid" by Professor A. E. Landry, Catholic University of America.

2. "The periodic motion of a dynamical system near an equilibrium point of stable type" by Dr. D. C. Lewis, Jr., Johns Hopkins University (Introduced by Professor Abraham Cohen).

3. "A simple geometric configuration" by Professor F. E. Johnston, George Washington University.

4. "An irreducible complete system of Euclidean concomitants for the line and the conic" by T. L. Wade, Jr., University of Virginia.

5. "On a property of contact transformations" by Professor Tobias Dantzig, University of Maryland.

6. "State systems of plane coordinates" by Dr. O. S. Adams, U. S. Coast and Geodetic Survey.

7. "An explicit summation formula for the finite series $\sum_{n=1}^{n=x} n^p$" by Janny Yates, University of Maryland (Introduced by Professor Tobias Dantzig).

8. "On general aspects of the calculus of variations" by Professor Arnold Dresden, Swarthmore College.

Abstracts of four of the papers follow:

1. Designate the cusps by $A_i (i=1, 2, 3)$ and the parameter at any point of the curve by t; let $f(t)$ be the residual parameter on the tangent from t, $\theta_i(t)$ the residuals on the joins of t to the respective cusps. Drawing the tangent to the curve at its intersection (other than the cusp) with that line of symmetry which passes through the cusp A_3, and letting t_0 be the residual which lies on the arc A_2A_3, the property in question is expressed by the equation $\theta_2\theta_3\theta_2 f(t_0) = t_0$.

(Note that it makes no difference whether the symbols of operation are read from left to right or in the reverse direction, also that by rotation and reflexion six angles are obtained.)

3. In general coordinates in the plane, the coefficients of $a_1x_1+a_2x_2+a_3x_3=0$ are subjected to the six permutations of the symmetric group on a_1, a_2, a_3. The six resulting lines satisfy Brianchon's condition for a circumscribing hexagon. Certain of the sixty points of concurrence of Brianchon's figure are seen to be invariant for all sets of values of a_1, a_2, a_3; and the twenty lines on which these sixty points usually lie by threes reduce to thirteen lines, four of which are invariant.

4. This irreducible complete system is found to consist of eight invariants, ten covariants, fifteen contravariants, and nine mixed concomitants. Of these forty-two irreducible concomitants only twelve are algebraically independent, and a set of thirty independent syzygies are found connecting them. One or more geometric interpretations are found for each of the irreducible concomitants.

6. The demand has arisen among the engineers of the country for systems of plane coordinates that can be carried through as much as a state or through a large part of a state. In response to this demand the U. S. Coast and Geodetic Survey is computing systems of plane coordinates for the various states of the union which will be used as a basis for the computation of the control surveys of that organization on the respective grids of the states. These coordinates will then be used by the surveyors throughout the various states as starting data for the computation of such regional surveys as they may make. This paper is a report on such activities along this line as are now in progress in the U. S. Coast and Geodetic Survey.

F. M. WEIDA, *Secretary*

SOME FREQUENTLY OVERLOOKED MATHEMATICAL PRINCIPLES OF DESCRIPTIVE GEOMETRY[1]

By W. H. ROEVER, Washington University

1. *Introduction*

The purpose of this paper is, as its title suggests, to call attention to some of those fundamental mathematical principles of Descriptive Geometry which, if not actually overlooked, are not sufficiently stressed.

This oversight is, perhaps, not surprising in view of the fact that although this subject can be regarded as one of the mathematical disciplines, it is almost entirely ignored, and even regarded as trivial, by most of our mathematicians. For this reason its instruction is left to drawing teachers, who, as a class, are not interested in the mathematical aspects of the subject.

Perhaps also this state of affairs is natural for the reason that the subject was developed for technical needs and is thus taught (almost) exclusively in

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[1] Read at Joint Session of the Mathematical Association of America with the Society for the Promotion of Engineering Education at Minneapolis on Sept. 4, 1931.

technical schools. However, many other mathematical disciplines have had practical origins and were evolved from what at first might have been regarded as merely an art.

But it is in the separation of the principles of such an art from their technical applications that the foundations of a science may be laid. After this science has been developed along theoretical lines it can be re-applied and then it becomes a much more valuable instrument for practical purposes than was the art from which it sprang.

Thus Descriptive Geometry sprang from the art of stereotomy (stone and wood cutting) which was developed in the Middle Ages. Frézier was the man who, in 1738, made the separation of the geometric constructions of this art from their technical applications and used them to solve the problems of geometry of space. It remained for Gaspard Monge (1746–1818), however, to further develop this new constructive geometry of space and to elevate it to the dignity of a science, and it was he who gave to it the name *Descriptive Geometry.*

2. *The Problem of Descriptive Geometry and means of Attaining its Solution*

The process of drawing is the method at our disposal for solving graphically the problems of geometry of space.

Now this process must of necessity be executed in the plane (or, more generally, upon a surface). Hence the first essential of any constructive geometry of space consists in the representation of the objects of space by means of figures which lie in a plane.

This is also the aim of the artist. But his particular desire is to make a plane representative which shall resemble (i.e., convey an adequate notion of the form of) the space object shown by his picture.

To state the matter more precisely, *the artist's criterion of representation* is that the plane representative, when properly placed with respect to the eye, shall produce a retinal image differing but little from that produced by the object itself. That this *criterion is satisfied by a projection* (at least approximately) follows from the optical properties of the eye (which are similar to those of the camera).[2]

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[2] In order to see this let us recall the following well-known facts:

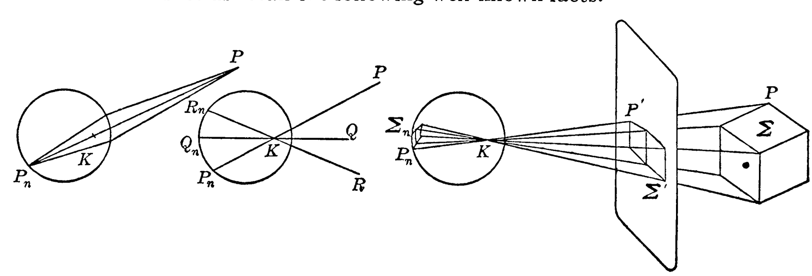


Fig. i Fig. ii Fig. iii

(1) All the rays of light which emanate from a point P (Fig. i) and pass through the pupil into the eye undergo several refractions and then unite again in a point P_n of the retinal surface of the

The different types of projection are defined as follows:

By a *central projection*, from a point C, of the points P of space upon a plane α, is meant those points P' in which the lines CP pierce the plane α (see Fig. 1). If the projecting lines, instead of emanating from the point C, are parallel to a straight line r, the projection is said to be *parallel*. If the line r is *not* perpendicular to α, the parallel projection is called *oblique* (see Fig. 2); while if r is perpendicular to α, the projection is called *orthographic* (see Fig. 3).

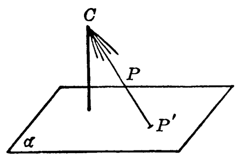


FIG. 1

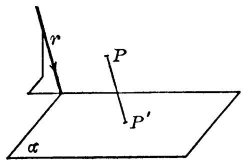


FIG. 2

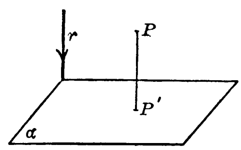


FIG. 3

While a *single projection* may satisfy the needs of the artist, it is not a sufficiently complete representation for the purpose of solving the problems of space by means of constructions in the plane. For, in order to solve such problems in this way, it must not only be possible to pass from an object in space to its plane representative, but it must also be possible to pass back again from this representative to the object in space. Now, it is evident (see Figs. 1, 2, 3) that a single projection is a singular transformation: i.e., while it enables one to pass from a point P of space to its projection P' in the plane of projection, it

eye. This point is called the retinal image of the point P. There is a particular ray PP_n which passes through the eye practically unbroken in direction. This ray may be taken as a substitute for that slim bundle of rays which come from P and pass through the pupil into the eye. Its direction is that in which the eye perceives the point P to lie. We will call it the *sight-line of P*.

(2) The sight-lines from the various points P, Q, R of a body Σ all pass through a fixed point K, which is called the *optical center* of the eye (Fig. ii). Thus the sight-lines belong to a geometric bundle of lines of center K. The retinal image Σ_n, composed of the images P_n, Q_n, R_n of the points P, Q, R of Σ, is thus seen to be the geometrical intersection of the retinal surface of the eye with those lines of this bundle which come from the points of Σ.

It follows from fact 1 that any light emitting point P', which lies on the sight-line of P, has the same retinal image as P. Hence to set up a plane picture Σ' which shall produce the same retinal image as that which is produced by Σ, it is sufficient to replace the points P of Σ by the points P' in which the sight-lines of the points P pierce the plane of the picture (Fig. iii).

By fact 2, all of the sight-lines which come from the points of Σ belong to a bundle of lines of center K. Therefore it follows that the picture Σ' is simply the geometrical intersection of the picture plane with those lines of this bundle which come from the points of Σ. In other words, Σ' is a central projection of the object Σ, the center of projection being the optical center of the eye and the plane of projection the plane of the picture.

On the other hand, when the central projection of an object is properly placed with respect to the eye, i.e., when the eye is placed at the center of projection, a retinal image will be produced by the picture which is the same as that which would have been produced by the object. However, when such a picture is not properly placed with respect to the eye, or when such a picture is replaced by one which is obtained by parallel instead of converging rays (whether these be perpendicular or oblique to the picture plane) the eye still recognizes in the picture the object represented. Thus we are led to the conclusion stated above.

does not admit an inverse, that is, it fails to reproduce uniquely P from P', since P' may be the projection of any point whatever on the projecting ray of P.

In other words, what is needed for the graphical solution of space problems is *the requirement of an unambiguous correspondence between space and the plane.* In order that we may appreciate that two projections are sufficient for this purpose, let us suppose that we have two photographs (central projections) of the same object (or landscape) taken from slightly different points of view, and that these photographs are placed in a stereoscope so that a person may simultaneously observe one photograph with one eye and the other photograph with the other eye, and thus obtain the impression of seeing the object in relief. The reason that a single projection, like a photograph or an artist's drawing, may convey an adequate notion of the form of a space object, such as a building, is because one has some information about this object such, for instance, as the perpendicularity of its edges.

We have already stated that the graphical process of drawing can be carried out only upon a plane (or curved) surface.

The usual instruments (of the geometer) by means of which this process is actually performed are *pencil*, *ruler* and *compasses*. With these instruments it is possible to solve graphically the three fundamental problems of plane geometry:

Group I
- (1) To find the (straight) line connecting two points.
- (2) To find the point of intersection of two lines.
- (3) To construct a circle of given center and radius.

In addition to these problems, we should like to be able to solve also the following fundamental problems of space:

Group II
- (1) To find the line connecting two points of space.
- (2) To find the line of intersection of two planes.
- (3) To find the plane determined by a point and a line.
- (4) To find the point in which a line pierces a plane.
- (5) To construct a sphere of given center and radius.[3]

[3] These problems have been further subdivided as follows: (A) Problems of Geometry of Position, (B) Perpendicularity Problems, (C) Metrical Problems. In each of these subdivisions there is an indefinitely great number of problems, all of which may be solved, however, by means of a few thereof, which are called the fundamental problems.

The problems of Subdivision (A), which do not involve the notions of perpendicularity, length, or angular magnitude, have for fundamental problems the first four problems of Group II.

In particular, one of the given elements may lie at infinity. Thus if, in Problem 1, one of the points lies at infinity, this problem becomes the following: (1a) Through a point pass a line parallel to another line.

In Problem 3, either the given point or the given line may lie at infinity. In these cases the problem assumes, respectively, the following forms:

(3a) Through a given line pass a plane parallel to another given line.

(3b) Through a given point pass a plane parallel to a given plane.

As an illustration of the way in which the given problem may be resolved into fundamental problems, we consider the example: Find the point X common to three planes α, β, γ. A solution

It is by virtue of the possibility of representing the elements of space (such as points, lines, planes, etc.) by means of elements of the plane (such as pairs of points or pairs of lines, etc.) that the problems of space (i.e., of the type of Group II) can be solved by means of constructions in the plane (i.e., by those of Group I).

3. *The Mongean Method of Double Orthographic Projection*

For the purpose of representing in a plane an object of space, Monge chose the natural, and already ancient, method of constructing, what the architect calls, plan and elevation, i.e., the orthographic projections of this space object on each of two mutually perpendicular planes π_1 and π_2 (Fig. 4), one of which (π_1) is later revolved about their intersection (the ground line, g) into coincidence with the other. He thus represented the objects of space by pairs of figures in the (drawing) plane (Fig. 5).

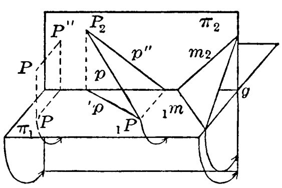


FIG. 4

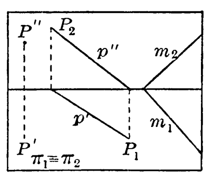


FIG. 5

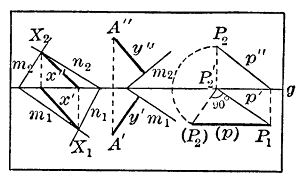


FIG. 6

The objects of space being bounded and defined by points, lines, planes, etc., the plane representatives of each of these elements is obtained. Thus

(1) for a point P of space there are in the drawing plane two points P', P'' which lie on the same perpendicular to the ground line;

(2) for a line p of space there are two lines p', p'' in the drawing plane (or the two traces P_1, P_2, where the line pierces the planes of projection π_1, π_2, respectively);

---

consists in finding, by means of Problem 2, the line of intersection l of the given planes α and β, and then, by Problem 4, the point X in which l pierces γ.

Of Subdivision (B), the following five problems are regarded as fundamental:

1. Through a point pass a line perpendicular to a plane.
2. Through a point pass a plane perpendicular to a line.
3. Through a point pass a line perpendicular to and touching a line.
4. Through a line pass a plane perpendicular to a plane.
5. Find the common perpendicular to two non-intersecting lines.

The fundamental problems of Subdivision (C) are the following:

1. To determine the length of the segment connecting two points.
2. To determine the magnitude of the angle formed by two lines.
3. To determine the magnitude of the dihedral angle formed by two planes.

In the solution of the problems of Subdivisions (B) and (C) use is made of the possibility of constructing circles and spheres (as is assumed in the statements of the final problems of Groups I and II), and hence ultimately of circles only.

See Introduction to Volume I, G. Loria, *Vorlesungen über Darstellendegeometrie* (1907), B. G. Teubner (Leipzig). Also W. H. Roever, *The Mongean Method of Descriptive Geometry*, The Macmillan Company.

(3) for a plane μ of space there are the two traces m_1, m_2 meeting on the ground line g (see Figs. 4 and 5).

This method of representation evidently possesses *the property of unambiguous correspondence* between space and the plane, and hence by means of it the problems of space may be solved by constructions in the plane.

From what has been said it is evident that (a) the condition that a point be on a line is that the projections of the point lie on the corresponding projections of the line; (b) the condition that a line lie in a plane is that the traces of the line lie in the corresponding traces of the plane. It can also easily be shown that (c) the condition that a line and a plane be perpendicular to each other is that the projections of the line be perpendicular to the corresponding traces of the plane.

The last condition makes possible the solution of perpendicularity problems. To solve the metrical problems use is made of a process called *revolvement.* This consists in revolving a plane figure around the horizontal or vertical trace of its plane until it coincides with the picture plane. In Figure 6 are given the solutions, by the Mongean method, of each of the three problems:

Problem I. To find the line of intersection $x \equiv (x', x'')$ of the two planes $\mu \equiv [m_1, m_2]$, $\nu \equiv [n_1, n_2]$.

Problem II. To find the line $y \equiv (y', y'')$ which passes through the point $A \equiv (A', A'')$ and is perpendicular to the plane $\mu \equiv [m_1, m_2]$.

Problem III. To find the length of that portion of the line $p \equiv (p', p'')$ which is contained between its traces P_1 and P_2.

The solution of Problem I depends upon condition b and the fact that line x lies in both of the given planes. The solution of Problem II depends upon conditions a and c. To solve Problem III the right triangle $P_2 P_2' \, {}_1P$ (Fig. 4) is revolved around $'p$ into coincidence with π_1. Hence in Figure 6 the segment $\overline{P_2' (P_2)}$ is drawn through P_2' perpendicular to p' and equal to $\overline{P_2 P_2}$. Thus we have solved by constructions in a plane three problems of space. Of these, I is a problem of geometry of position, II is a perpendicularity problem, and III is a metrical problem.

The method of Monge, thus briefly described, is the method most frequently used in architecture and technology, and is, in general, the only method taught in our (American) technical schools. It is, however, not the only method of Descriptive Geometry. Since the time of Monge, Descriptive Geometry has made much progress. In the sections which follow we shall describe some of the more recent methods and advancements.

4. *Axonometry: Another Double-Projection Method*

In order to introduce this subject, various methods of procedure are possible. Let us begin, however, by supposing that in the photograph (central projection) of a landscape there appears, besides the image of a bird in flight, that of its shadow upon the ground (or a wall).

To simplify matters, let us suppose further that the camera with which the photograph was taken was so far distant from the landscape under consideration that the rays of light which passed into it from the points of this landscape were practically parallel, and that therefore the projection represented by the photograph was a parallel (instead of a central) projection.

More particularly, let us assume that the sun, when it cast the shadow of the bird, was in the zenith. Thus we have for a definite point of space (here the position of the bird when the picture was taken) two points in the picture plane, namely, the image of the bird and that of its shadow.

We already know that the image of the bird alone is not sufficient to determine the position of the bird in space. That the images of the bird and its shadow together are sufficient to determine the position of the bird in space will now be shown.

To this end, let us suppose the existence of a set of rectangular axes Ox, y, z in space, such that the origin O and two of these axes Ox, Oy shall lie on the ground (assumed to be a horizontal plane) and the third axis Oz shall be a vertical line. Then, in the picture, the images of these axes (or rather the images of their positive portions) would appear as three rays $O^*\,x^*$, $O^*\,y^*$, $O^*\,z^*$ emanating from the image O^* of the origin O (see Fig. 7).

The image of the bird would appear at P^* and that of its shadow at P'''^*, these two images lying on a line parallel to the (axonometric) axis $O^*\,z^*$. If now the axes in space had carried number scales, their projections in the picture would also carry such scales. If further we had in the picture the image of the so-called *projecting parallelepiped* (shown by lines parallel to the axonometric axes) we could read off from the picture the coordinates of the position P of the bird in space.

Thus the axonometric axes O^*–x^*, y^*, z^*, together with two points which lie on the same parallel to one of these axes, are sufficient to determine the position of a point in space. Hence we have another method of double projection.

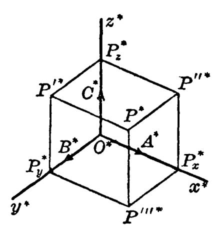


FIG. 7

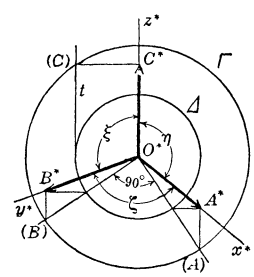


FIG. 8

This method would be without significance if it were not possible to choose in a plane three vectors

$$\overrightarrow{O^*A^*}, \quad \overrightarrow{O^*B^*}, \quad \overrightarrow{O^*C^*}$$

of common origin O^* in such a way that they may be regarded as the projections of three mutually perpendicular space vectors of common origin and equal lengths.

In the case of orthographic projection, the possibility of selecting such a set of three vectors in a plane is contained in the

Theorem of Schwarz:[4] *Any two conjugate diameters and the minor axis of an ellipse may be regarded, as to directions, as the orthographic projection, on the plane of the ellipse, of three mutually perpendicular axes of space. If these space axes be taken as cartesian axes and the plane of the ellipse as the picture plane, then the ratios borne by the halves of the chosen conjugate diameters and the focal distance of the ellipse to the half major axis of the ellipse are the fore-shortening ratios, i.e., the numbers by which the scales on the space axes must be multiplied (respectively) in order to obtain those on the axonometric axes.*

However, without constructing the ellipse, the vectors

$$\overrightarrow{O^*A^*}, \quad \overrightarrow{O^*B^*}, \quad \overrightarrow{O^*C^*}$$

in the plane may be properly determined by the following construction (see Fig. 8). In this construction Δ and Γ are circles of common center O^*, while $O^*(A)$ and $O^*(B)$ are perpendicular radii lying below the horizontal line which passes through O^*. Through the points where $O^*(A)$ cuts Δ and Γ, horizontal and vertical lines are drawn, respectively, meeting in A^*. In a similar manner B^* is determined. A vertical tangent t is drawn to Δ meeting Γ in (C) above O^*, and then through (C) a horizontal line is drawn cutting the vertical line through O^* in C^*. The three segments $\overline{O^*A^*}$, $\overline{O^*B^*}$, $\overline{O^*C^*}$ thus found are of proper lengths and directions.

Thus we see that for orthographic projection the lengths and directions of the vectors

$$\overrightarrow{O^*A^*}, \quad \overrightarrow{O^*B^*}, \quad \overrightarrow{O^*C^*}$$

are not independent. However, if the parallel projection is not required to be orthographic, it is possible to choose at pleasure both the lengths and directions of these vectors. The truth of this assertion is contained in the famous

Theorem of Pohlke (1853): Three straight line segments of arbitrary lengths in a plane, drawn from a point and making arbitrary angles with each other, can be regarded as a parallel projection of three equal segments drawn from the origin on three rectangular coordinate axes; however, not more than one of the given segments, or one of the given angles, can vanish.

Thus we see that in the axonometric method, as in the Mongean, it takes two points in the picture plane to represent unambiguously a point in space. Similarly, it takes two lines p^*, p'''^* (or any of five other pairs) of the picture plane

[4] Really not due to Schwarz.

to represent unambiguously a line p of space, while it takes two lines of the picture plane m_1^*, m_2^* which intersect on one of the axonometric axes O^*x^*, O^*y^*, O^*z^* to represent unambiguously a plane μ of space (see Fig. 9).

Of particular interest are the traces t_1, t_2, t_3 of the picture plane which form the axonometric triangle $T_xT_yT_z$. The sides of this triangle meet the corresponding traces m_1^*, m_2^*, m_3^* of a general plane μ in the points L, M, N of a straight line m, which is called the axonometric trace of the plane μ. If, in particular, the parallel projection is orthographic, the sides of the axonometric triangle are perpendicular, respectively, to the axonometric axes (see Fig. 9).

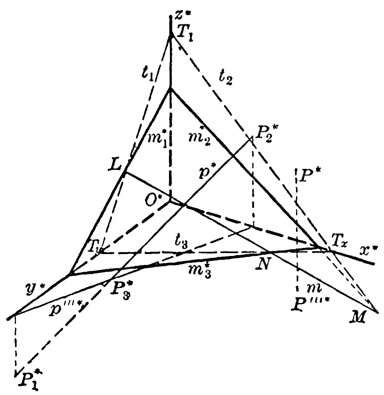


FIG. 9

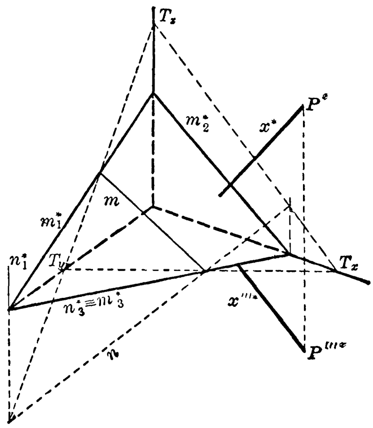


FIG. 10

Statements (a), (b), (c) given under the Mongean method are true here also if we understand by projections, *the projections on the picture plane*, and by traces, *the traces in the picture plane.*[5] With this information we are in a position to solve the

Problem: To find the line $x\equiv(x^*, x'''^*)$ which shall pass through the point $P\equiv(P^*, P'''^*)$ and be perpendicular to the plane $\mu\equiv[m_1^*, m_2^*]$.

The solution for orthographic axonometry is as follows (see Fig. 10). Draw the axonometric triangle $T_xT_yT_z$ with sides perpendicular to O^*x^*, O^*y^*, O^*z^*. Then the axonometric trace of the plane μ is the line m; the perpendicular to m through P^* is x^*. Next represent a plane $\nu\equiv[n_1^*, n_2^*]$ which shall pass through m_3 and be perpendicular to the plane xOy. Then n_3^* coincides with m_3^* while n_1^* is parallel to O^*z^*. The axonometric trace of this plane is the line n. Hence since x''' lies in the plane xOy, it follows that x'''^* is perpendicular to n, and it passes through P'''^* because x must pass through P. The required line x has thus been found since its representatives in the picture x^*, x'''^* have been found.

The solution of this problem suggests, what is true, namely, that by the axonometric method, as well as by the Mongean, all the problems of space can

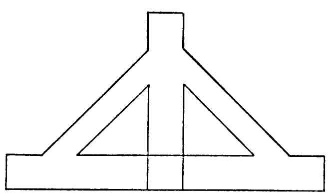
[5] Statement (c), thus modified, is true only for orthographic axonometry, i.e., for the case where the projecting rays are perpendicular to the picture plane.

be solved, whether these be problems of geometry of position, perpendicularity problems or metrical problems.

Furthermore, the pictures produced by this method (i.e., for orthographic or oblique axonometry) may also be used as working drawings by the builder, since, like the Mongean pictures, they are capable of being "scaled off." In addition to this property and that of unambiguous correspondence, which this method has in common with the Mongean method, it has a characteristic which, for many purposes, makes it preferable to the Mongean.

To see this, let it first be observed that most of the objects of architecture and technology contain, among their bounding surfaces, planes which are mutually perpendicular and which intersect in edges of the object. It is on these planes, or on planes parallel to these, that the objects are generally projected (orthographically) in order to obtain their Mongean representatives (i.e., their pictures). Thus certain planes (namely, those perpendicular to both of the planes of projection) are shown merely by lines, and therefore the Mongean pictures of such objects fail to convey a satisfactory notion of the space forms of these objects. This point is well illustrated by Figures 11 and 12, which are the plan and elevation, respectively, and together form the Mongean representative of a bracket-shaped object. On the other hand, Figure 13 is a projection (also orthographic) of the same object on a plane which is not parallel to one of the mutually perpendicular planes of the object. Evidently Figure 13 conveys to the mind a much better notion of the space form of the object which it represents than do Figures 11 and 12.

FIG. 12



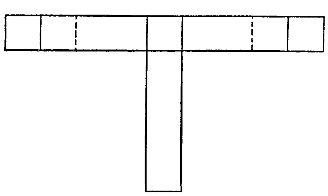


FIG. 11

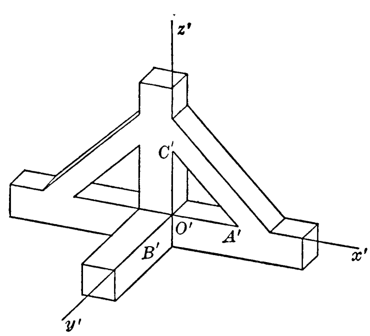


FIG. 13

5. *Free Perspective: A Method of Double-Trace*

We have just considered two methods of double-projection, namely, the Mongean and axonometric methods. In both of these methods, however, a (straight) line might just as well have been given by two traces (i.e., the points

where it pierces the picture, or coordinate, planes) as by two projections. On the other hand, a plane is necessarily given by its traces.

In a slightly different manner we shall now consider, in this and the following section, two methods in which lines and planes are represented by pairs of points and pairs of lines, respectively. The first of these is the so-called *Method of Free Perspective.*

Now, just as parallel projection (oblique as well as orthographic) was modified, or, rather, augmented, in a manner exemplified by considering in a photograph the image of the shadow of a bird as well as that of the bird itself, in order to make of it a method possessing the property of unambiguous correspondence, so central projection may also be modified in order to give to it this property. In the method of free perspective

(1) a *line of space* g is represented by two points in the picture plane π, namely, by *its trace* T (i.e., the point where it pierces the picture plane) and its vanishing point I' (i.e., the point where the line through the center of projection C and parallel to the given line pierces the picture plane) (Fig. 14), and denoted by the symbol $g \equiv (TI')$;

(2) a *plane of space* τ is represented by its trace t and its vanishing line i' (these two lines being parallel) (Fig. 15), and is denoted by the symbol $\tau \equiv [ti']$;

(3) a *point of space* P is represented by its projection P' and a line, or plane, on which it lies (Figs. 14, 15), and is denoted by the symbol $P \equiv (TI', P')$, or $P \equiv (ti', P')$.

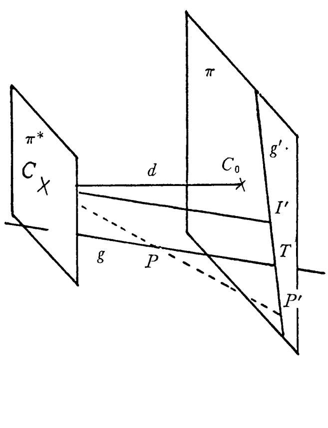


Fig. 14

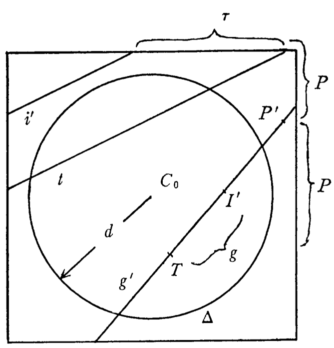


Fig. 16

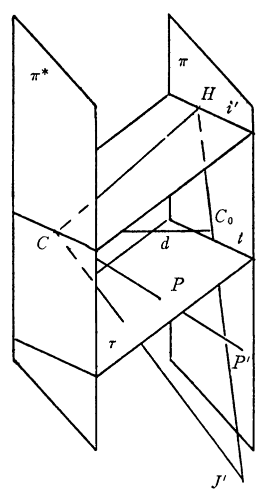


Fig. 15

The plane of the paper with a circle Δ of center C_0 and radius $d = \overline{CC}_0$ is the drawing plane π (Fig. 16).

With these plane representatives of the elements of space, the solution of a space problem is reduced to a construction in the plane (just as for the Mongean

and axonometric methods). As examples, we give here solutions of the following problems:

(1) *Find the line of intersection* (TI') *of two given planes* $\tau_1 \equiv [t_1 i_1']$, $\tau_2 \equiv [t_2 i_2']$, Figure 17.

(2) *Through a point* $P \equiv (TI', P')$ *pass a line* $x \equiv (UJ')$ *parallel to a line* g (of vanishing point J'). (See Fig. 18). Since J' is also vanishing point of x, it follows that $x' \equiv J'P'$. Since both x and (TI') contain P, they lie in one plane. Hence the trace TU of this plane must be parallel to its vanishing line $J'I'$. Thus the trace U of x is determined, and $x \equiv (UJ')$.

(3) *Find the vanishing point* J' *of a line perpendicular to a given plane* $\tau \equiv [uj']$. To solve this problem revolve right triangle HCJ' of Figure 15 into $H(C)J'$ of the drawing plane of Figure 18. By combining solutions of Problems 2 and 3, we obtain a solution of the problem:

(4) *Through a point* P *pass a line* x *perpendicular to a given plane* $\tau \equiv [uj']$ (Fig. 18).

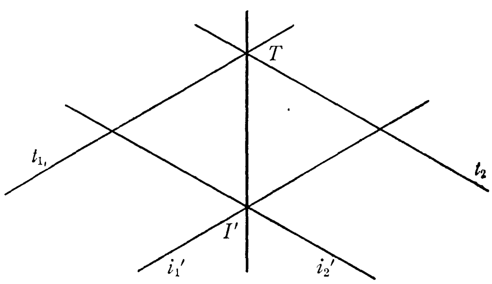


FIG. 17

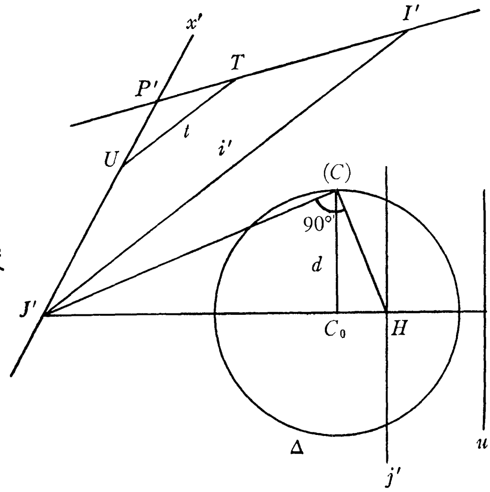


FIG 18

6. *Topographical Representation: Another Method of Double-Trace*

In the treatment of the method of free perspective, it was convenient to make use of the vanishing plane, i.e., the plane (π^*, Fig. 14) which passed though the center of projection and was parallel to the picture plane.

Let us now suppose that this plane be replaced by a more general plane ι which cuts the picture plane π in a line f. Then a general line g pierces the planes π and ι in points T and I, respectively, and the projection of I, from the center C, on the plane π, is a point I' (Fig. 19). The two points T and I' of the picture plane π are then the plane representatives of the line g. In a similar manner a plane τ of space is represented in the picture plane by two lines t and i' which

intersect on the line f (Fig. 20). If now, in particular, the center of projection C moves off to infinity in a direction given by a line l, and the plane ι, remaining finite, is taken parallel to π, then the method becomes, what Peshka has called, *Clinographic Representation.*

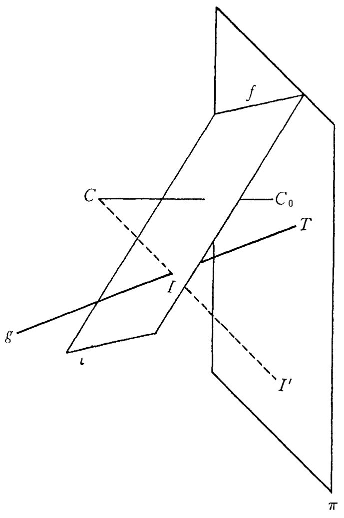


FIG. 19

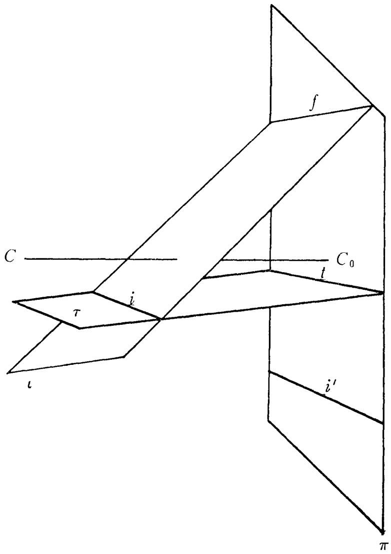


FIG. 20

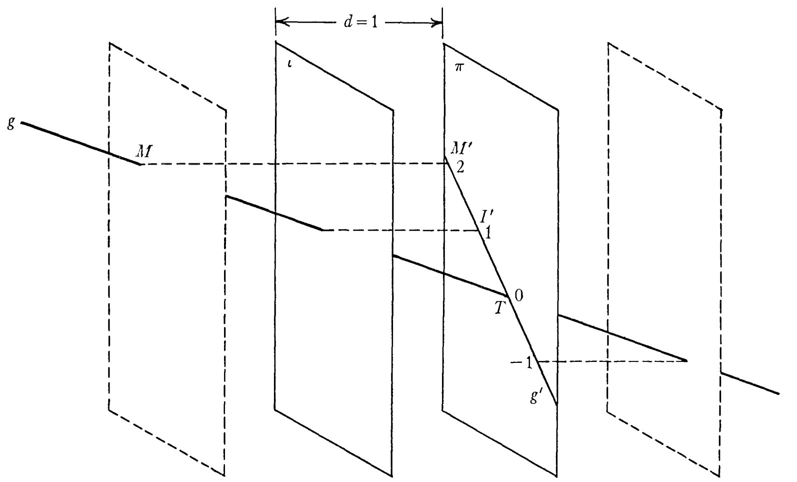


FIG 21

If, more particularly, the line l is taken perpendicular to the plane π, the representation is fully determined by the distance d between the planes π and ι, and is called *Topographic Representation.* In this case

(1) a *line* $g \equiv (TI')$ can be represented by *its orthographic projection on the*

picture plane and a scale on this projection (the points of division of which are the points into which are projected the intersections of the line g by equally spaced planes, distance d apart, which are parallel to the picture, or datum, plane, Fig. 21);

(2) a *plane* is represented by one of its lines of greeatst *slope* (drawn double so as to distinguish it from the representative of a line, Fig. 22);

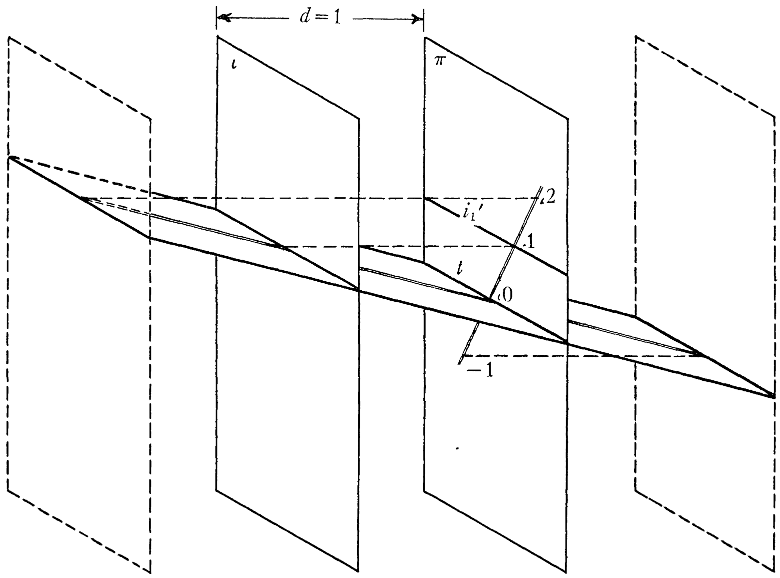


FIG. 22

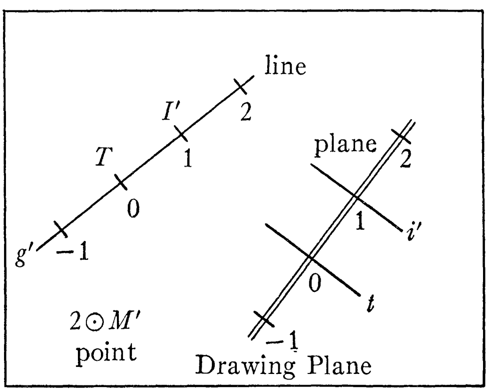


FIG. 23

(3) finally, a *point* M may be represented by its orthographic projection M' and a number which represents its distance from the datum (picture) plane, Figure 21. In Figure 23 the drawing plane is shown.

With such a representation, in the plane, of the elements of space, the prob-

lems of space may now be solved by constructions in the plane. As examples of such we cite the following problems:

(1) *Find the line of intersection x of two planes, τ_1 and τ_2* (Fig. 24).

(2) *Find the point X in which a given line l pierces a given plane τ* (Fig. 25). Pass through the given line a plane (in particular, the one of which this is a line of greatest slope). Then by the previous problem, find the line of intersection S of this plane with the given plane. This line intersects the given line in the required point $X(X', 3.7)$.

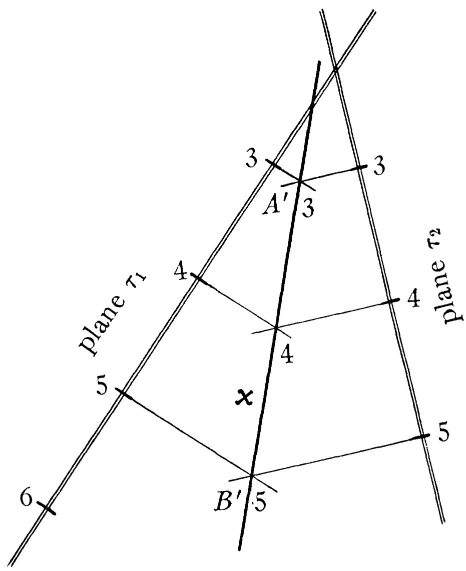


FIG. 24

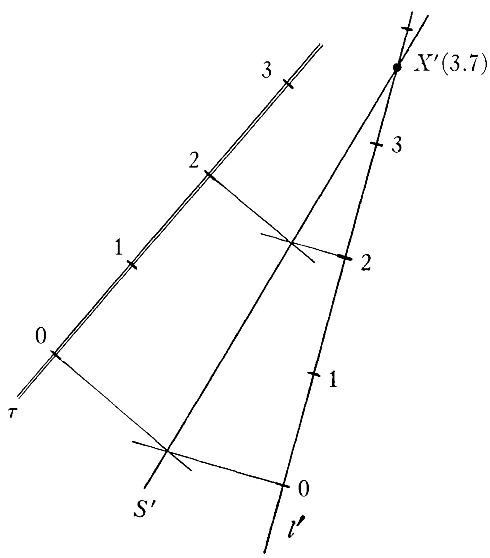


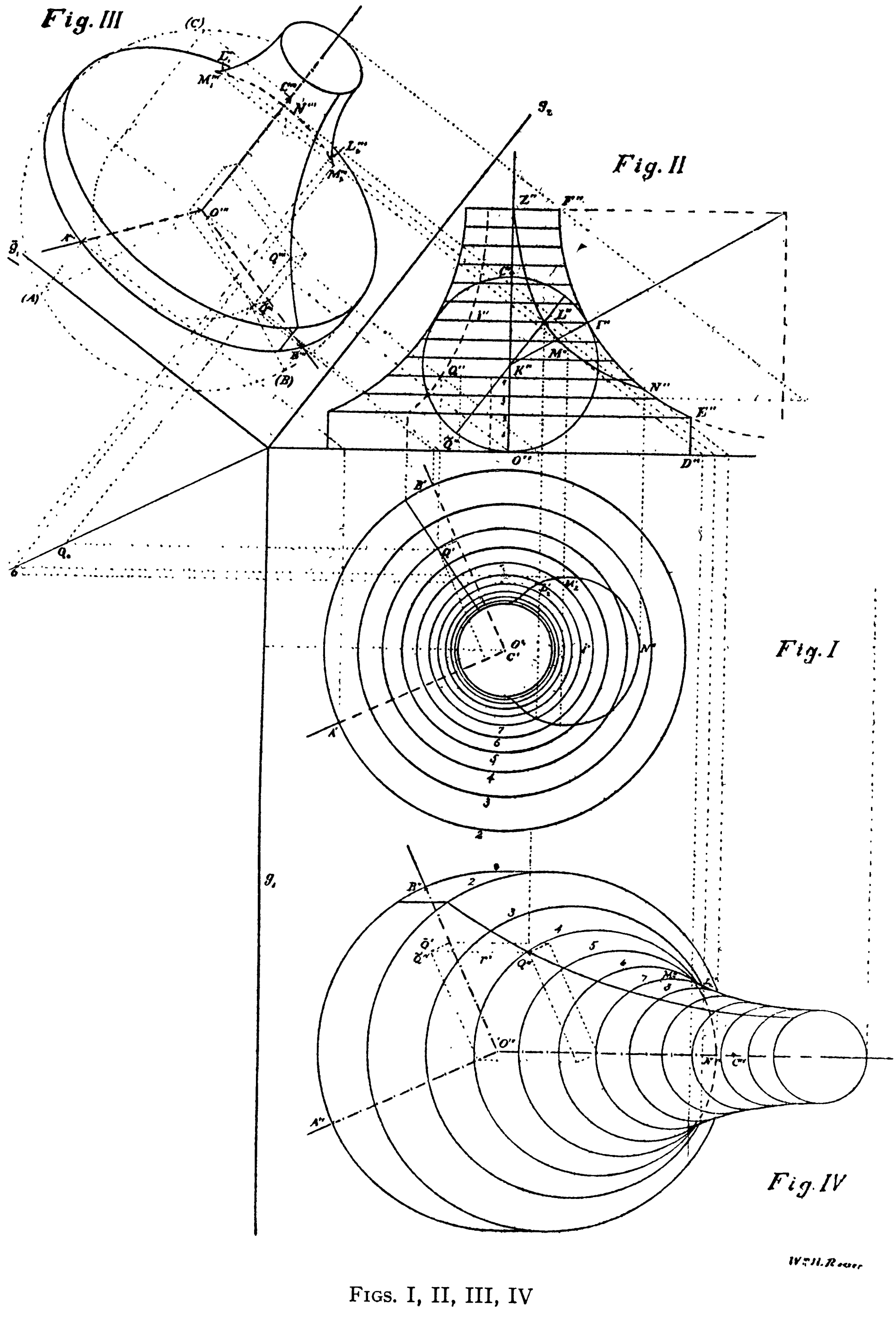
FIG. 25

This method is used by the military engineer in solving problems concerning the representation and construction of fortifications, and it is also employed in the representation of the natural surface of the ground in topographical and hydrographical maps.

7. *Representations of a Certain Surface by Various Methods*

In the accompanying plate, representations are given, by several of the methods described above, of a certain solid of revolution resembling a pedestal (the bounding surface of a portion of which is a *torus*). Here Figures I and II, taken together, form the *Mongean* representative. Figure III, with its axes and scales, constitutes an *orthographic axonometric* representative. Figure I, with its numbered circles (or contour lines) gives the *topographic* representative, and Figure IV, with its numbered *level lines*, forms a *clinographic* representative,[6]

[6] In preparing this plate several years ago, it occurred to the author that, instead of representing the natural surface of the ground by means of a contour map of the type shown in Figure I, it might be better to represent this surface by means of a map of the type of Figure IV, in which the contour, or level, lines are shifted laterally. Such a map, in addition to furnishing all the infor-



Figs. I, II, III, IV

which may also be regarded as a military perspective (although for the latter the angle of inclination of the projecting rays is usually taken to be 45°).

Important in the representation of a surface is that of its *bounding curve.* In Figure 26 is shown that of a hyperbolic paraboloid in parallel projection. In order to obtain the bounding curve in Figure III, let us observe that if we regard the rays which project our solid on the plane of projection π as rays of light, this bounding curve is the projection on π of the *line of shade*, i.e., the curve which separates the illuminated portion of our solid from the unilluminated portion.

Points of such a line of shade may be determined for a surface of revolution by the following well-known construction. A surface of revolution may be regarded as the envelope of a one-parameter family of spheres of which the centers lie on the axis of revolution. Each sphere of this family has for its line of shade a great circle whose plane is perpendicular to the direction of the illumi-

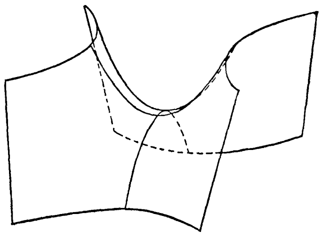


FIG. 26

nating rays. On the other hand, each such sphere has contact with the envelope along a common circle of latitude of these two surfaces. The points in which these two circles intersect are points of the line of shade of the envelope. To find the center of the sphere having contact with our surface along the circle of latitude generated by the point I'', we erect at I'' in the drawing plane a normal to the meridian curve $E''\ I''\ F''$. This cuts the axis $O''Z''$ in the required center K''. Through K'' we then draw a line parallel to g_2, since the rays of light are perpendicular to the plane π. This is the projection on π_2 of the line of shade of the sphere of center K, and cuts the corresponding projection i'' of the circle of contact in the point L''. This point is the vertical projection of two points of our line of shade. The horizontal projections of these points are found by drawing through L'' a line perpendicular to g_{12} cutting the circle i' in the points L_1' and L_2'. The corresponding projections L_1''', L_2''' on the plane π (Fig. III) are then found by the construction given in the figure.

mation which is given by a contour map, would have the further advantage of giving the impression of a relief map.

It was therefore very gratifying to find in Zurich, last September, while attending the International Mathematical Congress there, a publishing house which constructs a map of this type, of various regions of Switzerland, and calls such a map an *Aerovue.* The name of this firm is "Conzett and Huber." In visiting this publishing house, the author was shown every courtesy by Dr. Basile Giŭrkowsky.

Thus the bounding curves of Figure III corresponding to the toroidal surface of our solid may be found point by point. Apparently these curves stop abruptly at the points M_1''' and M_2'''. However, these points are not "points d'arret," but they are *cusps* of the complete curve of which the remaining portion $M_1''' N''' M_2'''$ is the bounding curve of the hidden (back) side of the torus. In fact this curve is a *parellel to the ellipse* into which the axial circle of the torus is projected on the plane π.

ON CYCLIC NUMBERS

By SOLOMON GUTTMAN, Minneapolis, Minn.

Let $N_{(k)}$ represent a number of k digits,[1] $d_1 d_2 \cdots d_k$ in a scale[2] with "radix" S; that is

$$N_{(k)} = d_1 S^{k-1} + d_2 S^{k-2} + \cdots + d_{k-1} S + d_k.$$

Also let ${}_mN_{(k)}$ denote the number of k digits, $d_{m+1} d_{m+2} \cdots d_k d_1 d_2 \cdots d_m$, obtained by transferring the first m digits of $N_{(k)}$ cyclically to the right. If $M_{(m)}$ is the number represented by the first m digits of $N_{(k)}$, $d_1 d_2 \cdots d_m$, then

$$(1) \qquad {}_mN_{(k)} = N_{(k)} S^m - M_{(m)}(S^k - 1).$$

It is to be understood that any of the digits may be zero, including the first one on the left. Thus if $N_{(4)}$ is 0032, ${}_3N_{(4)}$ is 2003; and if $N_{(5)}$ is 21053, ${}_2N_{(5)}$ is 05321. If $N_{(6)} = 857603$,

$${}_2N_{(6)} = 760385 = (857603 \times 10^2) - 85(10^6 - 1),$$

and this would hold not only in the denary scale of notation but in any notation in which S is greater than any of the digits appearing.

I. *The Fundamental Property of Cyclic Numbers*

If p is a prime number not a factor of S, then by Fermat's theorem $S^{p-1} - 1$ is divisible by p; or, in the congruence notation, $S^{p-1} \equiv 1 \pmod{p}$. Letting S^k represent the *lowest* power of S such that $S^k - 1$ is divisible by p, we may write

$$(2) \qquad S^k - 1 = pC_{(k)}$$

where $C_{(k)}$ is to be a number of k digits. If $p > S$, $C_{(k)}$ will have one or more zeros on the left. Thus $10^6 - 1$ is divisible by 13, and we have $10^6 - 1 = 13 \times 076923$.

It is easily seen that $C_{(k)}$ is the "repetend" when the fraction $1/p$ is reduced to a repeating decimal, and that the digits of $C_{(k)}$ in order are the first digits obtained in the quotient when any power of S is divided by p.

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[1] Throughout the paper, a subscript enclosed in parentheses will indicate the number of digits in a number.

[2] In the ordinary denary scale of notation S is ten. While the paper is written for the general case, the reader will lose nothing by assuming that S is ten throughout. Note that *in any scale of notation*, S would be written as 10.

Now divide S^i, where $0<i\leqq k$, by p, obtaining a quotient *of i digits*, $Q_{(i)}$, and a remainder $r_i<p$; and we have

$$r_i = S^i - pQ_{(i)}. \tag{3}$$

We shall say that r_i *belongs to* i, and that i is the *index* of r_i. Note that when $i=k$, $Q_{(i)}$ is exactly $C_{(k)}$, and $r_i=1$. Multiplying both sides of (3) by $C_{(k)}$, we have

$$r_iC_{(k)} = C_{(k)}S^i - Q_{(i)}pC_{(k)},$$

or, by (2)

$$r_iC_{(k)} = C_{(k)}S^i - Q_{(i)}(S^k - 1).$$

But since $Q_{(i)}$ consists of the first i digits of $C_{(k)}$, we have, from (1),

$$r_iC_{(k)} = {}_iC_{(k)}; \tag{4}$$

and hence the

THEOREM: *If S^k is the lowest power of S such that S^k-1 is divisible by the prime p, $S^k-1=pC_{(k)}$, and if r_i is the remainder when S^i is divided by p, $0<i\leqq k$; then the effect of multiplying $C_{(k)}$ by r_i is to transfer i digits cyclically from the left of $C_{(k)}$ to the right.*

Because of the relationship[3] expressed in this theorem, we may call $C_{(k)}$ a *cyclic number*.

For example, if S is ten and $p=7$, we have $k=6$, and $C_{(6)}=(10^6-1)/7=142857$; $r_1=3$ and hence $3\times142857=428571={}_1C_{(6)}$; $r_3=6$ and hence $6\times142857=857142={}_3C_{(6)}$; etc. Again in the usual notation, with $p=13$, $k=6$, $C_{(6)}=076923$, $r_1=10$ and $10\times076923=769230={}_1C_{(6)}$, $r_2=9$ and $9\times076923=692307={}_2C_{(6)}$, etc. In the septenary scale of notation ($S=7$), for $p=5$ we have $k=4$, $C_{(4)}=(10^4-1)/5=1254$, $r_2=4$ and $4\times1254=5412$.

II. *Complete and Incomplete Cyclic Numbers*

When S is a primitive root,[4] mod p, k will be $p-1$, and the set of remainders, r_i, will consist of the $p-1$ integers $1, 2, \cdots, (p-1)$. In this case, $C_{(k)}=C_{(p-1)}$ will be called a *complete* cyclic number, and a cyclic permutation of its digits will result when it is multiplied by any positive integer less than p. But when S is not a primitive root, mod p, k will be less than, and a factor of, $p-1$, say $p-1=nk$. The set of remainders will then consist of only k of the positive integers less than p. In this case $C_{(k)}$ may be called an *incomplete* cyclic number. Now let $r^{(2)}$ be some positive integer less than p and not in the set of remainders r_i. (It will be convenient to take $r^{(2)}$ as the *smallest* such integer, though this is

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[3] This theorem was proved by a different method by R. E. Moritz in his paper *On products whose digits are cyclical permutations of the digits of the multiplicand*, this MONTHLY, vol. 34 (1927), p. 33.

[4] See Carmichael's *Theory of Numbers*, p. 65.

immaterial.) Multiplying $C_{(k)}$ by $r^{(2)}$ the result will be a new number, $C^{(2)}_{(k)}$ which is *not* a cyclic permutation of $C_{(k)}$, i.e.,

(5) $$r^{(2)}C_{(k)} = C^{(2)}_{(k)}.$$

Now divide $r^{(2)}S^i$, where $0<i\leqq k$, by p, denoting the quotient by $Q^{(2)}_{(i)}$ and the remainder by $r^{(2)}_i$, so that

(6) $$r_i^{(2)} = r^{(2)}S^i - Q^{(2)}_{(i)}p.$$

Note that when $i=k$, $Q^{(2)}_{(i)}=Q^{(2)}_{(k)}=C^{(2)}_{(k)}$, and $r^{(2)}_{(k)}=r^{(2)}$; and that $Q^{(2)}_{(i)}$ consists of the first i digits of $C^2_{(k)}$. Multiplying (6) by $C_{(k)}$, and using (1), (2), and (5), we have

$$r_i^{(2)}C_{(k)} = {}_iC^{(2)}_{(k)},$$

or

(7) $$\frac{r_i^{(2)}}{r_k^{(2)}}C^{(2)}_{(k)} = {}_iC^{(2)}_{(k)}.$$

We may call $C^{(2)}_{(k)}$ a *co-cyclic* number. It has the property that if it be multiplied by the fraction $r^{(2)}_i/r^{(2)}_k$ the result will be to transfer its first i digits cyclically to the right. It will now be convenient in the case of incomplete cyclic numbers to use the superscript (1) to indicate the first set of remainders and the cyclic number, i.e., $r_i=r^{(1)}_i$ and $C_{(k)}=C^{(1)}_{(k)}$.

If the $2k$ integers $r^{(1)}_i$ and $r^{(2)}_i$ do not exhaust all the integers less than p, we may take $r^{(3)}_k$ as the smallest integer not in the two sets and, proceeding as before, obtain a third set $r^{(3)}_i$ and a new co-cyclic number $C^{(3)}_{(k)}$. Ultimately one obtains n sets of k integers each, $r^{(j)}_i$, $i=1, 2, \cdots, k$ and $j=1, 2, \cdots, n$; and n co-cyclic numbers $C^{(j)}_{(k)}$ of k digits each; and for any i and j we have

(8) $$r_i^{(j)}C^{(1)}_{(k)} = {}_iC^{(j)}_{(k)},$$

or

(9) $$\frac{r_i^{(j)}}{r_k^{(j)}}C^{(j)}_{(k)} = {}_iC^{(j)}_{(k)}.$$

As an example of a complete cyclic number, $(10^{16}-1)/17=0588235294117647$. When this number is multiplied by any integer less than 17 its digits are permuted cyclically.

Since $(10^6-1)/13=076923$, we have in this case an incomplete cyclic number. The numbers 1, 3, 4, 9, 10, and 12 are the remainders of the first set. Of the remaining integers less than 13, the smallest is 2. Hence $C^{(2)}_{(6)}=2\times 076923$ $=153846$, and the integers 2, 5, 6, 7, 8, and 11 make up the second set. Arranged in the order of indices, we have the table

$i=$	1	2	3	4	5	6		
$r^{(1)}_{(i)}=$	10	9	12	3	4	1	$C^{(1)}_{(k)}=$	076923
$r^{(2)}_{(i)}=$	7	5	11	6	8	2	$C^{(2)}_{(k)}=$	153846

The index of 9 is 2, hence $9\times 076923=692307$, 2 digits being transferred. The index of 11 is 3, hence 3 digits are transferred when 153846 is multiplied by 11/2.

As another example, $(10^5-1)/41=02439$. Here $k=5$ and $n=8$, and we obtain the following table:

$i=$	1	2	3	4	5	
$j=1$	10	18	16	37	1	02439
2	20	36	32	33	2	04878
3	30	13	7	29	3	07317
4	40	31	23	25	4	09756
5	9	8	39	21	5	12195
6	19	26	14	17	6	14634
7	28	34	12	38	11	26829
8	27	24	35	22	15	36585

By means of such a table and formula (8) we can multiply the cyclic number $C^{(1)}_{(k)}$ by any integer less than p. For instance, to multiply 02439 by 37 we note that the index of 37 is 4 and transfer 4 digits, $37\times 02439=90243$. To multiply 02439 by 26, we see that 26 is in the 6th row with the index 2, and we transfer 2 digits of 14634, $26\times 02439=63414$. If $C^{(7)}_{(5)}=26829$ is multiplied by the fraction $r^{(7)}_4/r^{(7)}_{(5)}=38/11$, 4 digits will be transferred, $(38/11)\times 26829=92682$.

Another group of *sub-cyclic* numbers may be formed by dividing $C^{(1)}_{(k)}$ by some factor p', or dividing S^k-1 by a composite factor. These sub-cyclic numbers are essentially the same as the co-cyclic numbers described above.

III. *Multiplication by Cyclic Numbers*

Let $C_{(k)}$ be the cyclic number $(S^k-1)/p$, and let N be any positive integer. Dividing N by p we have $N=Qp+r$, $r<p$. Then $NC_{(k)}=QpC_{(k)}+rC_{(k)}$ $=Q(S^k-1)+rC_k=QS^k+rC_k-Q$. The product $rC_{(k)}$ can be found as in the preceding section, and we have the following

Rule: *To multiply any number N by a cyclic number $C_{(k)}=(S^k-1)/p$, divide N by p; to the quotient Q annex the product of $C_{(k)}$ and the remainder r; then subtract Q.*

For example, to multiply 83495 by 142857, where $142857=(10^6-1)/7$, divide 83495 by 7. The quotient is 11927, and the remainder 6 has the index 3. Hence we write $83495\times 142857=11927857142-11927=11927845215$. Again, $(10^6-1)/13=076923$. To multiply 5362832 by 076923, we divide 5362832 by 13 obtaining the quotient 412525, and the remainder 7. Using the table previously given for this case, we write $5362832\times 076923=412525538461-412525$ $=412525125936$.

In the duodecimal scale (S is twelve, and t and e represent ten and eleven respectively) we have $(10^6-1)/7=186t35$. To multiply $38e956$ by $186t35$, we divide $38e956$ by 7, obtaining the quotient 65142 and the remainder 4 with index 2. Hence $38e956\times 186t35=651426t3518-65142=6514263t396$.

To multiply N by any cyclic permutation of $C_{(k)}$ or by any cyclic permutation of any of the associated co-cyclic numbers, we make use of (8), i.e., to multiply N by ${}_iC_{(k)}^{(j)}$ we first multiply N by $C_{(k)}^{(1)}$ by the rule given above, and then the result by $r_i^{(j)}$. Thus for $p=13$, $k=6$, we have ${}_4C_{(6)}^{(2)}=461538=076923\times 6$. Hence $5362832\times 461538=5362832\times 076923\times 6=412525125936\times 6$ $=2475150755616$.

IV. *Division by Cyclic Numbers*

Let N be any positive integer to be divided by $C_{(k)}=(S^k-1)/p$. Write $N=AS^k+B_{(k)}$, where $B_{(k)}$ consists of the k right hand digits of N and A the remaining digits. Then

$$N = A(S^k - 1) + A + B_{(k)} = ApC_{(k)} + (A + B_{(k)}).$$

We may consider three cases.

Case I: $A+B_{(k)}<C_{(k)}$. Then obviously the quotient is Ap and the remainder is $A+B_{(k)}$. For example, we may divide 42938076583 by 142857, where $142857 =(10^6-1)/7$. Here $A=42938$, $B_{(6)}=076583$, and $A+B_{(6)}=119521<142857$. Hence the quotient is $42938\times 7=300566$, and the remainder is 119521.

Case II: $C_{(k)}<A+B_{(k)}<S^k$. Divide $A+B_{(k)}$ by $C_{(k)}$, $A+B_{(k)}=C_{(k)}Q+R$, where $Q<p$ and $R<C_{(k)}$. Then $N=ApC_{(k)}+QC_{(k)}+R=(Ap+Q)C_{(k)}+R$, the quotient is $Ap+Q$ and the remainder is R.

For example, to divide 86493756082 by 076923, where $076923=(10^6-1)/13$, we have $A=86493$, $B_{(6)}=756082$, $A+B_{(6)}=842575=10\times 076923+73345$. Hence the quotient is $(86493\times 13)+10=1124419$ and the remainder is 73345.

Case III: $A+B_{(k)}>S^k$. In this case we may write $A+B_{(k)}=A'S^k+B'_{(k)}$ $=A'pC_{(k)}+A'+B'_{(k)}$ and $N=(Ap+A'p)C_{(k)}+A'+B'_{(k)}$. This process may be repeated as often as necessary, say t times, until $A^{(t)}+B_{(k)}^{(t)}<S^k$, and the problem is then reduced to Case I or II.

Thus, to divide 4836358743967215 by 076923, we have $A=4836358743$, $B_{(k)}=967215$, $A+B_{(k)}=4837325958$, $A'=4837$, $B'_{(k)}=325958$, $A'+B'_{(k)}=330795$ $=4\times 076923+23103$. Hence the quotient is $(4836358743+4837)\times 13+4$ $=62872726544$ and the remainder is 23103.

The mode of multiplication and division described above can be extended to factors of S^k+1 as well by changing certain signs.

V. *The Even Cyclic Numbers*

A cyclic number that consists of an even number of digits possesses certain special properties of which we shall consider here only one or two that we believe are not generally known.

In any number $N_{(2k)}$ of the $2k$ digits $d_1d_2\cdots d_{2k}$, let $E_{(k)}$ denote the number

that consists of the first k digits $d_1d_2 \cdots d_k$, and $E'_{(k)}$ the number that consists of the last k digits $d_{k+1}d_{k+2} \cdots d_{2k}$, so that

$$N_{(2k)} = E_{(k)}S^k + E'_{(k)}.$$

If $\overline{F}_{(k)} = (S^k - 1) - F_{(k)}$, we may call $\overline{F}_{(k)}$ the *complement* of $F_{(k)}$. Note that *each digit* of $\overline{F}_{(k)}$ is the complement of the corresponding digit of $F_{(k)}$, i.e., $\bar{d}_i = (S-1) - d_i$.

Now let S^{2k} be the lowest power of S such that $S^{2k}-1$ is divisible by p, and let $C^{(1)}_{(2k)} = (S^{2k}-1)/p$. Let ${}_iC^{(j)}_{(2k)}$ be any cyclic permutation of $C^{(1)}_{(2k)}$ or of any of the associated co-cyclic numbers, and let

$${}_iC^{(j)}_{(2k)} = E_{(k)}S^k + E'_{(k)}.$$

By (8), ${}_iC^{(j)}_{(2k)} = rC^{(1)}_{(2k)}$, where r is some positive integer less than p. Since $(S^{2k}-1) = (S^k+1)(S^k-1)$, and $S^k - 1$ is not divisible by p by hypothesis, S^k+1 must be divisible by p. Let $S^k+1 = pQ_{(k)}$, so that $C^{(1)}_{(2k)} = Q_{(k)}(S^k-1)$. Then

$$(10) \qquad {}_iC^{(j)}_{(2k)} = rC^{(1)}_{(2k)} = rQ_{(k)}(S^k - 1) = E_{(k)}S^k + E'_{(k)}$$

or

$$S^k(rQ_{(k)} - E_{(k)}) = rQ_{(k)} + E'_{(k)} > 0.$$

Hence

$$0 < rQ_{(k)} - E_{(k)} = (rQ_{(k)} + E'_{(k)})/S^k.$$

Since $pQ_{(k)} = S^k+1$ and $r < p$, $rQ_{(k)} \leqq S^k$; and also $E'_{k)} < S^k$. Therefore $(rQ_{(k)} + E'_{(k)})/S^k < 2$, and

$$0 < rQ_{(k)} - E_{(k)} < 2.$$

It follows that $rQ_{(k)} - E_{(k)} = 1$, or

$$(11) \qquad rQ_{(k)} = E_{(k)} + 1.$$

Then, by (10),

$$(E_{(k)} + 1)(S^k - 1) = E_{(k)}S^k + E'_{(k)},$$

or

$$E'_{(k)} = (S^k - 1) - E_{(k)} = \overline{E}_{(k)};$$

and hence the

THEOREM: *In any even cyclic number $C^{(1)}_{(2k)}$, or any cyclic permutation of such a number or of any of its associated co-cyclic numbers, the last k digits are the complements of the first k digits.*

As examples, note the even cyclic and co-cyclic numbers already cited such as 076923, 153846, 0588235294117647, etc.

From (11) it is seen that $E_{(k)}+1$ is divisible by $Q_{(k)}$, or $E_{(k)}\equiv -1 \pmod{Q_k}$. Hence the following

THEOREM: *In an even cyclic or co-cyclic number, $C_{(2k)}^{(j)}$, or in any cyclic permutation thereof, any number consisting of k adjacent digits is congruent to* -1, *modulo* $Q_{(k)}$, *or modulo any divisor of* $Q_{(k)}$, *where* $Q_{(k)}=(S^k+1)/p$.

For instance, for $p=7$, $C_{(6)}=142857$ and $Q_{(3)}=(10^3+1)/7=143=13\times 11$. Hence

$$142 \equiv 428 \equiv 285 \equiv 857 \equiv 571 \equiv 714 \equiv -1 \pmod{143}.$$

For $p=13$, $C_{(6)}^{(1)}=076923$, $C_{(6)}^{(2)}=153846$, and $Q_{(3)}=(10^3+1)/13=77$. Hence

$$076 \equiv 769 \equiv 692 \equiv 923 \equiv 230 \equiv 307 \equiv 153 \equiv 538 \equiv \cdots \equiv 615 \equiv -1 \pmod{77}.$$

VI. *Cyclic Congruences*

From (1) it follows that if x is any factor of S^k-1, and $N_{(k)}$ is also divisible by x, then any cyclic permutation of $N_{(k)}$ is divisible by x.

Thus $10^5-1\equiv 0 \pmod{369}$ and $33579\equiv 0 \pmod{369}$, and therefore $35793 \equiv 79335\equiv 57933\equiv 0 \pmod{369}$. Also $10^5-1\equiv 0 \pmod{41}$, and $00041\equiv 0 \pmod{41}$, hence $10004\equiv 0 \pmod{41}$. Since 7 is a factor of 10^6-1, we have $21\equiv 100002\equiv 0 \pmod{7}$.

Since S^k-1 is a divisor of $S^{nk}-1$, any factor of S^k-1 is a factor of $S^{nk}-1$. Hence we have, more generally, that if x is a factor of S^k-1 and of $N_{(nk)}$, it is a factor of any cyclic permutation of $N_{(nk)}$. For instance, 13 is a factor of 10^6-1 and of 104. Hence $104\equiv 40001\equiv 40000000001\equiv (4\times 10^{6n+4})+1\equiv 0 \pmod{13}$.

VII. *Cyclic Remainders*

Returning to (3) of Section I, we have, for $i=m$, n, and $m+n$ respectively

$$S^m = pQ_{(m)} + r_m, \tag{12a}$$

$$S^n = pQ_{(n)} + r_n, \tag{12b}$$

$$S^{m+n} = pQ_{(m+n)} + r_{m+n}. \tag{12c}$$

Multiplying (12*a*) and (12*b*) we have

$$S^{m+n} = p[pQ_{(m)}Q_{(n)} + r_mQ_{(n)} + r_nQ_{(m)}] + r_mr_n$$

and comparing this with (12*c*) we see that $r_mr_n\equiv r_{m+n} \pmod{p}$, or

$$r_mr_n = pt + r_{m+n}, \tag{13}$$

where

$$t = Q_{(m+n)} - pQ_{(m)}Q_{(n)} - r_mQ_{(n)} - r_nQ_{(m)}. \tag{14}$$

From (13), since $r_{m+n}<p$, we see that t is the quotient when r_mr_n is divided by p. From (14) we have

$$Q_{(m+n)} = (pQ_{(m)} + r_m)Q_{(n)} + r_nQ_{(m)} + t$$
$$= (pQ_{(n)} + r_n)Q_{(m)} + r_mQ_{(m)} + t,$$

or

$$Q_{(m+n)} = Q_{(n)}S^m + r_nQ_{(m)} + t = Q_{(m)}S^n + r_mQ_{(n)} + t. \tag{15}$$

Expressed in words it means the following: If in the process of finding a cyclic number C_k by dividing 10^k by p, we have one partial quotient of m digits $Q_{(m)}$ with the remainder r_m, and a second partial quotient of n digits $Q_{(n)}$ with the remainder r_n, we can multiply the second quotient $Q_{(n)}$ by the first remainder r_m and annex the product to the first quotient $Q_{(m)}$, or vice versa; and then adding to this the quotient t when the product of the remainders r_mr_n is divided by p, we get a new partial quotient of $m+n$ digits, $Q_{(m+n)}$.

If in (15) we put $m=n$, we have

$$Q_{(2m)} = Q_{(m)}S^m + r_mQ_{(m)} + t, \tag{16}$$

i.e., if we multiply the partial quotient $Q_{(m)}$ by the remainder r_m, and annex this product to this same quotient $Q_{(m)}$, and then add the quotient t obtained by dividing r_m^2 by p, the result will be the partial quotient $Q_{(2m)}$ of twice as many digits. This, together with the results of Section V about even cyclic numbers, will greatly facilitate the writing down of cyclic numbers that consist of a large number of digits.

A few examples will suffice to illustrate the foregoing.

In computing $C_{(6)}=(10^6-1)/7$ by dividing 10^6 by 7, we have $Q_{(2)}=14$, $r_2=2$. We then annex $2\times 14=28$ to 14, obtaining 1428, and this is $Q_{(4)}$ since $t=0$ in this case. Also $r_4=r_2^2=4$. Now multiplying $Q_{(2)}$ by r_4, and annexing the product to $Q_{(4)}$, we have 142856; and since $r_2r_4=8=(1\times 7)+1$, we have $t=1$, and adding this we have $Q_{(6)}=C_{(6)}=142857$.

As an example in which there will be real saving in computation, let us find $C_{(96)}=(10^{96}-1)/97$. Here $Q_{(2)}=01$, $r_2=3$, and hence $Q_{(4)}=0103$ and $r_4=r_2^2=9$; $Q_{(8)}=01030927$, $r_8=r_4^2=81$. Now $r_8=81$ is too large a number to operate with easily, and so we choose to multiply $Q_{(8)}$ by $r_4=9$. Since $r_4r_8=729=(97\times 7)+50$, we have $r_{12}=50$, $t=7$, and hence $Q_{(12)}=010309278350$. It is easily seen that if $r_{12}=50$, $r_{11}=5$ since $r_1=10$. Hence we use $Q_{(11)}$ and $r_{11}=5$, and obtain $r_{22}=25$, and, since $t=0$ in this case, $Q_{(22)}=0103092783505154639175$. Since $r_{22}^2=625$ $=(97\times 6)+43$, we have $r_{44}=43$, and $Q_{(44)}$ will be obtained by annexing $25\times Q_{(22)}$ $+6$ to $Q_{(22)}$ giving $Q_{(44)}=01030927835051546391752577319587628865979381$. Since $r_4r_{44}=387=(97\times 3)+96$, we annex $r_{44}Q_{(4)}+3$ to $Q_{(44)}$ and have $Q_{(48)}$ $=010309278350515463917525773195876288659793814432$. And then, using the property of even cyclic numbers we finally have $Q_{(96)}=C_{(96)}=01030927835$-$05154639175257731958762886597938144329896907216494845360824742268041$-$2371134020618556 7$.

THE LAG IN MATHEMATICS BEHIND LITERATURE AND ART IN THE EARLY CENTURIES[1]

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Among the various characteristics which distinguish man from the lower animals, there are two which stand out in bold relief; namely, the language concept and the number concept. These will be made the basis of study and contrast in the present discussion. It will be found that these two concepts did not develop in the human mind with equal ease or with equal speed, and it will be of more than passing interest to examine some of the reasons for such great disparity and possibly to suggest and support an explanation which has not been dwelt upon by the historians. In a word we shall try to find out some of the reasons why mathematics lagged so far behind literature and art in the early centuries.

The case for language is comparatively simple to state, though its interpretation and full understanding may often-times be obscure and difficult. The fact is that every nation or tribe of men, ancient or modern, pagan or civilized, has a language which may be crude or well developed but, at any rate, is adequate for their needs of intercommunication. Witness the North American Indian tribes among whom there were some sixty different language stocks; or the inhabitants of Asiatic India who are said to have developed more than one hundred dialects; or the great group of Indo-European languages of which there are eight main divisions and fifty or more subdivisions spread over a large part of Europe and of Asia. In complete bewilderment one exclaims: truly man is a language producing animal.

As a leading example of language development of the highest type, let us look more closely into the case of the Greek language which is one of the eight main divisions of the Indo-European family. The first essential in the development of a written language is an adequate alphabet. Perhaps the earliest attempt at alphabetic structure is to be found in the hieratic writings of the Egyptian priests three to five thousand years ago, as for instance in the Ahmes Papyrus of about 1700 B.C. However, such writings were in the nature of a mystery to the common people and little used except by the priests. But it may be that these early attempts at alphabet making inspired the Phoenicians to strive for mastery along this line. At any rate, by the end of the ninth century B.C. they were in possession of a fairly complete alphabet which was in common use, and from this source the Greeks undoubtedly imported the first draft of their own alphabet. Then began a long period, perhaps two or three hundred years, during which the Greeks gradually modified and perfected their alphabet. For instance, the Phoenicians had too many consonants and not enough vowels, they had too many aspirates but no symbols for a number of sounds used by the Greeks. As a result of this long and careful development, the Greeks

[1] A paper read before the Mathematical Association of America at the joint meeting with Section A of the A.A.A.S. held in Chicago, June 20, 1933.

produced an alphabet which came to be *an accepted model* for most of the subsequent languages of Europe, and which deservedly derived its name from a combination of the two Greek letters, alpha and beta.

With the first necessary condition thus fulfilled, namely the possession of a well-organized alphabet, the Greeks began to build a written language, the ultimate perfection of which has never been excelled or equaled. For example, the Greek verb system has no rival unless possibly in the old Sanskrit literature which was already in decay when the Greeks began their development. In no language of modern times can the fine and delicate distinctions of meaning be so accurately displayed as was possible in the Greek of the Classic period. As examples of highly developed Greek language we might select at random almost any of the works of scores of their poets, epic, tragic, comic or lyric; or of their orators, philosophers, and historians; but we shall content ourselves with one famous example, namely, the orator Demosthenes. "His ancient fame," says one writer, "can be compared only with the fame of the poet Homer." Dionysius, the most penetrating of his many critics, both ancient and modern, exhausts the language of admiration in showing how "Demosthenes unified and elevated whatever had been best in the earlier masters of the Greek idiom and how he perfected Greek prose by fusing into a glorious harmony all the elements which had hitherto marked the culmination of separate types of excellence."

Our purpose here is merely to call attention to certain historical facts in Greek language development which we shall use for comparison later on in this discussion. We have thus seen that the Greeks perfected an alphabet and thereby were able to develop a written language which also exhibited a very high degree of perfection. Moreover, through the medium of this language they were able to produce a literature which has been at once the inspiration and the despair of later writers and has commanded the respect and admiration of historians and critics of all succeeding generations. But there are other channels also through which the genius of the ancient Greeks sought and attained a high degree of perfection. One of these was in the realm of art as expressed in architecture, sculpture, and all forms of plastic and pictorial artistry. Great as has been the influence on mankind of Greek literature, still more pervasive and profound has been the influence of Greek art. It is as though all the modern world had sat at the feet of the Greek masters.

We have thus sketched one of the most striking language developments in all history which took place among the Greeks in the centuries immediately preceding and following the Christian era; and we wish now to make a corresponding study of the development of number symbolism among these same people and also elsewhere in the world during these same centuries. Some time in the fifth or sixth century B.C. the Greeks had selected certain capital letters from their imported and perfected alphabet to represent numbers, I, Π, Δ, H, X, M, standing respectively for 1, 5, 10, 100, 1000, 10,000. By means of such symbols and various other marks and devices they were able to write any desired number. For instance, the number 5348 would be written,

$$\underset{\mathrm{X}}{\Pi}\,\mathrm{H}\,\mathrm{H}\,\mathrm{H}\,\Delta\,\Delta\,\Delta\,\Pi\,\mathrm{I}\,\mathrm{I}\,\mathrm{I}$$

where the symbol for 5 placed over X changes it from 1000 to 5000. There is no historical evidence that these number symbols were ever seriously used for computation purposes, as may well be imagined if one attempts to perform any operation with them except, perhaps, simple addition. We are not surprised, therefore, to find that the fertile-minded Greeks were not content to continue indefinitely such a fruitless number symbolism. It may be that Thales (640–550 B.C.) and Pythagoras (569–500 B.C.) had agitated for a change in their times since they both must have keenly felt the serious limitations of their system whenever they deviated from purely geometric processes. Pythagoras especially attempted to study some of the properties of numbers and must have been badly handicapped by the existing notation, though it must be said that most of his investigations in number theory were made by the aid of geometry and that his calculations were doubtless carried out by means of the abacus. At any rate, soon after the death of Pythagoras in 500 B.C., the Greeks inaugurated what they must have considered a reform in their number symbolism, and again they borrowed the plan from Asia Minor as they had done earlier in the case of their alphabet. The Syrians in this instance set the example by using all the letters of their alphabet to stand for numbers, and the Greeks imitated them as shown in the table below, where the numbers for which the letters stand are placed above them:

1	2	3	4	5	6	7	8	9
α	β	γ	δ	ε	ϛ	ζ	η	θ
10	20	30	40	50	60	70	80	90
ι	κ	λ	μ	ν	ξ	ο	π	ϙ
100	200	300	400	500	600	700	800	900
ρ	σ	τ	υ	φ	χ	ψ	ω	ϡ

1,000	2,000	3,000	4,000	
,α	,β	,γ	,δ	etc.
10,000	20,000	30,000	40,000	
M	$\overset{\beta}{\mathrm{M}}$	$\overset{\gamma}{\mathrm{M}}$	$\overset{\delta}{\mathrm{M}}$	etc.

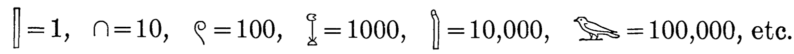
This scheme is theoretically quite perfect and apparently much simpler than the old plan. For example, the number 5348 now becomes

,ετμη.

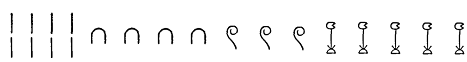
However, there is no evidence that this symbolism ever came into common use and, as remarked before, the best reason for its failure is found when one attempts to do a little computation by means of it. The abacus continued to be the practical tool for computation and the mathematically minded Greeks

turned their attention for the most part to pure geometry or to numerical processes treated by geometric methods. The one brilliant exception to this statement was the case of Diophantus, some hundreds of years later, who, about 250 A.D., wrote an extensive book on arithmetic treated by purely algebraic methods, in which he used the alphabetic numerals. He was an expert in the use of this notation, but evidently the intricacies arising from the use of letters in such a double capacity, and other confusing symbolisms, proved to be too difficult for the ordinary people. Diophantus apparently had no prominent predecessors in the cult and no followers to carry on where he left off.

We have witnessed the unparalleled success of the Grecians in building an alphabet, a language, and a literature in a land embellished by them in all forms of artistic creation, while they suffered comparative defeat in producing a number symbolism which could be used widely and successfully in originating and developing the science of algebra. However, they might have been able to derive a good deal of consolation from the fact that their neighbors and eventual conquerors, the Romans, who also attained very high standards in language, literature, and art, failed utterly to contribute anything toward number symbolism beyond combinations of the characters, I, V, X, L, C, D, M, whose use for computation was impracticable. Again, the Greeks may have found some satisfaction in contemplating the clumsy system developed by the Egyptians many centuries earlier. Their numbers were written by means of the following hieroglyphic symbols:



In this notation, reading from right to left, our number 5348 would be written thus:



The Egyptian priests actually used this symbolism to make many computations,[1] but it is easy to see that such a number system could never have formed the basis of modern arithmetic and algebra.

Viewing the matter now, in the light of historical experience, we see that several prerequisites were necessary in order to produce an adequate and successful number symbolism.

(1) The number of symbols in the system should be less by one than the number of units in the base employed. For example, exactly nine symbols if the base is ten, nineteen if the base is twenty.

(2) The base chosen should not be too small, such as five and not too large, such as twenty.

(3) Each symbol should consist of a single character and should be distinct from all the others.

(4) Alphabetic symbols are undesirable on account of the confusion arising

[1] See the "*Rhind Mathematical Papyrus,*" Vol. II, by Arnold Buffum Chace, The Open Court Publishing Co.

from double use of the letters. For example, see the algebraic work of Diophantus.

(5) A zero symbol is needed to indicate place value so that the original symbols may be repeated again and again for units of higher order.

A careful study of these specifications shows that not one of the systems which we have thus far considered meets all the requirements, and in particular not one of them has the zero symbol. Before examining the system developed in the Orient, it is interesting to note that the Mayans of Central America had a system which included a zero and in this respect may have antedated all other similar attempts in the world. They used the base twenty but made up the nineteen symbols by combinations of dots and bars, thus abrogating requirements (2) and (3) above. They understood the crucial requirement (5) but failed to safeguard (3), which was needed for convenience of computation.

If we turn now to the Orient, we find that some time early in the Christian era, somewhere in Asia, probably in India, possibly in Babylonia, the so-called Hindu-Arabic numerals, including zero, had been invented and were slowly coming into use during the sixth to the tenth centuries A.D. There is much obscurity as to their origin and development, but certainly by the beginning of the twelfth century Bhâskara, an Indian mathematician, used these numerals, including zero, and gave a systematic exposition of the decimal notation as a system already well recognized and long used. Whatever the origin of these number symbols may be, the Hindus doubtless deserve the credit for developing and perfecting the decimal system based on nine symbols with a zero to denote place value; and the Arabs deserve credit for putting this system into circulation, not only in Asia, but also eventually in Europe. This Hindu-Arabic system possesses all the requirements, mentioned above, for successful use both in computation and in scientific investigations growing out of the number concept. The use of these numerals spread wherever the conquering Arabs held sway, from Bagdad in the East to Cordova in the West. They held commercial intercourse with far-flung nations. They established universities where the text-books were mostly Arabic translations of Grecian science, mathematics, and philosophy. They became the conservators of learning while Europe was submerged in the coma of the dark ages. Eventually, the Hindu-Arabic number symbolism became known in Europe largely through the influence of Leonardo of Pisa who had traveled and studied extensively and had become convinced that the archaic number systems of Greece and Rome were hopeless and should be replaced by an up-to-date system. In 1202 he wrote a book called Liber Abaci in which he gave an exposition of the Hindu-Arabic system and showed its great superiority over anything Europe had ever produced. Leonardo was the first and greatest exemplar of mathematics in the Middle Ages, but in spite of his great ability and influence the adoption of our modern number symbolism in Europe was slow and halting, extending over three or four hundred years. However, the conquest was eventually complete and thus the foundation was laid for the development of mathematical analysis which began very slowly in the 15th and

16th centuries, blossomed out in the 17th and 18th centuries, and burst forth into full effulgence in the 19th century, thus marking a period of mathematical splendor fully comparable to the glory of Grecian literature and art attained two thousand years earlier.

Herein we seem to have discovered an historical enigma. Why were the Greeks able to excel in language symbolism but unable to do so in number symbolism? Why were they able to inaugurate and perfect postulational reasoning to such a high degree in geometry but unable to develop algebraic science to any notable extent? Why were the scholars of India and Arabia able to lay the foundation of mathematical analysis by inventing an adequate and workable number symbolism, while the fertile minded and highly intellectual Greeks were unable to make such a contribution to world progress?

There have been many conjectures and attempted explanations of this unsolved enigma. With respect to the great contributions made by the Greeks in geometry in comparison to the almost exclusive algebraic trend of the Hindu and Arabic writers, some have supposed that these were respectively inborn traits and hence were natural developments on the one hand of the Grecian love for form and symmetry and on the other hand of the Hindu devotion to contemplative abstraction. On this score, one immediately wonders to what additional heights Euclid and Apollonius would have risen if they had possessed an adequate number system and corresponding algebraic methods with which to work. On the other hand, what might a Bhâskara or an Al-Khowarizmi have done with their algebraic symbolism if they had possessed the geometric insight of Euclid? After the second attempt and failure of the Greeks to devise a workable number symbolism, and especially in view of the long series of conquests in geometry begun by Thales and Pythagoras and brought to a glorious triumph by Euclid and Apollonius, it would seem that the Greeks gave up all hope of ever treating numbers and algebra on the same high plane of logical rigor which they had been able to give to geometry. They surely recognized the difficulties, especially in connection with irrational quantities and incommensurable ratios, difficulties which were destined to remain unsolved for nearly two thousand years and the background for whose solution, namely, an adequate number symbolism, was to be found not in Europe but in Asia, not through Grecian channels but by way of Arabia, Spain and Italy.

No one doubts the transcendent intellectual powers of the ancient Greeks. For example, if we put the Asiatic mathematicians Āryabhata, Brahmagupta, Bhâskara, Al-Khowarizmi and Omar Khayyam into comparison with Pythagoras, Aristotle, Euclid, Apollonius, Archimedes and Diophantus, we are ready to assert that in our belief no intellectual feat within the grasp of individuals in the first group could conceivably be beyond the powers of those in the second group. And yet we are obliged to admit that one of the most important inventions of all time, and certainly the greatest boon ever conferred by the Orient upon the Occident, namely, our Hindu-Arabic decimal number system, was the product of Asiatic minds, typified by the names in the first group above. There

is evidence that outstanding Greek mathematicians like Archimedes and Apollonius were sorely exercised over the short-comings of their alphabetic number system. They both devised special schemes for writing large numbers and presumably this was in connection with efforts made by them to improve their number notation which apparently they did not succeed in doing, so far as any records show.

There remains one explanation of this historical enigma which is based neither upon the superior capacity of the Asiatics nor upon the inferior ability of the Greeks. To use a familiar illustration borrowed from the football field which has been invoked hundreds of times to account for the winning of a game, the Asiatics were favored with a "lucky break" and the Greeks were not thus fortunate. That sort of thing has happened over and over again in other fields. It was just such a stroke of luck when Edison found that a delicate carbon filament in a glass vacuum bulb would produce the glow of an electric lamp, and again when he stumbled upon the principle that led eventually to the phonograph. It was a freak of fortune when the wireless telegraph emerged from the laboratory as an unexpected by-product in the course of other electrical experiments.

It requires no great stretch of imagination to see what might have happened if Archimedes, for example, had been lucky enough to hit upon the idea of nine number symbols (non alphabetic) and a zero. There would seem to be little question but that this added tool in his hands might have been a wonder-worker. Even without this tool, he was a giant in his generation, anticipating by hundreds of years some of the ideas of Newton, and laying down the original foundations of mechanics and hydrostatics; with this tool he might have been a pioneer in algebraic science and might have paved the way for a continuous development leading up to a high algebraic level on which Diophantus might have built a superstructure capable of antedating by fifteen centuries achievements which were finally made in other parts of Europe and which had their early beginnings in Asia. Standing in time as Archimedes did between the two geometers, Euclid and Apollonius, it would have been almost certain that with a full-fledged decimal number system in hand, he might have put algebra on a basis coordinate with geometry and might have antedated Descartes by eighteen hundred years in the founding of analytic geometry.

We may agree that all of these matters are highly speculative in character and lead us into the realm of the metaphysical, but if we once grant the hypothesis that the invention of the Hindu-Arabic number symbolism was just a "lucky break" which might have occurred in Greece as well as in Asia, then all the above conclusions, and even greater achievements by the Greeks, were apparently within possible realization.

However, on the other side of the question there remains the fact that such outstanding men as the Hindu, Bhâskara, and the Arab, Al-Khowarizmi, with their decimal number system, based on nine symbols and zero, well in hand, were unable to develop the science of algebra beyond the most elementary

stages. This fact leads us to hesitate in our assurance that the Greeks would, under similar conditions, have pushed the algebraic fronteers much further out than did the Asiatics; and our assurance is still further diminished when we recall that Leonardo of Pisa, who was the first great mathematician in Europe to advocate the adoption of the Hindu-Arabic decimal notation, and whose books set forth complete methods of calculation with integers and fractions by means of this notation, nevertheless was unable to make much impression on the sluggish Europe of the Middle Ages. It was still roughly two hundred years from his time to Regiomontanus and Copernicus; three hundred to Tartaglia, Cardano and Vieta; four hundred to Kepler, Fermat, Descartes, Pascal, Newton and Leibniz; five hundred to De Moivre, Maclaurin, D'Alembert, Euler and La Grange; and six hundred to Gauss, Laplace, Legendre, Fourier, Abel, Galois, Cauchy, Riemann, Sylvester, Weierstrass and Cayley.

As we survey this partial list of names of great men taken at random from the galaxy of mathematicians of the fifteenth to the nineteenth centuries, we are led to realize that the time was not ripe for either the Greeks or the Asiatics in their day to make great advances in the science of mathematics. In the nature of the case the development of such a science had to be cumulative over a very long period. This was especially true of algebraic analysis, both because of its dependence on the invention of a workable number symbolism, as we have set forth in this discussion, and because of its abstract nature and inherent difficulty as compared, for example, with geometry. But the time element was also involved in the development of Greek geometry—Euclid collated as well as invented, though his geometry was the cumulation of a few hundred years while the perfection of algebraic science took nearly two thousand years.

In conclusion, then, it seems clear that the number concept and its development in algebraic science have caused the human race far greater trouble than the language concept and its development in literature; and that the lag of mathematics behind literature was an inevitable consequence of the relative difficulties involved in these two concepts.

ON SYMMETRIC DETERMINANTS

By W. V. PARKER, Mississippi Woman's College

1. *Introduction.* In a paper presented to the American Mathematical Society, L. M. Blumenthal[1] has shown by an example that a theorem, announced by H. W. Richmond, fails in certain cases. The theorem of Richmond is as follows:

If in a non-vanishing, symmetric determinant of order six, the six elements in the principal diagonal are all zero, and the complementary minors of five of these elements are also zero, then the complementary minor of the remaining element must be zero.

[1] *An Application of Metric Geometry to Determinants*, Bulletin of the American Mathematical Society, vol. 37 (1931), pp. 752–758. See this paper for other references to the theorem.

Blumenthal also shows that the analogous theorem for fourth and eighth order determinants as extended by Segre is not true. The main purpose of his paper is to establish a "counter theorem" for determinants of order five. The theorem of Blumenthal may be stated as follows:

If in a non-vanishing, symmetric determinant of order five, the five elements in the principal diagonal are all zero, the remaining elements of the first row (column) all ones and the remaining elements all positive, then if four of the fourth order principal minors are zero, the remaining fourth order principal minor is not zero.

The writer[2] extended this theorem by purely algebraic methods to the case when the elements outside the principal diagonal are real. The methods used there are not suitable for investigating determinants of higher order.

The purpose of the present note is to establish this theorem by certain considerations from geometry, and also to show when the theorem analogous to that of Richmond fails in the case of determinants of order four.

2. *A Transformation on the Determinant.* Let $D_n = |a_{ij}|$ be a *symmetric determinant* of order n whose principal diagonal elements are all zero, and denote by M_i the principal minor obtained from D_n by striking out the ith row and column. If D_n is non-vanishing and $n-1$ of the minors M_i are zero, we may arrange D_n so that $a_{12} \neq 0$ and so that all principal minors of order $n-1$, except M_1, are zero. We may assume[3] without loss of generality that $a_{12}=1$. If now we subtract a_{1j} times the second column from the jth $(j=3, 4, \cdots, n)$, and then subtract a_{i1} times the second row from the ith $(i=3, 4, \cdots, n)$, we get a determinant of order n in which the elements of the first row and column are all zero, except the second which is 1. If now we expand D_n according to the first row and column, we get $D_n = -\Delta_{n-2}$ where Δ_{n-2} is a symmetric determinant of order $n-2$. If K_i denotes the principal minor obtained from Δ_{n-2} by striking out the ith row and column, the minors of Δ_{n-2} and D_n are connected by the relation $K_i = -M_{i+2}$ $(i=1, 2, \cdots, n-2)$. The two remaining minors of D_n may be written as the bordered determinants

$$M_1 = \begin{vmatrix} 0 & a_{23} & a_{24} \cdots a_{2n} \\ a_{32} & & \\ a_{42} & \Delta_{n-2} & \\ \cdot & & \\ \cdot & & \\ \cdot & & \\ a_{n2} & & \end{vmatrix}, \qquad M_2 = \begin{vmatrix} 0 & a_{13} & a_{14} \cdots a_{1n} \\ a_{31} & & \\ a_{41} & \Delta_{n-2} & \\ \cdot & & \\ \cdot & & \\ \cdot & & \\ a_{n1} & & \end{vmatrix}.$$

[2] *On Real Symmetric Determinants Whose Principal Diagonal Elements are Zero,* Bulletin of the American Mathematical Society, vol. 38 (1932), pp. 259–262.

[3] See *A Theorem on Symmetric Determinants,* Bulletin of the American Mathematical Society, vol. 38 (1932), foot-note on p. 547.

3. *The Case* $n=4$. Let

$$D_4 = \begin{vmatrix} 0 & 1 & a & b \\ 1 & 0 & m & n \\ a & m & 0 & p \\ b & n & p & 0 \end{vmatrix}, \quad \text{then } \Delta_2 = \begin{vmatrix} -2am & p-an-bm \\ p-an-bm & -2bn \end{vmatrix},$$

and

$$M_1 = \begin{vmatrix} 0 & m & n \\ m & \multicolumn{2}{c}{\Delta_2} \\ n & & \end{vmatrix}, \qquad M_2 = \begin{vmatrix} 0 & a & b \\ a & \multicolumn{2}{c}{\Delta_2} \\ b & & \end{vmatrix}.$$

Denote by F_2 the quadratic form in x_1, x_2 whose discriminant is Δ_2 and by π the pair of points obtained by equating F_2 to zero. Then $M_1=0$ if and only if the point, P_1, $mx_1+nx_2=0$ belongs to π, and $M_2=0$ if and only if the point, P_2, $ax_1+bx_2=0$ belongs to π. If now $M_2=M_3=M_4=0$, $K_1=K_2=0$ and π is the pair of points $(0, 1)$, $(1, 0)$ and since P_2 belongs to π either a or b must be zero. If both a and b are zero P_2 is arbitrary and hence belongs to π. P_1 belongs to π if and only if one of the elements m or n is zero. Hence we have the following Theorem. *If in a non-vanishing symmetric determinant of order four, the four elements in the principal diagonal are all zero and the complementary minors of three of these elements are also zero, then the complementary minor of the remaining element is not zero if in any row the elements outside the principal diagonal are all different from zero.*

We have, therefore, the

Corollary. *The symmetric determinant,*

$$D_4 = \begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & m & n \\ 1 & m & 0 & p \\ 1 & n & p & 0 \end{vmatrix}$$

vanishes if all its third order principal minors are zero.

4. *The Case* $n=5$. Let

$$D_5 = \begin{vmatrix} 0 & 1 & \alpha & \beta & \gamma \\ 1 & 0 & a & b & c \\ \alpha & a & 0 & m & n \\ \beta & b & m & 0 & p \\ \gamma & c & n & p & 0 \end{vmatrix},$$

then

$$\Delta_3 = \begin{vmatrix} -2a\alpha & m-\beta a-\alpha b & n-\gamma a-\alpha c \\ m-\beta a-\alpha b & -2b\beta & p-\gamma b-\beta c \\ n-\gamma a-\alpha c & p-\gamma b-\beta c & -2c\gamma \end{vmatrix}.$$

Suppose $D_5 \neq 0$ and $M_2 = M_3 = M_4 = M_5 = 0$, then $K_1 = K_2 = K_3 = 0$ and hence α, β, γ, a, b, c, must all be different from zero since any one of these being zero would make every element in one row and column of Δ_3 zero. We may then without loss of generality assume $\alpha = \beta = \gamma = 1$. Similarly m, n, p must all be different from zero. We have, therefore, the following

THEOREM. *If in a non-vanishing symmetric determinant of order five, the five elements in the principal diagonal are all zero, and the complementary minors of four of these elements are also zero, then the elements outside the principal diagonal are all different from zero.*

Let us suppose now that the elements of D_5 are all real and we have

$$\Delta_3 = \begin{vmatrix} -2a & m-a-b & n-a-c \\ m-a-b & -2b & p-b-c \\ n-a-c & p-b-c & -2c \end{vmatrix},$$

and

$$M_1 = \begin{vmatrix} 0 & a & b & c \\ a & & & \\ b & & \Delta_3 & \\ c & & & \end{vmatrix}, \qquad M_2 = \begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & & & \\ 1 & & \Delta_3 & \\ 1 & & & \end{vmatrix}.$$

Denote by F_3 the quadratic form in the variables x_1, x_2, x_3 whose discriminant is Δ_3, and by C_1 the conic obtained by equating F_3 to zero. Then $M_1 = 0$ if and only if the line L_1,

$$ax_1 + bx_2 + cx_3 = 0$$

is tangent to C_1, and $M_2 = 0$ if and only if the line L_2,

$$x_1 + x_2 + x_3 = 0$$

is tangent to C_1. Let C_2 be the degenerate conic

$$(x_1 + x_2 + x_3)(ax_1 + bx_2 + cx_3) = 0,$$

and let C_3 be the conic, $(C_1 + 2C_2)/2$,

$$mx_1x_2 + nx_1x_3 + px_2x_3 = 0.$$

If now L_1 and L_2 are tangent to C_1, they are tangent at the same points to every conic of the pencil determined by C_1 and C_2, and hence are tangent to C_3. Since $K_1 = K_2 = K_3 = 0$, C_1 is tangent to each side of the triangle of reference

while C_3 goes through each vertex of this triangle. Hence C_1 and C_2 cannot have double contact, with real tangents at their points of contact.[4] That is, not both L_1 and L_2 are tangent to C_1 if they are distinct.

In case L_1 coincides with $L_2, a=b=c$ and since $K_1=K_2=K_3=0, m=n=p=4a$. But in this case L_2 is not tangent to C_1, that is $M_2\neq 0$. We have, therefore, the following

Theorem. *If in a real non-vanishing symmetric determinant of order five, the elements in the principal diagonal are all zero, and the complementary minors of four of these elements are also zero, then the complementary minor of the remaining element is not zero.*

QUESTIONS, DISCUSSIONS AND NOTES

EDITED BY R. E. GILMAN, Brown University, Providence, Rhode Island

The department of Questions and Discussions in the Monthly is open to all forms of activity in collegiate mathematics, including the teaching of mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.

PARAMETRIC SOLUTIONS OF CERTAIN DIOPHANTINE EQUATIONS

By R. B. THOMPSON, University of Nebraska

The purpose of this note is to present parametric solutions of two very general diophantine problems.[1]

The first equation considered is

$$X_1^2 + a_1X_2^2 + a_2X_3^2 + \cdots + a_{n-1}X_n^2 = Y_1^2 + b_1Y_2^2 + b_2Y_3^2 + \cdots + b_{m-1}Y_m^2.$$

The solution of this equation is as follows,

$$\begin{aligned}
X_1 &= h^2 - a_1x_2^2 - a_2x_3^2 - \cdots - a_{n-1}x_n^2 + b_1y_2^2 + b_2y_3^2 + \cdots + b_{m-1}y_m^2,\\
X_2 &= 2hx_2, \qquad X_3 = 2hx_3, \cdots, X_n = 2hx_n,\\
Y_1 &= h^2 + a_1x_2^2 + a_2x_3^2 + \cdots + a_{n-1}x_n^2 - b_1y_2^2 - b_2y_3^2 - \cdots - b_{m-1}y_m^2,\\
Y_2 &= 2hy_2, \qquad Y_3 = 2hy_3, \cdots, Y_m = 2hy_m,
\end{aligned}$$

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[4] If C_1 is the conic $x_1^2+x_2^2+x_3^2-2x_1x_2-2x_1x_3-2x_2x_3=0$, the only conic through the vertices of the triangle of reference and having double contact with C_1 is $x_1x_2+x_1x_3+x_2x_3=0$. The common tangents at their points of contact are $x_1+\omega x_2+\omega^2x_3=0$ and $x_1+\omega^2x_2+\omega x_3=0$, where ω is a primitive cube root of unity. These are the conics and lines corresponding to the determinant

$$D_5 = \begin{vmatrix} 0 & 1 & 1 & \omega^2 & \omega \\ 1 & 0 & 1 & \omega & \omega^2 \\ 1 & 1 & 0 & 1 & 1 \\ \omega^2 & \omega & 1 & 0 & 1 \\ \omega & \omega^2 & 1 & 1 & 0 \end{vmatrix}.$$

[1] Although the solutions given involve several parameters, there is no reason to believe that either is a completely general solution. Indeed the first cannot be general since X_1 and Y_1 alone can be odd numbers and the solution $X_1=1, X_2=3, X_3=5, X_4=7, Y_1=2, Y_2=4, Y_3=8$ of the equation $X_1^2+X_2^2+X_3^2+X_4^2=Y_1^2+Y_2^2+Y_3^2$ is not included. (EDITOR.)

where $h, x_2, x_3, \cdots, x_n, y_2, y_3, \cdots, y_m$ are arbitrary parameters giving an $(m+n-1)$-fold infinitude of solutions. This solution may easily be verified by direct substitution.

For example, in the following equation

$$X_1^2 + X_2^2 + X_3^2 = Y_1^2 + Y_2^2 + Y_3^2 + Y_4^2 + Y_5^2 + Y_6^2 + Y_7^2$$

if we set $h=1$, $x_2=2$, $x_3=1$, $y_2=2$, $y_3=1$, $y_4=3$, $y_5=2$, $y_6=1$, $y_7=3$, we have

$$(24)^2 + 4^2 + 2^2 = (-22)^2 + 4^2 + 2^2 + 6^2 + 4^2 + 2^2 + 6^2.$$

The second equation is,

$$X_1^2 + X_2^2 + X_3^2 + \cdots + X_p^2 = W^n,$$

p and n being any integers.[2] The solution of this equation may be written as follows,

$$X_1 = \left[x_1^n - \binom{n}{2} x_1^{n-2}\sigma + \binom{n}{4} x_1^{n-4}\sigma^2 - \binom{n}{6} x_1^{n-6}\sigma^3 + \binom{n}{8} x_1^{n-8}\sigma^4 \cdots \right]$$

$$X_2 = x_2\left[-\binom{n}{1} x_1^{n-1} + \binom{n}{3} x_1^{n-3}\sigma - \binom{n}{5} x_1^{n-5}\sigma^2 + \binom{n}{7} x_1^{n-7}\sigma^3 - \cdots \right]$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$X_p = x_p\left[-\binom{n}{1} x_1^{n-1} + \binom{n}{3} x_1^{n-3}\sigma - \binom{n}{5} x_1^{n-5}\sigma^2 + \binom{n}{7} x_1^{n-7}\sigma^3 - \cdots \right]$$

$$W = x_1^2 + \sigma,$$

where $\sigma = x_2^2 + x_3^2 + \cdots + x_p^2$ and where the last term in X_1 is $\pm\sigma^{n/2}$ or $\pm\binom{n}{1} x_1 \sigma^{n-1/2}$ as n is even or odd, and where the last term in X_i $(i=2, 3 \cdot, \cdots p)$ is $\pm\binom{n}{1} x_1 x_i \sigma^{(n-2)/2}$ or $\pm x_i \sigma^{(n-2)/2}$ as n is even or odd.

To verify that these formulas give a solution we have

$$\begin{aligned} X_1^2 + X_2^2 + \cdots + X_p^2 = x_1^{2n} &+ \left[\binom{n}{1}^2 - 2\binom{n}{2}\binom{n}{0}\right] x_1^{2n-2}\sigma \\ &+ \left[\binom{n}{2}^2 - 2\binom{n}{3}\binom{n}{1} + 2\binom{n}{4}\binom{n}{0}\right] x_1^{2n-4}\sigma^2 \\ &+ \left[\binom{n}{3}^2 - 2\binom{n}{4}\binom{n}{2} + 2\binom{n}{5}\binom{n}{1} \right. \\ &\qquad \left. - 2\binom{n}{6}\binom{n}{0}\right] x_1^{2\ \ -6}\sigma^3 \cdots, \end{aligned}$$

and

[2] A method for obtaining solutions of this equation is given by G. Candido in Periodico di Mathematica, Serie III, vol. 12, Anno XXX, pp. 45–47. The solution however is not given in explicit form. A solution in explicit form of a similar but much less general equation is given in Dickson's *History of the Theory of Numbers*, vol. 2, p. 429. (EDITOR.)

$$W^n = (x_1^2 + \sigma)^n = x_1^{2n} + \binom{n}{1} x_1^{2n-2}\sigma + \binom{n}{2} x_1^{2n-4}\sigma^2 + \binom{n}{3} x_1^{2n-6}\sigma^3 + \cdots.$$

If our solution is correct the sums or the right hand sides of the last two equations must be equal. In order to show this we show that the coefficients of corresponding powers of σ are equal. That is we have

$$\binom{n}{r/2} = \binom{n}{r/2}^2 - 2\binom{n}{r/2+1}\binom{n}{r/2-1} + 2\binom{n}{r/2+2}\binom{n}{r/2-2}$$
$$- 2\binom{n}{r/3+3}\binom{n}{r/3-3} + \cdots \pm 2\binom{n}{r}\binom{n}{0}.$$

This relation is easily proved by expanding both members of the equation $(1-x)^n(1+x)^n = (1-x^2)^n$, and equating the coefficients of corresponding powers of x.

As an example in the equation $X_1^2+X_2^2+X_3^2 = W^7$, if we set $x_1=1$, $x_2=2$, $x_3=1$, we obtain $(-104)^2+(-464)^2+(-232)^2=6^7$.

On an Orthopole-Locus

By R. Goormaghtigh, Bruges, Belgium

Professor J. H. Weaver has proved (this Monthly, vol. 40 (1933), p. 88) that, if a variable triangle is inscribed in a fixed circle and has a fixed orthocenter, the locus of the orthopole of a fixed tangent to the circumcircle is a circle equal to the nine-point circle.

In Mathesis, vol. 2 (1914), p. 151 and the Tohoku Mathematical Journal, vol. 27 (1926), p. 90, we have given the following more general theorem:

The locus of the orthopoles of a given straight line in a system of triangles inscribed in a circle and described about a conic is a circle.[1]

In the special case when the triangles $A_1A_2A_3$ are inscribed in a circle with center O and have a fixed orthocenter H, let T be the orthopole of a fixed straight line Δ, $\bar{\omega}$ the orthopole of the circumdiameter δ parallel to Δ. The locus of $\bar{\omega}$ is the nine-point circle O_9 of the triangles considered, and the locus of T is therefore a circle with center O' equal to the nine-point circle, and the distance O_9O' is equal to $\bar{\omega}T$ or the distance from δ to Δ. Hence we have the following theorem:

The locus of the orthopoles of a given straight line Δ in a system of triangles inscribed in a circle O and having a fixed orthocenter H, is a circle O' equal to the nine-point circle O_9; the circle O' is the reflexion of the circle O_9 in a straight line passing through the mid-point of the distance from H to Δ; if Δ remains tangent to a circle with center O, the locus of the center of O' is a circle having its center at O_9 and as radius the distance from O to Δ.

[1] When the given conic is a parabola, the locus of the orthopoles is a straight line.

The theorem given by Professor Weaver will be obtained when the distance from δ to Δ is equal to the radius of the circle $A_1A_2A_3$; the locus is then a circle equal to the nine-point circle and tangent to this circle at the mid-point of the segment from H to the contact point of the given fixed straight line with the circumcircle.

On a Relation in the Geometry of the Triangle

By R. Goormaghtigh, Bruges, Belgium

A. A. Bennett has given (this Monthly, vol. 38 (1931), p. 577) a proof of the following theorem:

In a triangle $B_1B_2B_3$ the outer common tangents to the excircles form a triangle whose incenter coincides with the circumcenter ω of the excenters I_1, I_2, I_3 and the radius of whose incircle is equal to twice the circumradius R plus the inradius r of $B_1B_2B_3$. (Johnson, *Modern Geometry*, 1929, p. 194.)

The following proof is also very simple.

The common tangents considered are the reflexions of the lines B_2B_3, B_3B_1, B_1B_2 respectively in the lines I_2I_3, I_3I_1, I_1I_2. Therefore it will be enough to prove that, if ω_1 is the reflexion of ω in I_2I_3, the distance ω_1C_1 from ω_1 to B_2B_3 is $2R+r$. But I being the orthocenter of $I_1I_2I_3$, ω_1I is parallel and equal[1] to ωI_1 or $2R$; hence ω_1I is perpendicular[2] to B_2B_3 and ω_1C_1 is equal to ω_1I+IC, or $2R+r$.

RECENT PUBLICATIONS

Edited by R. A. Johnson, Brooklyn College of the City of New York

All books for review should be sent directly to the editor of this department, at Brooklyn College, 66 Court Street, Brooklyn, N. Y., and not to any of the other editors or officers of the Association.

NEW BOOKS RECEIVED

An Elementary Treatise on Differential Equations. By Abraham Cohen. Second Edition, Completely Revised. New York, D. C. Heath and Company, 1933. viii+338 pages. $2.40.

Theory of Functions as applied to Engineering Problems. Edited by R. Rothe, F. Ollendorf, and K. Pohlhausen; translated by Alfred Herzenberg. Cambridge, Technology Press, 1933. x+190 pages.

Mathematical Tables, Volume III. Minimum Decompositions into Fifth Powers. Prepared by L. E. Dickson, London, The British Association for the Advancement of Science, 1933.

The Administration of Mathematics in Secondary Schools. By Ernst R. Breslich. Chicago, The University of Chicago Press, 1931. viii+408 pages. $3.00.

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[1] Since, as in the proof of the existence of the Nine Point Circle, the segment joining the orthocenter of a triangle to a vertex is parallel to and is twice as long as the perpendicular from the circumcenter to the side opposite the given vertex. Compare Johnson, *Modern Geometry*, Fig. 65, page 195 (Editor).

[2] Compare Johnson, *loc. cit.*, page 163, section 253, e. (Editor).

The Elements of Euclid. Edited by Isaac Todhunter, with introduction by Sir Thomas L. Heath. Everyman's Library, no. 891. New York, E. P. Dutton, 1933. xx+298 pages. 70 cents.

The Nature of Mathematics. By Max Black. New York, Harcourt, Brace and Company, 1934. xiv+220 pages. $3.50.

Principles of Geometry. By H. F. Baker. Cambridge University Press, 1933. Volume V, Analytical Principles of the Theory of Curves. x+248 pages. $4.50. Volume VI, Introduction to the Theory of Algebraic Surfaces and Higher Loci. x+308 pages. $4.75.

Differential Equations. By N. B. Conkwright. New York, The Macmillan Company, 1934. xiv+234 pages. $1.90.

Tables of the Higher Mathematical Functions. Computed and Compiled under the direction of Harold T. Davis. Bloomington, Indiana, The Principia Press, 1933. xiv+378 pages.

Tables of Functions. By Eugen Jahnke and Fritz Emde. Second (Revised) Edition. (Text in German and English.) Berlin, B. G. Teubner, 1933. xviii+330 pages. RM 16.

Algebraic Functions. By G. A. Bliss. New York, American Mathematical Society, 1933. Colloquium Publication, Volume 16. x+220 pages.

Tables of Integrals and Other Mathematical Data. By Herbert B. Dwight. New York, The Macmillan Company, 1934. x+222 pages. $1.50.

Differential Equations. By Lester R. Ford. New York, McGraw-Hill Book Company, 1933. x+264 pages. $2.50.

Plane Trigonometry. By W. A. Granville, revised by P. F. Smith and J. S. Mikesh. Boston, Ginn and Company, 1934. xii+212+iv+44 pages. $1.60.

Galois and the Theory of Groups. By H. G. and L. R. Leiber. Lancaster, Pennsylvania, The Science Press, 1932. 66 pages.

REVIEWS

An Elementary Treatise on Differential Equations. By Abraham Cohen. Second edition, completely revised. New York, D. C. Heath and Co., 1933. viii+337 pages. $2.40.

The second edition of this well known textbook follows the general outline of the first edition[1] but differs from it in numerous details. We find the same fourteen chapters, but they are arranged in a different order. The chapters on Total Differential Equations, and on Systems of Simultaneous Equations now immediately precede the chapters on Partial Differential Equations, an arrangement which seems a more logical one than that of the first edition. In comparing each chapter of the new edition with the corresponding chapter of the first edition, it is observed that most of the section headings are the same. Hence the general plan of the new edition is that of the first edition.

[1] Published in 1906. Reviewed by C. R. MacInnes in the Bulletin of the American Mathematical Society, vol. 13 (1906–7), pp. 515–6.

But it does not follow that no essential changes have been made in the book. Many of the chapters, and almost all of the individual sections have been rewritten, either expanded or condensed as has seemed best to the author in the light of the experience of users of the book during the past quarter of a century.

Among new topics introduced in this edition may be noted: isobaric equations, Legendre's and Bessel's equations (under Integration in Series), and Fourier Series (under Higher Partial Equations). An additional "Note" has been added to the appendix, giving the condition that n functions of a single variable may be linearly dependent. The sets of exercises have been extensively revised and many new exercises have been introduced, chiefly those concerned with applications. A long section (VI.13) on applications to physical problems has been added to the chapter on linear equations with constant coefficients.

This edition adopts the custom of numbering sections by chapter number and section number, rather than consecutively. This tends to increase the usefulness of the book for reference purposes. The increased page size and larger type give the book an attractive appearance from a typographical standpoint. The present edition is a worthy successor to the original edition.

H. M. GEHMAN

Differential Equations. By Max Morris and O. E. Brown. New York, Prentice-Hall, 1933. xii+409 pages. $2.50.

This book covers the ground usually covered by texts on the subject of differential equations. After an introductory chapter in which the elementary concepts and definitions are discussed in adequate fashion, there are two chapters on differential equations of the first order, single chapters on linear equations, numerical approximation to solutions, and integration in series. The seventh chapter takes up ordinary equations in more than two variables, and the three concluding chapters are on partial differential equations.

The philosophy of the authors is expressed in the preface. They point out that the student taking the course in differential equations has usually had only a first year course in calculus, and hence "has mastered only the merely manipulative aspects of the calculus. Under such conditions, a searching and serious study of the more theoretical aspects of the subject of differential equations must of necessity be left in abeyance. On the other hand, the opportunity, as the subject is developed, for implanting in the student something like a feeling for mathematical rigor, presents itself frequently enough—and should be improved when met with. A course in which a generous amount of drill in integrating the various types of differential equations is accompanied by an exposure of the student, on as wide a front as possible, to the more exacting and searching aspects of mathematics, is a desideratum with every teacher of the subject. This text has been prepared with the aim of assisting the teacher to come measurably near that end."

The reviewer believes that the authors have succeeded in constructing a text which will fulfill the needs of many teachers. For the teacher interested in

drilling his students in the solution of problems, there are numerous and lengthy sets of exercises. Illustrative examples are worked out in great detail. Besides the type of exercise which serves to illustrate a formula or method of solution, there are exercises which are designed to lead the student to discover the method of treatment of exceptional cases not covered by the discussion in the text. Under applied problems, the authors have not merely given the differential equation to which the problem gives rise, but have attempted to give the background of the problem as well.

The text will be particularly suitable for the teacher who is interested in making the first course in differential equations serve as an introduction to further work in analysis. The student is referred to other texts (chiefly to Wilson, Dickson, and Bôcher) for a discussion of such matters as linear dependence, and rank of a matrix, topics which are apt to be ignored in a text of this type. Among topics introduced are Legendre polynomials, Bessel functions, hypergeometric series, Jacobian, and Wronskian. Chapter 5 on numerical approximation by various methods is worthy of special mention, as this topic is not ordinarily discussed. The text considers the subject of differential operators and their inverses in some detail.

The authors are to be commended for writing a text in which the student is given such an insight into the field of analysis in general and of differential equations in particular.

H. M. GEHMAN

College Algebra. By R. W. Brink. New York, The Century Co., 1933. xvii +445 pages. $2.25.

This text contains a generous supply of material for the review of elementary algebra. It is designed for students who have had either two or three semesters of high school algebra and contains almost as many examples on this review material as one would find in an elementary text.

The arrangement of the text into relatively short sections with accompanying exercises would seem to give the book a certain adaptability to courses of various lengths. The problems, which are drawn from many fields, are well graded and should develop in the student a moderate technical skill. On the whole the treatment of the various topics is conservative and excellent. The chapters on Determinants and on Probability seem to the reviewer to be especially well written. The author has in many instances added considerable interest by his graphical interpretation of algebraic ideas.

The book is well illustrated and neat and attractive in appearance.

MALCOLM FOSTER

MATHEMATICS CLUBS

EDITED BY F. M. WEIDA, The George Washington University, Washington, D.C.

All reports of club activities, suggestions and topics for club programs should be sent to F. M. Weida. The George Washington University, Washington, D.C. All manuscript should be typewritten with double spacing, and with margins at least one inch wide. All club activity manuscripts for the academic year 1933–1934 should be submitted for publication not later than June 1, 1934.

CLUB ACTIVITIES

1932–1933

THE PI MU EPSILON MATHEMATICAL FRATERNITY

Pi Mu Epsilon is an academic fraternity in institutions of university grade. Its primary aim is the advancement of mathematics and scholarship. It is a living, active, working fraternity of scholars in which the members are actively engaged in study and research and the exchange of ideas in the field of mathematical science.

Pi Mu Epsilon of Pennsylvania State College

The year 1932–1933 proved to be a successful one for our chapter, particularly from the standpoint of the number of interesting papers that were presented. Dr. F. W. Owens, head of the Department of Mathematics at this college and an active member of Pi Mu Epsilon, has recently been elected National President of Pi Mu Epsilon.

The officers of our chapter, elected at a meeting of the chapter held May 17, 1933, are: Dr. C. C. Wagner, Director; Mr. F. Brink, Vice Director; Mr. W. C. Johnson, Secretary; Miss G. M. Quigg, Treasurer; Mr. H. J. Herring, Librarian.

The number of active members was seventy-two. Initiations are held during May and December of each year. The number of initiates varies between five and fifteen.

The meetings and programs were as follows:

November 17, 1932: Business meeting and election of new members.
December 13, 1932: Business meeting and initiation of new members.
March 16, 1933: "Complex numbers and some generalizations" by Dr. H. B. Curry.
March 29, 1933: "Symmetrical components" by Mr. C. B. Holt; election of members.
May 1, 1933: Business meeting; initiation of pledges.
May 17, 1933: "The group concept" by Dr. F. W. Owens; election of officers.

The following officers have guided Pi Mu Epsilon during the past year: Dr. O. Frink, Director; Mr. J. B. Pearson, Vice Director; Mr. A. F. Pechter, Secretary; Miss G. M. Quigg, Treasurer; Dr. C. C. Wagner, Librarian.

W. C. JOHNSON, *Recording Secretary*

Pi Mu Epsilon of the University of Illinois

Illinois Alpha Chapter of Pi Mu Epsilon reports a very successful year on the Illinois campus

The officers for 1932–1933 were: W. Taylor Waterhouse, Graduate in mathematics, Director; Byron T. Darling, Senior in engineering, Vice Director; Alma Carson, Senior in arts, Recording Secretary; J. M. Glass, Junior in engineering, Corresponding Secretary; John F. Kinder, Senior in arts, Treasurer.

The Executive Committee was composed of R. A. Baumgartner, Elizabeth Ford, Mildred Norval and Ted Martin.

All officers were elected at the regular spring election held May 24, 1932, except Mr. Waterhouse, who was elected February 14, 1933, to fill the vacancy left by R. E. Lane, and Mr. Darling, who was elected at the same time to fill Mr. Waterhouse's position as Vice Director.

The chapter had thirty-nine active members, thirteen of which were initiated December 20,

1932. This year the chapter presented to the mathematics library a set of fundamental mathematical books.

The meetings and programs were as follows:

October 11, 1932: "Imaginaries are real" by Dr. Olive Hazlett.
October 25, 1932: "What does an actuary do?" by Professor A. R. Crathorne.
November 8, 1932: "Analyticity and some of its properties" by Mr. B. T. Darling.
November 22, 1932: "Trigonometric solutions of the cubic" by Professor L. L. Steimley.
December 6, 1932: "A theorem in partitions" by Donald Brown.
December 9, 1932: Pledging luncheon held in The Green Tea Pot in Champaign.
December 20, 1932: Initiation banquet held at the Union Arcade in Champaign. Professor Arnold Emch gave the address on "Mathematical reminiscences of Europe."
January 10, 1933: "Finding a place in the world of mathematics" by Professor A. B. Coble.
February 14, 1933: "Monogenic non-analytic functions" by Mr. B. T. Darling.
February 28, 1933: "Configurations connected with a symmetric group in S_3" by Mr. C. A. Jacokes.
March 14, 1933: "A crusade against the use of negative numbers" by Professor G. A. Miller.
March 28, 1933: "Continued fractions" by Professor H. W. Bailey.
April 11, 1933: "Magic squares" by Miss Le Vona Voigt.
April 25, 1933: "An (m, m) correspondence" by Dr. Miles C. Hartley.

J. M. GLASS, *Corresponding Secretary*

LOCAL MATHEMATICS CLUBS

The Mathematics Seminar of the University of Maryland

The officers for 1932–1933 were: J. Yates, President; C. Tompkins, Secretary.

The meetings and programs were as follows:

October 5, 1932: "Unusual geometric constructions" by Dr. Dantzig.
October 19, 1932: "Discriminant of the cubic" by Mr. Tompkins.
October 26, 1932: "Pascal's hexagram" by Mr. Kent.
November 2, 1932: "Plücker's numbers" by Dr. Yates.
November 9, 1932: "Prime numbers" by Mr. Burger.
November 16, 1932: "Fourth dimension" by Mr. Aldrich.
November 23, 1932: "A theorem of Morley's" by Mr. Dantzig.
December 7, 1932: "Graphical solution of the quartic" by Mr. Yates.
December 14, 1932: "A summer trip" by Mr. Gwinner.
January 4, 1933: "The golden section" by Mr. Smith.
January 11, 1933: "Inversion" by Mr. Hamilton.
January 18, 1933: "The problem of pursuit" by Dr. Yates.
February 8, 1933: "Old and new dynamics" by Mr. Tompkins.
February 15, 1933: "Nine point circle" by Mr. Park.
March 1, 1933: "Ruler and compass constructions" by Dr. Dantzig.
March 15, 1933: "Famous constructions" by Mr. Yates.
March 29, 1933: "Differential equations reducible to Clairaut's equation."
April 5, 1933: "e" by Mr. Kent.
April 12, 1933: "Cubic curves" by Mr. Burger.
April 19, 1933: "Folding geometry" by Mr. Shipley.
April 26, 1933: "Pre-historic mathematics" by Mr. Stinson.
May 3, 1933: "Introduction to relativity" by Mr. McClurg.
May 24, 1933: "Mathematics in Physics" by Mr. Eichlin.
May 31, 1933: "Platonian solids" by Mr. Smith.

The prize offered by the faculty for the best paper by a student was awarded to Mr. Smith for his paper on "The golden section."

C. TOMPKINS, *Secretary*

The Mathematics Club of the University of Kansas

This club is a students' club. It includes thirteen faculty members. We had a membership of fifty-two students, graduate and undergraduate. Usually about twenty-five of the students are active in the organization.

The officers for the following year are elected by the club at the last regular meeting of each school year. The officers for 1932–1933 were: Oleta Markham, President; Otis Brubaker, Vice President; Walt Simmons, Secretary-Treasurer; Carol Hunter, Social Chairman; Professor J. J. Wheeler, Faculty Adviser.

We have two standing committees: The executive committee which consists of the officers, and the program committee which consists of three members appointed by the President.

The eligibility for membership consists of having studied through the calculus (six to eight hours of calculus) and of having given evidence of an interest in mathematics.

The primary aim of the club, as stated in the constitution, is "to increase interest in mathematics by considering unusual problems or investigating subjects not generally discussed in regular courses."

Following each program, there is a half-hour social gathering.

The meetings and programs were as follows:

October 3, 1932: Business meeting, chiefly for the election of new members.
October 17, 1932: "On the square" by Professor G. W. Smith.
October 31, 1932: "Primitive methods for measuring time" by Ralph L. Scott.
November 14, 1932: "Mathematics and astronomy" by Professor W. H. Garrett, Baker University.
December 5, 1932: "Ballistics and gunnery" by Major W. C. Koenig.
January 9, 1933: "Dimensional analysis" by Daniel P. Johnson.
January 23, 1933: Some chemical applications of mathematics" by Professor A. W. Davidson.
February 13, 1933: "The nine-point circle" by Lilly E. Somers.
February 27, 1933: "Higher plane curves" by Dorothy E. Doering.
March 13, 1933: "Correlation connected with forecasting" by Professor E. B. Dade.
March 27, 1933: "Mathematics in music" by Professor C. S. Skilton.
April 10, 1933: "Finite geometry" by Eula Johnson.
April 24, 1933: "Mathematics in scientific museums" by Professor U. G. Mitchell.
May 15, 1933: Picnic with mathematics clubs of neighboring colleges as invited guests.

WALT SIMMONS, *Secretary-Treasurer*

The Mathematics-Science Club of Drake University

Our club is a combination of mathematics, physics and astronomy. The aim of the club is to give the students a chance to hear interesting topics reported on and discussed and also to promote friendship between its members.

Students having completed nine semester hours and having the intention of majoring in one of the above mentioned subjects are eligible to membership. Selection is made by unanimous vote of the members from those having the above qualifications. Grades are considered in the selection of candidates but there is no specific requirement. This year we had twenty-nine members including seven faculty members.

The officers for 1932–1933 were: Adair Baker, President; Donald Stewart, Vice President; Walter Stillwell, Secretary-Treasurer; Virginia Schall, Corresponding Secretary. The officers are elected at the May meeting.

The meetings and programs were as follows:

October 11, 1932: Picnic meeting. Names of prospective members were proposed.
November 8, 1932: Election of new members.
December 13, 1932: Initiation of new members. Reading and revision of constitution.
January 10, 1933: Meeting held at KSO broadcasting station, Des Moines, Iowa. Mr. Greer, chief engineer, explained and demonstrated some of the workings of Radio Broadcasting.

February 14, 1933: "Survey of U. S. Public lands" by Professor Neff.
March 13, 1933: "Building an observatory at Bloemfontein, South Africa" by Professor Jessup.
April 11, 1933: Meeting was held at the Des Moines Municipal Airport. Mr. Combs told interesting facts about aircraft and took us through the hangar.
May 9, 1933: Picnic at the Drake observatory. Election of new officers.

VIRGINIA SCHALL, *Corresponding Secretary*

PROBLEMS AND SOLUTIONS

EDITED BY OTTO DUNKEL, H. L. OLSON, AND W. F. CHENEY, JR.

ELEMENTARY PROBLEMS

Send all communications about Elementary Problems and Solutions to W. F. Cheney, Jr., Dept. Box 35, Storrs, Connecticut.

The Department of Elementary Problems welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. Problems may be submitted unaccompanied by their solutions.

PROBLEMS FOR SOLUTION

E 81. *Proposed by Rev. Clete Adams, Quincy College, Quincy, Ill.*

A "Poosh-up" game with eleven balls contains pockets marked "100," "200," "300," "400," "500," "600," "700," "800" and "Double." Every ball in the "Double" pocket doubles the entire score in all the other pockets. What is the greatest score that can be rolled if the numbered pockets can each hold only one ball and the "Double" pocket any number? What is the greatest score possible if each pocket is large enough to hold all the balls?

E 82. *Proposed by L. S. Johnston, University of Detroit.*

If the points, $P, A, B, C, D, \cdots, K$, lie on a straight line, and the point O is off that line, show that if the segments, $PA, PB, PC, PD, \cdots, PK$ form a harmonic sequence, then the tangents of the angles, $AOP, BOP, COP, DOP, \cdots$ KOP, also form a harmonic sequence.

E 83. *Proposed by Morgan Ward, California Institute of Technology.*

Show that in any arithmetic progression of integers with common difference less than two thousand, at most ten consecutive integers can be primes.

E 84. *Proposed by J. D. Leith, University of North Dakota.*

A given parabola, $x^2=4py$, envelopes an infinite family of circles centered on the Y-axis. Determine the ordinate of the center of such a circle as a function of its radius and p.

E 85. *Proposed by R. E. Moritz, University of Washington, Seattle, Wash.*

Show analytically that the two polar coordinate equations, $r=k/(1+m\cos\theta)$ and $r=-k/(1-m\cos\theta)$, represent the same curve.

E 86. *Proposed by Roy MacKay, Albuquerque High School, Albuquerque, N. M.*

If the faces of a tetrahedron are congruent triangles, prove that the circumcenter and the centroid are coincident.

SOLUTIONS

E 54 [1933, 491]. *Proposed by P. R. Hill, University of Georgia.*

If the probability that an event will occur in a single trial is $1/N$, then as N increases without limit the probability that the event will occur at least once in N trials approaches the limit, $1-1/e$.

Solution by Elizabeth Taylor, University of Iowa.

Since the probability that the event will occur in a single trial is $1/N$, the probability that the event will not occur in a single trial is $1-1/N$. Then the probability that the event will not occur in any one of N trials is $(1-1/N)^N$. But $\text{limit}_{N\to\infty}\ (1-1/N)^N=1/e$. Therefore the probability that the event will occur at least once in N trials, must approach the limit $1-1/e$, or $.632+$, as N increases without limit.

Also solved by A. E. Andersen, L. M. Bauer, Churchill Eisenhart, Morris Levenson, Hy Marcus, B. D. Roberts, C. W. Trigg, Simon Vatriquant, Maud Willey and the proposer.

E 55 [1933, 491]. *Proposed by J. Rosenbaum, Milford, Connecticut.*

Obtain the general solution in positive integers of

$$2z^3 = x + (x^2 - 4y^3)^{1/2}.$$

Solution by E. P. Starke, Rutgers University, New Brunswick, N. J.

Transposing x and squaring gives $z^6-xz^3+y^3=0$, which is equivalent to the given equation with the ambiguous sign before the radical. Since y is obviously divisible by z, we set $y=uz$. Then substitution and the cancellation of z^3 gives $x=u^3+z^3$.

If the radical in the given equation must be positive, as convention would require, $2z^3\geqq x$, and $z\geqq u$. We may thus choose any two positive integers whatsoever, using the sum of their cubes for x, their product for y and the larger of the two chosen integers for z, and the three numbers thus obtained must satisfy the given equation.

Also solved by Wm. Douglas, Daniel Finkel, Theodore Lindquist, Roy MacKay, Hy Marcus, A. A. Rood, T. L. Smith, Simon Vatriquant and the proposer.

E 56 [1933, 491]. *Proposed by Otto Dunkel, Washington University, Saint Louis, Missouri.*

From the base vertices A and B of an isosceles triangle ABC, segments of straight lines AL and BM of equal length are drawn to the opposite equal sides. Determine by plane geometry the locus of P, the intersection of AL and BM.

Solution by Roy MacKay, Albuquerque High School, Albuquerque, N. M.

Suppose that AL is the segment of an arbitrary line through A, not perpendicular to BC, and let BM and BN be the two equal segments from B to M and N on AC, cutting AL at P and Q respectively. In labeling M and N on AC, make $CM=CL$.

Since triangles ABM and ABL are congruent, triangles APM and BPL are congruent and $PA=PB$. Consequently the locus of P is the perpendicular bisector of AB, one part of the required locus.

Angles BNM, BMN and BLA are equal, and therefore angles QNA and CLA are equal. Furthermore, angles NAQ and CAL are equal, and angles AQN and ACB are equal. Finally, angles AQB and ACB are either equal or supplementary, according as AL is more or less than AB. Therefore Q lies on the circle in which the chord AB subtends an angle equal to the angle ACB on the side of AB remote from C. It is easy to show that this circle passes through O, the orthocenter of the triangle ABC, for angle AOB is the supplement of angle ACB. Hence the required locus consists of the perpendicular bisector of AB and the circle passing through A, B and O.

Note by the proposer. The above locus is a degenerate cubic curve which appears when the same construction is made with a scalene triangle which approaches the isosceles as a limit. In a trivial sense, the base AB is also a part of the locus in the case of the isosceles triangle.

Also solved by Hy Marcus, B. D. Roberts, C. W. Trigg, Simon Vatriquant and the proposer.

E 57 [1933, 491]. *Proposed by W. F. Cheney, Jr., Connecticut State College.*

Tom, Dick and Harry were comparing notes on their fishing experiences of the previous summer, and found that one had caught only perch, one only trout, and one only salmon. They remarked that the number of fish Tom had caught was seven more than three-fifths of the number of perch that were caught; that the number of fish Dick had caught was three more than five-sevenths of the number of salmon that were caught; and that the total number of all the fish caught was a three-place prime number. Determine how many and what kind of fish each caught, and show that the solution is unique.

Solution by A. C. Maddox, Louisiana State Normal College, Natchitoches, Louisiana.

It is obvious that the number of perch that were caught is a multiple of five, and that the number of salmon that were caught is a multiple of seven. Since seven is not two-fifths of any integer, Tom did not catch the perch. Likewise, since three is not two-sevenths of any integer, Dick did not catch the salmon. Also, since three more than five-sevenths of an integer can not be an integer multiple of five, Dick did not catch the perch. Consequently, Dick caught the trout, Tom caught the salmon and Harry the perch.

Now, since three-fifths of the number of perch that were caught is a multiple of seven (being seven less than the number of salmon Tom caught), the number of perch Harry caught is a multiple of thirty-five. Call this number of perch, $35x$.

Then the number of salmon Tom caught was $21x+7$, and the number of trout Dick caught was $15x+8$. Consequently $71x+15$ denotes the total number of fish caught, which must be a three place prime number.

Since any odd value of x makes $71x+15$ a multiple of 2, x must be even. Nor can x be a multiple of 3 or 5 without making $71x+15$ also a multiple of 3 or 5. And since $71x+15$ is a three-place prime, x is less than 14. This leaves for examination the values, $x=2$, 4 and 8. When $x=2$, $71x+15=157$ which is prime; but if $x=4$ or 8, $71x+15=299$ $(=13\cdot 23)$ or $583(=11\cdot 53)$ respectively.

Consequently the unique solution is: Tom caught 49 salmon, Dick caught 38 trout, and Harry caught 70 perch; making altogether a total of 157 fish caught.

Also solved by A. E. Andersen, W. E. Buker, Daniel Finkel, W. Hurry, Roy MacKay, Hy Marcus, A. A. Rood, W. R. Ransom, E. P. Starke, C. W. Trigg, Simon Vatriquant, B. C. Zimmerman and the proposer.

E 58 [1933, 491]. *Proposed by R. M. Sutton, Haverford College, Pa.*

In the following division of a three-place number into a five-place number each digit has been replaced by a code letter. Assuming only that the remainder, Y, is not zero, reconstruct the problem and show that the solution is unique.

$$
\begin{array}{r}
L\ M\ N)\,R\ S\ T\ U\ N(U\ X \\
\underline{R\ T\ Y\ X} \\
\underline{T\ Y\ Y\ N} \\
\underline{T\ Y\ Y\ J} \\
Y
\end{array}
$$

I. *Solution by C. E. Shannon, University of Michigan.*

1. If N is even, J and X are even from the multiplications, Y is even from the last subtraction, and U is even from the first subtraction. Each of these five letters is obviously different from zero, which is impossible if all are even, since there are only four even digits greater than zero. Therefore N must be odd.

2. If J is even, X must be even from the second product, and then U must be even from the first product. This makes Y odd in the second subtraction, but even in the first subtraction, which is impossible. Therefore J must be odd.

3. Since J and N are both odd, X is odd from the last product, and then U is odd from the first product. Since from the multiplications, none of these four can be 5, one of them must equal 1. Again from the multiplications, none of U, X or N can be 1, so J must equal 1.

4. In the last multiplication, $N(=3$, 7 or 9) times $X(=3$, 7 or 9) gives a product ending in 1. Since neither N nor X can be 9 and make this true, they must be 3 and 7, one way or the other, and U must take the remaining value of 9.

5. If $N=7$, $X=3$ and $Y=6$ from the last subtraction. Then in the last multiplication, $3\cdot 7=21$, carrying 2 into the adjacent $Y(=6)$, so $3M$ ends in 4 and M must $=8$. Also $8\cdot 3=24$, carrying 2 again into the next Y, so $3L$ ends in 4,

and L must also $=8$. But this is contrary to the hypothesis that the ten letters were distinct, so N can not be 7.

6. The only remaining value for N is 3. Then $X=7$ and $Y=2$ from the last subtraction. We have already found that $J=1$ and $U=9$. The last product is now $T221=7\cdot LM3$, and following through the multiplication shows that $M=0$, $L=6$ and $T=4$. The dividend is now $97\cdot 603+2=58493$ and the entire division is

$$\begin{array}{rrrrrrrr}
6&0&3)5&8&4&9&3(9&7\\
&&5&4&2&7&&\\
\cline{3-6}
&&&4&2&2&3&\\
&&&4&2&2&1&\\
\cline{4-7}
&&&&&&2&
\end{array}$$

II. *Solution by Rev. M. A. Scheier, Catholic University, Washington, D.C.*

Examining each letter in the multiplications and subtractions shows that none but M can be zero. Therefore $M=0$.

Then $N\cdot U=10Y+X$ and $N\cdot X=10Y+J$. Trying all possible values for N and U in the multiplication table, the only ones that fit these two equations are $N=2$ and $N=3$.

If $N=2$, $Y=1$, $U=9$, $X=8$ and $J=6$ are the only distinct values fitting the two equations above. But no value of T makes $T116$ a multiple of 98, so this hypothesis must be abandoned.

If $N=3$, $U=9$, $Y=2$, $X=7$ and $J=1$ are the only distinct values that fit the above two equations. From the last product we now have $L=6$ and $T=4$, whence $S=8$ in the first subtraction, and the remaining digit 5 must be the value of R. Substitution and division verifies these values.

Also solved by W. E. Buker, Daniel Finkel, C. G. Killen, H. R. Leifer, A. C. Maddox, C. W. Munshower, R. S. Park, W. R. Ransom, E. P. Starke, C. W. Trigg, Simon Vatriquant, B. C. Zimmerman and the proposer.

E 59 [1933, 492]. *Proposed by J. H. Butchart, Indianapolis, Indiana.*

In the angle ACB of triangle ABC circles are inscribed tangent respectively to AC at A and to BC to B. Prove that the chords intercepted on the side AB are equal.

Solution by T. L. Smith, Carnegie Institute of Technology, Pittsburgh, Pennsylvania.

Let the circle tangent to AC at A be tangent to BC at F and cut AB at D. Let the circle tangent to BC at B be tangent to AC at G and cut AB at E.

Since the square of the tangent to a circle from an external point equals the product of the segments of the secant to that point,

$$AB = BF^2/BD = AG^2/AE.$$

But $AG=BF$ by subtracting equal tangents from other equal tangents, so

$AE=BD$, and adding DE to each side of the last equation gives the desired result, that $AD=BE$.

Also solved by L. M. Bauer, E. M. Berry, Roy MacKay, B. D. Roberts, C. W. Trigg and Simon Vatriquant.

Advanced Problems

Send all communications about Advanced Problems and Solutions to Otto Dunkel, Washington University, St. Louis, Mo. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide.

Problems containing results believed to be new, or extensions of old results are especially sought. The editorial work would be greatly facilitated, if on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known textbooks or results found in readily accessible sources, will not be proposed as problems for solution in this department. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

PROBLEMS FOR SOLUTION

3667. *Proposed by Raphael Robinson, University of California at Berkeley.*

Show that $(-1)^{n-1}(n-1)2^{n-2}$ is the value of the n-rowed determinant for which $a_{ij}=|i-j|$.

3668. *Proposed by Elmer Schuyler, Bay Ridge High School, Brooklyn, N. Y.*

Construct a triangle ABC, given the circumcenter, the foot of the altitude from A, and the point common to BC produced and the bisector of the exterior angle at A.

3669. *Proposed by W. E. Buker, Leetsdale, Pa.*

Given a triangle with a point P on one side and a line l through P which bisects the area of the triangle. Find the envelope of l as P describes the perimeter of the triangle.

3670. *Proposed by N. A. Court, University of Oklahoma.*

The variable line equidistant from two given points in space and passing through a third fixed point generates a cone of the second degree.

3671. *Proposed by J. Rosenbaum, Milford, Conn.*

In a tetrahedron, if any two of the three centers, the circumcenter, the incenter, and the centroid, are coincident, then all three are coincident.

3672. *Proposed by W. V. Parker, Mississippi Woman's College.*

Prove that the sum of the primitive roots of $x^n-1=0$ is zero if n has a repeated prime factor and is $(-1)^k$ if n is the product of k distinct primes.

SOLUTIONS

3576 [1932, 549]. *Proposed by J. J. L. Hinrichsen, Iowa State College.*

Given any triangle with three line segments concurrent in a point interior to the triangle and joining each vertex to its opposite side; prove that the length of the longest of these three lines cannot be less than $\sqrt{3}/2$ times the length of

the opposite side of the triangle.

Partial Solution by F. Underwood, University College, Nottingham, England.

Let p, q, r be the lengths of the segments, respectively, from A, B, C to the opposite sides, the three segments meeting in P, a point inside the triangle. It will be assumed that the sides are in the order of magnitude $c \geqq b \geqq a$. Consider first the case in which $C \geqq 90°$. Then $p > b \geqq a$, or $p/a > 1 > 3^{1/2}/2$. If $q \geqq p$, then $q/b \geqq p/b > 1$. Since for P inside ABC, $r < b < p$, r cannot be the longest segment.

If $C < 90°$, then $C \geqq 60°$. If p is the longest segment, then $p \geqq b \sin C$, and

$$\frac{p}{a} \geqq \frac{b \sin C}{a} \geqq \frac{3^{1/2}}{2}.$$

If q is the longest segment, then $q \geqq p$, and

$$\frac{q}{b} \geqq \frac{p}{b} \geqq \sin C \geqq \frac{3^{1/2}}{2}.$$

If r is the longest segment, $r \geqq p \geqq c \sin B$, and $r/c \geqq \sin B$. If then $B \geqq 60°$, $r/c \geqq 3^{1/2}/2$. Hence the theorem is true if $C \geqq B \geqq 60°$.

There remains the case for which $A \leqq B < 60° < C < 90°$. In this case we need only to consider the segment r, and to prove that, if it is possible for it to be the longest segment, then $r/c \geqq 3^{1/2}/2$. This has not been accomplished.

Note by the Editors. It will be shown that for certain triangles for which $A \leqq B < 60° < C < 90°$ a region within the triangle can be found such that for its interior points r is the longest segment. Such a region is bounded by a part of a side, say b, and by parts of two curves defined respectively by $r = p$ and $r = q$. Only the part of the curve $r = p$ within the triangle is required; but, since the curve itself is of sufficient interest, its complete form will be determined.

Every point on the straight line which contains b satisfies the condition $r = p$, but this part of the locus is without interest and it will be excluded from the discussion. The curve passes through both A and C; for, if on CB the point $\overline{A} \neq C$ is located so that $A\overline{A} = b$, then for A, $r = p = b$. Moreover, $A\overline{A}$ is the tangent at A. In a similar manner we see that $C\overline{C}$ is tangent to the curve at C, where $\overline{C}$ lies on AB so that $C\overline{C} = b$. The only real point at infinity is given by r and p parallel to the median through B; hence this median gives the direction of the asymptote. Locate B_1 so that $ABCB_1$ is a parallelogram. Then B_1 is on the curve, for B_1 determines $r = p = \infty$. Let P be a point on the curve for which $p = AL$, $r = CN$; and set $\theta = \angle B_1AL = \angle BLA$, and $\phi = \angle B_1CN = \angle BNC$. Then $a \sin \theta = c \sin \phi$, for $r = p$. If T_1 and T_2 are the feet of the perpendiculars from B_1 to p and r, then $B_1T_1 = B_1T_2$. Hence when P moves so that p and r approach the position of b, P approaches B', where B' is the foot of the perpendicular from B_1 on b.

Consider a circle with B_1 as center, and the complete quadrilateral formed by the four tangents from A and C. The remaining four vertices P, P', Q, Q' are

points on the curve; let us assume for convenience that P is within AB_1C and that P' is its opposite vertex. If $a=c$, then P and P' describe the altitude from B of the resulting isosceles triangle, while Q and Q' describe the circle through A, C, B_1. This degeneration of the locus shows that the curve is of the third order. As P approaches B_1, the straight line B_1P approaches the bisector of the angle between the perpendiculars to B_1A and B_1C at B_1; thus the tangent at B_1 to this branch of the curve is the internal bisector of the angle B_1 of the triangle AB_1C. In a similar manner it follows that the external bisector is the tangent to a second branch at B_1. Let h_a, h_b, h_c denote the lengths of the altitudes of ABC. If on AB we locate the points C_1'' and C_2'' so that $CC_1''=CC_2''=h_a$, then CC_1'' and CC_2'' are tangents at the points where they are cut by the altitude AH, where H is the orthocenter of ABC. Thus no part of the curve lies within the angle $C_1''CC_2''$. The curve is a circular cubic, for the imaginary straight lines from A and C to one of the circular points at infinity determine $r=p=0$. The curve passes from ∞ to the following points in order C, B_1, A, B', B_1, ∞. If we take as the x-axis the external bisector of angle B_1 and for the y-axis the internal bisector, the equation of the curve is obtained by multiplying x^2+y^2 by the left side of the equation for the median B_1B and setting the sum of this product and kxy equal to zero. The value of k may be found by inserting the coordinates of either A or C. The result is

$$(x^2+y^2)\left[\sin\tfrac{1}{2}(C-A)\sin^2\tfrac{1}{2}By-\cos\tfrac{1}{2}(C-A)\cos^2\tfrac{1}{2}B\,x\right]+c\sin A\,xy=0.$$

We return now to the type of triangles specified at the beginning of this note, and we consider first the parts of the curve $r=p$ within the triangle ABC. Since the tangent at A enters the angle A for such triangles, one part of the loop on AB' lies entirely within ABC. The tangent at C does not enter the angle C. If $h_a=b\sin C<a$, this part of the curve cuts CB in two other points giving a non-loop part within the triangle; let A'' be the point nearer B in which the curve cuts CB, where $AA''=a$. Thus the curve $r=p$ cuts from the area of ABC two regions, one enclosed by the loop part within the area and its intercept AB' on the side b, and the other enclosed by the non-loop part within the area and its intercept on the side a, one end point of which is A''. For points within these two regions $r>p$.

For the curve $r=q$ the loop part on BA' lies entirely within the triangle, where A' is defined in the same manner as B'. If $A<B$, the tangent at C enters the angle C, and the curve enters the area of the triangle at C, and emerges at B'', where B'' is the nearer point to A on AC so that $BB''=b$. Within these two regions $r>q$. The altitude BH passes through the points of tangency of the two segments, each of length h_b, from C to the opposite side.

Now consider the possible overlapping of these two sets of regions. The loop regions on AB' and BA' are separated by CH and cannot overlap, since $h_a\geqq h_b>h_c$; similarly, the two non-loop regions terminating at B'' and A'', respectively, are also separated by CH and cannot overlap. Thus we obtain an overlapping if and only if B'' lies within the segment AB', or A'' lies within the

segment BA'. Consider the case where B'' lies within AB'; we then have a region bounded by the part $B''B'$ of b, an arc $B''E$ of the curve $r=q$, an arc EB' of $r=p$. For any point P within this region $B''B'E$ the segment r is the longest; for the point E we have $p=q=r$. A necessary and sufficient condition for a region $B''B'E$ in the case of the special triangles under consideration is

$$a \cos C > c \cos A - (b^2 - c^2 \sin^2 A)^{1/2},$$

$$\cos^2 C - \frac{4}{3} \frac{\sin B}{\sin A} \cos C + \frac{1}{3} < 0,$$

$$\tan A \tan C < 4,$$

where the second inequality results from the first, and the third from the second, since $c \cos A > a \cos C$. In order to have a corresponding region $A''A'E'$ on the side BC, we must have the condition mentioned above $b \sin C < a$ satisfied, and also the inequalities last written after interchanging A and B. We have thus the conditions

$$\sin B \sin C < \sin A, \qquad \tan B \tan C < 4.$$

Denote the values of r for E and E' by r_0 and r_0'. If it can be shown that the smaller of r_0 and r_0' is not less than $3^{1/2}c/2$, then the theorem of the problem will be completely established. It is not very difficult to obtain equations for this purpose, but they appear so involved that it is difficult to deduce from them the necessary inequality.

3597 [1933, 115]. *Proposed by J. E. Trevor, Cornell University.*

Continuous functions $f(x, y)$ and $F(x, y)$ of the independent variables x, y have continuous and non-vanishing first and second derivatives p, q, r, s, t and P, Q, R, S, T respectively. The surface $z=f(x, y)$ is tangent to the *developable surface* $Z=F(x, y)$ along a line whose projection on the x, y plane is a curve $\phi(x, y)=0$. On the line of contact x and y are functions of q. Writing dx/dq and dy/dq for the first derivatives of these functions, and putting $\Delta = rt - s^2$, show that, at points on the line of contact,

$$r - R = + T\Delta \left(\frac{dy}{dq}\right)^2$$

$$s - S = - T\Delta \frac{dx}{dq} \frac{dy}{dq}$$

$$t - T = + T\Delta \left(\frac{dx}{dq}\right)^2.$$

This problem is the mathematical aspect of a problem in thermodynamics.

Solution by J. H. Simester, University of Louisville.

Since the surfaces f and F are tangent along a curve C, we have $p=P$, $q=Q$

along C. Since $RT-S^2=0$, P and Q are functionally related, independent of x and y; hence, along C, p and q are functionally related, and p is thus a function of q only. If $\Delta\neq 0$ along C, p and q, as functions of x and y, can be solved for x and y as functions of p and q, and hence each as a function of q only. If $\Delta=0$ in a region containing a part of C, then the surfaces f and F coincide in at least a part of this region. For F is the envelope of the one parameter family of tangent planes along C for which the parameter may be taken as q; and similarly a part of f belongs to a developable surface which contains a part of C, and for this part of C the tangent planes coincide. If $\Delta\equiv 0$ then the two surfaces coincide completely. Both these cases are excluded by the problem. It appears, however, that it is given by the problem that the surfaces have contact along a curve C such that x and y for C are functions of q, and that their derivatives with respect to q exist.

Since q is to be regarded finally as the independent variable, we may say that $dq\neq 0$. For points on C for which neither dx/dq nor dy/dq is infinite, neither dx nor dy is infinite, and we shall consider only such points. Corresponding to increments dx, dy along C, $dp=dP$ and $dq=dQ$ along C; hence from

$$dP = R\,dx + S\,dy, \qquad dQ = S\,dx + T\,dy \tag{1}$$

result

$$\begin{aligned} T\,dp - S\,dq &= (RT - S^2)dx = 0,\\ R\,dq - S\,dp &= (RT - S^2)dy = 0, \end{aligned} \tag{2}$$

since dx and dy are both finite. Since R, S, T do not vanish, it follows from these equations that dp is not zero. From (1) we have

$$dp\,dq - S(dx\,dp + dy\,dq) = (RT - S^2)dx\,dy = 0. \tag{3}$$

The coefficient of S is not zero since $dpdq\neq 0$. Hence from (2) and (3) we have

$$\begin{aligned} &S = \frac{dp\,dq}{dp\,dx + dq\,dy}, \qquad R = \frac{dp^2}{dp\,dx + dq\,dy}, \qquad T = \frac{dq^2}{dp\,dx + dq\,dy}\\ &R = \frac{Tdp^2}{dq^2}, \qquad S = \frac{Tdp}{dq}\cdot \end{aligned} \tag{4}$$

From $dp=rdx+sdy$, $dq=sdx+tdy$ follow

$$tdp - sdq = \Delta dx, \qquad -sdp + rdq = \Delta dy. \tag{5}$$

Substituting the value of tdp from (5) in the expression for T in (4) we have in succession

$$\begin{aligned} T &= \frac{tdq^2}{\Delta dx^2 + (s\,dx + t\,dy)dq} = \frac{tdq^2}{\Delta dx^2 + dq^2},\\ t - T &= T\Delta\left(\frac{dx}{dq}\right)^2, \end{aligned} \tag{6}$$

where x is a function of q only. Substituting the value of rdq from (5) in the expression for R in (4) we have in succession

$$(7)\qquad \begin{aligned} R &= \frac{r\,dp^2}{\Delta\,dy^2 + (r\,dx + s\,dy)dp} = \frac{r\,dp^2}{\Delta\,dy^2 + dp^2}, \\ r - R &= R\Delta\frac{dy^2}{dp^2} = T\Delta\left(\frac{dy}{dq}\right)^2, \end{aligned}$$

where y is a function of q only. Substituting the value of sdp from (5) in the expression for S in (4) (sdq may be used instead) we have in succession

$$(8)\qquad \begin{aligned} S &= \frac{s\,dp\,dq}{-\Delta dx\,dy + (r\,dx + s\,dy)dq} = \frac{s\,dp\,dq}{-\Delta dx\,dy + dp\,dq}, \\ s - S &= -S\Delta\frac{dx\,dy}{dp\,dq} = -T\Delta\left(\frac{dx}{dq}\right)\left(\frac{dy}{dq}\right). \end{aligned}$$

These are the results desired.

If either dx/dq or dy/dq is infinite these relations are not true if $\Delta\neq 0$, owing to the continuity of r, s, t, R, S, T.

Solved also by the proposer.

3609 [1933, 243]. *Proposed by C. C. Carter, Bluffs, Illinois.*

Determine the area of a quadrilateral in terms of its sides, given in the order a, b, c, d, for which the diagonals are equal and at right angles. Is a solution possible if one of the sides, for example b, is not given?

Solution by S. Vatriquant, A. R. d'Ixelles, Brussels, Belgium.

Let $ABCD$ be the desired quadrilateral whose diagonals intersect at E. If we denote the segments AE, BE, CE, DE, respectively, by x, y, z, t, we have the relations

(1) $x^2 + y^2 = a^2$, (2) $y^2 + z^2 = b^2$,

(3) $z^2 + t^2 = c^2$, (4) $t^2 + x^2 = d^2$.

Adding (1) and (3), and then (2) and (4) we obtain

$$x^2 + y^2 + z^2 + t^2 = a^2 + c^2 = b^2 + d^2.$$

The equations are not independent. The value of b cannot be chosen arbitrarily, for it is determined by the values of the remaining three sides when they are chosen properly.

If u is the length of the diagonals, we have

$$(5)\qquad x + z = y + t = u, \qquad S = u^2/2,$$

where S is the area of the quadrilateral. Eliminating x, y, z and t between (1), (3), (4), (5) we find

$$(6) \qquad F(u) \equiv 2u^4 - 2u^2(a^2 + c^2) + (a^2 - d^2)^2 + (c^2 - d^2)^2 = 0,$$

and the area

$$S = \tfrac{1}{4}\{a^2 + c^2 \pm [(2ac)^2 - (a^2 + c^2 - 2d^2)^2]^{1/2}\}.$$

The particular case $a=c=d$ is easily handled, and if we exclude this case no root of (6) is zero and no one of the two values of S is zero.

The values of S will be both real, unequal, and positive under the conditions

$$(7) \qquad |a - c| < 2^{1/2}d < a + c.$$

The quadrilateral is possible only if

$$(8) \qquad |a - d| < u < a + d, \qquad |c - d| < u < c + d,$$

and in order to locate the positions of $a-d$ and $a+d$ with respect to the roots of (6) we compute

$$F(a - d) = [a^2 - c^2 - 2d(a - d)]^2, \qquad F(a + d) = [a^2 - c^2 + 2d(a + d)]^2.$$

On account of the symmetry of (6) with respect to a and c, we have also

$$F(c - d) = [c^2 - a^2 - 2d(c - d)]^2, \qquad F(c + d) = [c^2 - a^2 + 2d(c + d)]^2.$$

Hence the values of $(a-d)^2$ and $(a+d)^2$, as well as those of $(c-d)^2$ and $(c+d)^2$ do not lie inside the interval between the roots u^2 of the equation (6). The inequalities (8) will be satisfied if half the sum of the roots u^2 of equation (6) lies between $(a-d)^2$ and $(a+d)^2$ as well as between $(c-d)^2$ and $(c+d)^2$. This will be true if the following inequalities are satisfied

$$2(a - d)^2 < a^2 + c^2 < 2(a + d)^2,$$
$$2(c - d)^2 < a^2 + c^2 < 2(c + d)^2.$$

Note by the Editors. This discussion excludes certain cases, where one or more of the quantities $a-d$, $a+d$, $c-d$, $c+d$ are roots, as well as the case where the roots are equal.

Solved also by L. Aroian, A. D. Bradley, W. E. Buker, J. W. Clawson, J. Rosenbaum, F. Underwood, and R. C. Yates.

NEWS AND NOTICES

Readers are invited to contribute to the general interest of this department by sending items to Professor J. H. Weaver, Ohio State University, Columbus, Ohio.

At the next meeting of the International Conference of Mathematicians, to be held in Oslo, in 1936, two gold medals will be awarded to outstanding mathematicians. The award will be made by an international committee appointed for that purpose. The foundation of these medals is due to the efforts of the late Dr. J. C. Fields, F.R.S., Research Professor of Mathematics at the University of Toronto. Dr. Fields was president of the International Congress held at Toronto in 1924, and was the editor of its Proceedings. These Proceed-

ings consist of two large volumes published by the University of Toronto Press. With the proceeds remaining from the sale of these Proceedings, Dr. Fields suggested that a foundation for these medals be established, as a Canadian contribution to the cause of international scientific cooperation. In 1932 at Zurich, the Congress voted international approval of the foundation of these medals. The task of designing a suitable medal which, according to the wish of Dr. Fields, was to be international in character, was entrusted to the Canadian sculptor, R. Tait McKenzie, who has now completed his work. The medal is two and one-half inches in diameter and on the obverse side shows the head of Archimedes facing toward the right.

The annual meeting of the British Association for the Advancement of Science will be held in Aberdeen, Scotland, September 5–12. The president of the conference of delegates of corresponding societies, will be Colonel Sir Henry Lyons, from whom information may be obtained concerning the meeting.

A Mathematical Colloquium will be held in St. Andrews, July 18–28, 1934, under the auspices of the Edinburgh Mathematical Society. A number of eminent mathematicians will give lectures in this Colloquium. Fuller details will be given somewhat later.

Dr. George David Birkhoff, professor of mathematics at Harvard University has been awarded a prize of ten thousand lire ($825) donated by Pope Pius XI in an international competition for the best book on "Systems for the solution of differential equations." The award was made during the exercises inaugurating Pontifical Hall of Science at Vatican City, on December 17, 1933.

The degree of Doctor of Science has been conferred by Georgetown University on Dr. James Robertson, director of the Nautical Almanac Office of the United States Naval Observatory, in recognition of his work in theoretical astronomy.

Secretary-Treasurer W. D. Cairns is spending the second semester, on leave of absence, in visiting various universities of Europe, among these Paris, Rome, Cambridge, and Oxford. During his absence the business of the Mathematical Association will be conducted at Oberlin, Ohio, as usual.

In addition to those whose names were given in a note in the October issue of this bulletin, the following persons are in residence during the present academic year in the school of mathematics at the Institute for Advanced Study at Princeton: Dr. R. L. Echols, Dr. D. H. Lehmer, Dr. Mabel F. Schmeiser, Dr. I. J. Schoenberg, Dr. Anna A. Stafford.

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DIRECTORY

EDITORIAL CORRESPONDENCE should be addressed to the EDITOR-IN-CHIEF, W. B. CARVER, White Hall, Cornell University, Ithaca, N.Y.

BOOKS FOR REVIEW should be addressed to R. A. JOHNSON, Brooklyn College, 66 Court Street, Brooklyn, N.Y.

BUSINESS CORRESPONDENCE should be addressed to the SECRETARY-TREASURER of the Association, W. D. CAIRNS, 33 Peters Hall, Oberlin, Ohio.

CHANGE OF ADDRESS: Members should send notice of any change of address to the SECRETARY-TREASURER, W. D. CAIRNS, Oberlin, Ohio, before the 10th of each month.

MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Eighteenth Summer Meeting of the Association, Williamstown, Mass., Sept. 3-4, 1934.

The following is a list of the sections of the Association, with dates of those section meetings which have been scheduled for 1934 and reported to the Secretary.

ALLEGHENY MOUNTAIN, Pittsburgh, Pa., Feb. 10; Washington, Pa., May 5.
ILLINOIS, Jacksonville, May 4-5.
INDIANA, La Fayette, May 11-12.
IOWA, Des Moines, April 20-21.
KANSAS, Topeka, Mar. 17.
KENTUCKY, May.
LOUISIANA-MISSISSIPPI, Jackson, Miss., Mar. 23-24.
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Williamsburg, Va., May.
MICHIGAN, Ann Arbor, Mar. 17.
MINNESOTA, Northfield, May 12.
MISSOURI.
NEBRASKA, Crete, Apr. 27.
OHIO, Columbus, Apr. 5.
OKLAHOMA, Oklahoma City, Feb. 9.
PHILADELPHIA, Philadelphia, Dec. 1.
ROCKY MOUNTAIN, Colorado Springs, Apr. 20-21.
SOUTHEASTERN, University, Ala., Mar. 30-31.
SOUTHERN CALIFORNIA, Riverside, Mar. 3.
TEXAS, Apr. 17.
WISCONSIN, Oshkosh, May 5.

AFFILIATED ORGANIZATIONS: THE NEW ENGLAND ASSOCIATION OF TEACHERS OF MATHEMATICS. THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS.

The Carus Mathematical Monographs

The CARUS MONOGRAPHS are already fulfilling their mission as intended by the generous donor, MRS. MARY HEGELER CARUS, and her son, DR. EDWARD H. CARUS.

Somewhat more than one-half the members of the ASSOCIATION have taken advantage of the distribution at cost of the first four Monographs already published. Those who neglected to do so at the start may still have the privilege by applying to the Secretary. Each member is entitled to one copy of each Monograph at this special price.

It would be a great tribute to the donor and an honor to the ASSOCIATION if a large majority of the members would subscribe for the complete series.

It is believed that the ASSOCIATION is rendering a great service to mathematics by this enterprise, and a liberal support from the membership constitutes an appropriate vote of confidence in the undertaking.

MONOGRAPHS THUS FAR PUBLISHED

No. 1. *Calculus of Variations,* by PROFESSOR G. A. BLISS. (First Impression, 1925; Second Impression, 1927.)

No. 2. *Analytic Functions of a Complex Variable,* by PROFESSOR D. R. CURTISS. (First Impression, 1926; Second Impression, 1930.)

No. 3. *Mathematical Statistics,* by PROFESSOR H. L. RIETZ. (First Impression, 1927; Second Impression, 1929.)

No. 4. *Projective Geometry,* by PROFESSOR J. W. YOUNG. (First Impression, 1930.)

No. 5. *History of Mathematics in America before 1900,* by PROFESSOR DAVID EUGENE SMITH and DOCTOR JEKUTHIEL GINSBURG. (Ready for distribution, Mar. 1.) Special price to members, $1.25 each.

GEORGE BANTA PUBLISHING COMPANY, MENASHA, WISCONSIN

THE AMERICAN MATHEMATICAL MONTHLY

THE OFFICIAL JOURNAL OF THE
MATHEMATICAL ASSOCIATION OF AMERICA
(INCORPORATED)

DEVOTED TO THE INTERESTS OF COLLEGIATE MATHEMATICS

THE AMERICAN MATHEMATICAL MONTHLY, FOUNDED IN 1894 BY BENJAMIN F. FINKEL, WAS PUBLISHED BY HIM UNTIL 1913. FROM 1913 TO 1916 IT WAS OWNED AND PUBLISHED BY REPRESENTATIVES OF FOURTEEN UNIVERSITIES AND COLLEGES IN THE MIDDLE WEST

VOLUME XLI, 1934

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PUBLISHED TEN TIMES A YEAR

$4.00 a Year, Single Copies 45 cents, to Members
$5.00 a Year, Single Copies 60 cents, to Others

ACTIVITIES OF THE COMMISSION ON THE TRAINING AND UTILIZATION OF ADVANCED STUDENTS IN MATHEMATICS

At the meeting of the Mathematics Association of America in Chicago in June, 1933, the Trustees of the Association requested the President to appoint a commission on "The Training and Utilization of Advanced Students in Mathematics." President Dresden has appointed the following commission: William Betz (Rochester High School, Rochester, N.Y.), W. L. Hart (Minnesota), J. O. Hassler (Oklahoma), E. R. Hedrick (California), E. V. Huntington (Harvard), M. H. Ingraham (Wisconsin), R. G. D. Richardson (Brown), H. E. Slaught (Chicago), E. B. Stouffer (Kansas), E. J. Moulton (Northwestern), Chairman.

At the recent Christmas meeting of the Association at Cambridge the Commission made a report of progress. In view of the present scarcity of openings in colleges and universities for teachers of mathematics, it appears that a considerable number of highly trained students in mathematics will need to look for positions in other fields. One of these fields is that of the junior colleges and secondary schools. With regard to this latter field the Commission recommended the publication of the following statements, and the transmission of them to graduate students in mathematics:

(1) Advanced students in mathematics who may become candidates for positions in secondary schools should (a) prepare to meet the legal requirements of the state in which they hope to teach, which requirements include a certain number of credits in Education and Practice Teaching; and (b) give intelligent consideration to their whole program of study in preparation for such work.

(2) The Commission is undertaking a study which may lead to the formulation of recommendations concerning the training on both the professional and subject matter sides of teachers of mathematics in the secondary and college fields.

The Commission feels that there is a very important service to mathematics to be rendered by thoroughly trained teachers of mathematics in the field of secondary education. Members of the Commission have expressed their conviction that problems facing such teachers merit the best efforts of competent young mathematicians.

THE CARUS MATHEMATICAL MONOGRAPHS

The Carus Mathematical Monographs have registered a gratifying success since the series began in 1925 with "Calculus of Variations" by Professor G. A. Bliss. It was the hope at that time that at least one number could be issued each year, and in fact this hope was realized for the next two years; with "Analytic Functions of a Complex Variable" by Professor D. R. Curtiss in 1926, and "Mathematical Statistics" by Professor H. L. Rietz in 1927. However, the approach of the depression interrupted the series until 1930 when the fourth number on "Projective Geometry" by Professor J. W. Young was published, and

the continued depression again caused a delay of three years until the fifth number was made ready for publication in 1933 and was actually on the presses in February, 1934.

It was the desire of the donor that a "series of expository presentations of the best thought and keenest researches in pure and applied mathematics" should be made accessible to members of the Association at nominal cost, and the scope of the series was intended to "include also historical and biographical monographs." The first four monographs were strictly expository presentations, but the fifth is of a purely historical character, and quite appropriately, it would seem, it deals with the development of mathematics in our own country. It is entitled "A History of Mathematics in America before 1900," by Professors D. E. Smith and Jekuthiel Ginsburg. The scope of this monograph was limited to the period before 1900, (1) because the material available was more than ample for a volume of two hundred pages and (2) because the dividing line between the nineteenth and twentieth centuries marked a real line of cleavage in the development of mathematics in America. The history of mathematics in our country in the first third of the twentieth century will constitute a large volume by itself and should be written in 1950.

It is a matter of interest to know that the first three of these monographs have gone into second editions and the fourth one is very nearly at that stage. Well over ten thousand copies have been sold or used for review or advertising purposes, and of the total number 4852 have gone to members of the Association and 5213 to non-members. The uniform list price to the general public is \$2.00 per copy, and the price to members is \$1.25 substantially at cost; and the entire receipts from sales to members are returned to the Carus Publication Fund of the Association.

Members order their copies (one to each) through Secretary Cairns, and non-members through the Open Court Publishing Company, 149 East Huron Street, Chicago, Illinois. Copies for class use may be ordered on the customary trade discount basis.

It is believed that the Association is doing a real service to the cause of mathematics in America by this enterprise and a liberal support from the membership constitutes an appropriate vote of confidence in the undertaking as well as a tribute to the donor.

H. E. SLAUGHT

THE SEVENTEENTH ANNUAL MEETING OF THE KENTUCKY SECTION

The seventeenth annual meeting of the Kentucky Section of the Mathematical Association of America was held at Georgetown College, Georgetown, Kentucky, on Saturday, May 13, 1933. Professor Charles Hatfield presided at both morning and afternoon sessions.

The attendance was forty-three, including the following eighteen members of the Association: N. B. Allison, P. P. Boyd, L. W. Cohen, J. M. Davis, H. H.

Downing, A. R. Fehn, Charles Hatfield, W. R. Hutcherson, C. G. Latimer, F. Elizabeth LeStourgeon, Buena C. Mathis, W. L. Moore, Sister Charles Mary Morrison, Mabel I. Nowlan, Sallie E. Pence, D. W. Pugsley, Guy Stevenson, H. W. Wright.

Officers were elected for the year 1933–34 as follows: Chairman, L. A. Cohen, University of Kentucky and Secretary, A. R. Fehn, Centre College.

The following papers were presented:

1. "An example in cyclic involution of order seven" by Professor W. R. Hutcherson, Berea College.

2. "Infinite determinants" by Professor L. W. Cohen, University of Kentucky.

3. "Some geometric and trigonometric proofs of the Theorem of Morley" by Sister Charles Mary Morrison, Nazareth College.

4. "Genera of binary forms" by D. B. Palmeter, University of Kentucky, by invitation.

5. "Factorization of algebraic numbers and ideals" by Professor C. G. Latimer, University of Kentucky.

6. "Some adventures in the quadratic" by Mabel I. Nowlan, Bethel Woman's College.

7. "The equation of the normal correlation surface" by Buena C. Mathis, University of Kentucky.

8. "Some applications of the theory of probability" by Professor Guy Stevenson, University of Louisville.

9. "The modern trend in teaching arithmetic" by Professor F. A. Engle, Eastern Kentucky State Teachers College, by invitation.

10. "Fifty years experience in the teaching of high school mathematics" by Professor J. M. Guilliams, Berea College, by invitation.

11. "The place of mathematics in the liberal arts college" by Professor W. L. Moore, University of Louisville, and Professor H. A. Wright, Transylvania College.

Abstracts of some of these papers follow, numbered in accordance with their place on the program:

1. A cyclic involution I_n (n any positive prime) means that the points of an algebraic entity (curve, surface, variety) are arranged in sets of n, each point of a set determining the set. The cases for $n=2$, 3 have been extensively studied by Godeaux, Snyder, and others. For $n \geqq 5$, the theory is not so well established in space. The following illustration of the case $n=7$ was shown. A cubic surface

$$F_3(x_1x_2x_3x_4) \equiv ax_2^2x_3 + bx_3^2x_1 + cx_1x_2x_4 = 0$$

in ordinary three way space was exhibited as being invariant under the collineation $x_1':x_2':x_3':x_4' = x_1:\epsilon x_2:\epsilon^2x_3:\epsilon^3x_4$ where $\epsilon^7=1$. Four imperfect points were found and the properties of the two that were simple on F_3 were discussed very briefly.

3. The proofs for the Theorem of Morley were taken from the following sources: (a) an article by F. G. Taylor and W. L. Marr in the Proceedings of the Edinburgh Mathematical Society, vol. 32, 1914, pp. 119–150; (b) an article by Philip Franklin in Contributions of the Mathematics Department of the Massachusetts Institute of Technology, November, 1926; and (c) an article by J. O. Englehardt in the American Mathematical Monthly, November, 1930.

4. An introduction was given to the study of characters and genera of integral primitive binary quadratic forms, with special mention of some important properties of characters and genera. Attention was called to an equation which is satisfied by the character values of the forms of any given discriminant, and which suggests a method of proof of the theorem on the number of genera relative to the number of characters for any given discriminant.

6. The translation, in recent years, of papyri written in the hieroglyphics and hieratic characters of the Egyptians and the cuneiform writing of the Babylonians, has established the fact that these two nations were the mathematical forerunners of Greece. The testimony of the noted Greek writers themselves is to the same effect. A Graeco-Egyptian papyrus, acquired in Egypt in 1921 by the late Professor Kelsey, belongs to about the second century A.D. It shows considerable advancement over the problems of ancient Egypt, but the Egyptian influence, in subject matter and treatment, is unmistakable. This document helps to connect the mathematics of early Egypt and Greece with the algebra of the Hindus, the Arabs, and our own European forefathers. In Greece geometry was restricted to the ruler and compasses, but there were geometric solutions of the quadratic equation even before Euclid. The geometric version of the linear equation in one unknown is given in the first book of Euclid and the principles of its solution are elaborated in the sixth book and in Euclid's data, and lead to the solution of the three types of quadratic equations (the ancients did not recognize the possibility of transposition) and also to the construction of the pentagon. The Babylonians had a numerical solution of a complete quadratic equation. Many of their achievements were due to their remarkable development of scientific astronomy, notably their work in series, measurements connected with the circle, and measurement of angles. The series are also found in Egypt, as well as the Pythagorean theorem, linear and quadratic equations, and the symbol for square root. The area of the circle is given as $8/9d^2$. The formula for the volume of a truncated pyramid, $\frac{1}{3}h(a^2+ab+b^2)$, and that for the suface of a hemisphere, $2\pi r^2$, are given in the Moscow papyrus, thus relieving the Greeks of the responsibility for their discovery. The Hindus, and possibly the Chinese, knew the Pythagorean theorem before the day of Pythagoras. The Arabs gave the area of the circle as

$$d^2 - \frac{1}{2}d^2 - \frac{1}{7} \text{ of } \frac{1}{2}d^2,$$

showing clearly the influence of the Egyptian unit fractions. Al Khowarizmi was

the first to bring out sharply the parallelism between analytical and geometrical solutions of quadratic equations. The Arabs also trisected the angle and proved that the regular polygons of seven and nine sides reduce to cubic equations, which they solved by the intersection of conics. More recently mathematicians have been able to prove, by methods involving complex numbers, that these cubics are not reducible and that the constructions of the regular polygons of seven and nine sides, as well as the trisection of the angle, are not possible with ruler and compasses. Also, it is shown that the regular pentagon, constructible with ruler and compasses, corresponds to an algebraic equation which can be solved in terms of quadratic irrationalities. The great genius Gauss, when a youth of nineteen, showed that the equation $x^{17}-1=0$ can be solved by a series of quadratic equations and that this solution corresponds to the fact that a regular polygon of seventeen sides can be inscribed in a circle using only ruler and compasses.

(*Note*: The major portion of the material for this paper was reproduced from class-room notes taken under the instruction of Professor Louis C. Karpinski, and used by his permission.)

7. This paper dealt with the problem of the correlation of two variables, X and Y, in a given set of data. Instead of the well-known "normal curve," $y=f(x)$, which gives the probability of a deviation occurring between x and $x+\Delta x$, the "normal surface," $z=f(x, y)$, was derived. It gives the probability for the deviations between x and $x+\Delta x$ and y and $y+\Delta y$ occurring simultaneously. From the equation of the surface it was shown that a section through the surface perpendicular to the xy plane and in any direction would be a normal curve. A method for extending the idea to more variables and consequently to multiple correlation and hyper-space figures was indicated.

8. A method of determining the value of π by probability was given, and a small Galton machine illustrating the normal frequency curve was shown.

9. This article attempts to present the changes which have taken place in the aims and purposes of the arithmetic course, in time allotments, and in content and method of instruction from 1850 to the present. Progress made in the teaching of arithmetic in the last fifteen years has been significant and fundamental, in the methods of teaching and in the selection and organization of subject matter. In drill exercises there is a tendency for easy facts to recur much more frequently than the more difficult ones. Tests and test results should be closely connected with the teaching and learning process.

Several studies have been made to determine the amount of arithmetic at the command of high school graduates and college students. These studies show that most of them have neither the skill in computation, mastery of facts, nor ability to solve problems to fit them to teach arithmetic. Arithmetic is a professional subject that is necessary for the best preparation of the elementary-school teacher.

A. R. FEHN, *Secretary*

THE NINTH ANNUAL MEETING OF THE LOUISIANA-MISSISSIPPI SECTION

The ninth annual meeting of the Louisiana-Mississippi Section of the Mathematical Association of America was held at the Louisiana Polytechnic Institute, Ruston, Louisiana, on Friday and Saturday, March 3 and 4, 1933, in connection with the Sectional meeting of the National Council of Teachers of Mathematics. Professor C. D. Smith, Chairman of the Section, presided. Among those attending were the following fourteen members of the Association: V. W. Adkisson, T. A. Bickerstaff, A. M. Harding, H. M. Hosford, Dorothy McCoy, A. C. Maddox, Israel Maizlish, B. E. Mitchell, I. C. Nichols, C. D. Smith, H. L. Smith, P. K. Smith, B. A. Tucker, W. P. Webber.

The afternoon meeting of March 3 was given to organization with the following officers elected for the year 1934: Chairman, P. K. Smith, Louisiana Polytechnic Institute; Vice-Chairmen, J. A. Hardin, Centenary College, and W. V. Parker, Mississippi Woman's College; Secretary, Dorothy McCoy, Belhaven College. The 1935 meeting will be held at Louisiana College, Alexandria, Louisiana. A motion was made and carried to invite the State of Arkansas to join the Louisiana-Mississippi Section of the Association to make a tri-state section.

The Ruston Branch of the American Association of University Women and the Men's Faculty Club of the Louisiana Polytechnic Institute entertained with a joint coffee on Friday afternoon. A joint banquet was held on Friday evening.

Dr. A. M. Harding, Director of the Extension Division of the University of Arkansas, was the speaker at the public lecture on Friday evening. His subject was "The Depths of Space."

At the meeting on March 4 the following papers were presented:

1. "Education for the new era" by Dr. A. M. Harding, University of Arkansas.
2. "On a certain type of symmetric determinants" (by title) by Professor W. V. Parker, Mississippi Woman's College.
3. "The complex complete quadrangle" by Professor B. E. Mitchell, Millsaps College.
4. "Report on special problems in research" by Professor I. C. Nichols, Louisiana State University.

DEBORAH MAY HICKEY, *Secretary*

FUNDAMENTAL CONCEPTS IN THE THEORY OF PROBABILITY[1]

By T. C. FRY, Bell Telephone Laboratories

1. *Introduction.* The word "probability" is used in two senses: in ordinary

[1] A paper read before the Mathematical Association of America at the joint meeting with Section A of the A.A.A.S. held in Chicago, June 22, 1933.

life it refers to the degree of uncertainty one may feel about an event; in mathematics it refers to a number. There is nothing unusual about making a word do double duty in this way. In fact, the language is full of such verbal Siamese twins. We meet them whenever we talk about anything that can be measured. For example, length to a mathematician is a number; to the man on the street it is an abstract idea of extension in space. Time to the mathematician is a number; to the man on the street it is something vaguely related to the growth of his memory.

It would be difficult to say by just what bond these twin meanings are joined together, though there must be such a bond; otherwise the results of mathematical investigations could never be applied to every-day life. Whatever this bond may be, it belongs to the realm of philosophy. As mathematicians, we are accustomed to say that we have no interest in it and would not even care if it did not exist. Yet I think that statement must be interpreted in a rather Pickwickian sense. As mathematicians of course we are interested only in the relations among numbers. As mathematicians we are not interested in philosophy. Neither as mathematicians are we interested in food, wine, women or song. But no one respects us the less for occasionally functioning as something else than mathematicians. And when we do the bystander is apt to note a lively interest in any of these forbidden topics.

So I am sure I need not apologize for talking today about the three leading definitions of mathematical probability with a view to pointing out their relative advantages and disadvantages, not only from the logical standpoint, but also as regards their degree of resemblance to that other twin probability in the world of every-day affairs. And in order to make the latter part of my talk proceed as smoothly as may be it will be of advantage to begin with certain remarks of a rather definitely metaphysical sort.

First, let me point out that when we speak of defining length, time or temperature it is always the number twin of the Siamese pair that we are talking about, and never the physical concept itself. In the case of length, for example, we do not give a precise description of the notion of extension in space, but instead we give a set of instructions as to how to go about measuring an object. And this is true also of our definitions of probability. None of them tells us what the abstract idea of probability really means. They merely say, "Do such and such things and you will get a number called probability."

In the second place, I want to mention the fact that such definitions are seldom, if ever, "general" in one important sense. They may serve our purposes as mathematicians quite well enough; but when we try to interpret them in terms of the physical world we generally find plenty of situations in which the instructions simply cannot be carried out at all.

Take the idea of length, for example. The broad concept of extension in space is clear enough. It applies equally well to the wavelength of light, to a meter-stick or to the diameter of Betelgeuse. But to define the mathematical number which we call length in such a way as to apply to all three cases is not

so easy. We used to do it by saying, "Lay down some meter-sticks alongside the thing you are measuring; the number it takes to reach from one end to the other is the length of the object." Now, practically speaking, that is a good enough set of rules for measuring a room or a field, but you can't measure the length of a light wave that way, nor the distance to Jupiter either. More recently it has been much more polite to say, "Put a mirror at one end of the object to be measured and stand at the other; the number of times your watch ticks before your reflection gets back to you is the length of the object." But you can't measure the length of a light wave that way, nor even the length of a broom handle.

We don't abandon these definitions as worthless, however, merely because of their limited usefulness. For though one abstract differential relation might serve our purposes well enough in certain parts of pure mathematics, when we reach the applied field we find more than one definition of length not only convenient but absolutely necessary.

This is also true of probability. All the various ways of measuring it are useful, and no one of them could be dispensed with without appreciable loss, though from a purely logical standpoint some may stand out as definitely superior to others.

The third thing I want you to note is that in all such matters the idea of equality is a constant source of trouble. If you search carefully you will find either it, or something very like it, lurking unexplained in every measurement definition. When you define length by the meter-stick method it hides in the idea of reaching from one end to the other—what do you mean by an end of a meter-stick coinciding with an end of the measured object?—and when you define length by the ticks of a clock the undefined idea of equality appears under the name of simultaneity.

Even in mathematics we are none too clear about it. Take, for example, the simplest possible case of positive integers. I may say the number of fingers on my right hand equals the number on my left because I can put them in one-to-one correspondence; that is, because I can touch the fingers of either hand with the fingers of the other, so. But when I say I can match them against five light-years what in the world does the verb "can" mean? Or, to take another example, in just what sense *can* I match them against five points on a line?

The same is true of probability. Indeed the notion of equality has probably caused more headaches in the theory of probability than in any other scientific subject. I shall make no attempt to lead you along a smooth and easy path from which such stumbling blocks have been removed. On the contrary, it will be one of my major purposes to point out how the notion of equally likely events arises even in those definitions which are most often said to avoid it. But I think we may avoid the headache stage by being just a bit sensible and recognizing the fact that equality is no more unruly here than elsewhere in the thought domain.

The fourth philosophical remark, and this is the last, is this: while I can't

for the life of me tell you what the idea of length is, but only tell you how to measure it, I may be able to make illuminating comments about it. The same is true of any concept which we get by the process which the psychologists call abstraction, that is, by sorting out from a group of dissimilar ideas or objects those respects in which they are all alike. The illuminating remarks which people make about such concepts appeal to us because we just naturally feel the truth of them. They are things we knew all along, perhaps without knowing we knew them—like the man who talked prose all his life without knowing it.

For instance, if I say as I did a while ago that length is the concept of extension in space, you admit the sense of it, though it tells you exactly nothing at all about length unless you already know it. Or if I say, as some of our philosophers would do, "The length of an object is the thing it has in common with a lot of other objects that differ from it in every other respect," you get the point all right, and probably admit that it suggests mental processes by means of which such ideas arise, but it certainly does not tell you what length *is*. If that were a definition of length, then length and color would mean exactly the same thing; for you can replace the word "length" by "color" and the statement is just as true as before.

Now in its non-mathematical sense probability is just such an abstract concept. If you don't know what it means I can't hope to tell you. But if you do I may be able to say some sensible—perhaps even helpful—things about it. And I think we can proceed with a greater degree of assurance if we find we can agree, as I hope we shall, on three or four such statements.

To begin with, probability has something to do with ignorance. We don't ask, "What is the probability that it will rain today?" if we are dead sure it will. That, I think, is obvious—which merely means that so far we understand one another.

Next, the idea is a quantitative one. Some things are more probable than others. For example, the next man to enter this room is more likely to be an American than a foreigner. Indeed if the idea of probability were not quantitative it would not be a fit subject for mathematical investigation at all and we would be wise to call the mathematical theory of probability by some less misleading name.

Finally, it isn't the *extent* of our ignorance which is at issue, but something else. Here is a coin. What is the probability that it lies heads up? You admit rather complete ignorance I am sure. Well let me reduce your ignorance by telling you something about the coin. It is a quarter; it is about ten years old; it is rather more badly worn than one would expect for its age. Now you are less ignorant than you were before about this particular coin. What is the probability that it lies heads up? Just what it was before, to be sure. The additional information I have given you did not change matters at all. That is what I mean by saying that it is not the extent of our ignorance that is at issue, but something else. I have tried hard to find some incisive way of saying just what that something else is, but without very notable success. The best I have

ever been able to do is this: "Probability is a measure of the *importance* of our state of ignorance." It is just because the additional information which I gave you was of no importance whatever in determining which side was up that it had no influence on the probability.

Proceeding along this line we might say that two events are equally likely when our ignorance about each is equally important; which means much if you know what it means, and exactly nothing if you do not. Or we might say two events are equally likely when we have no reason for thinking they are not, about which the same remark might be made. Or again we might say two events are equally likely if they differ in no significant way, a form of speech which many people like better than the other two, though so far as I can see they all mean much the same thing. Whatever form of speech we might adopt, however, we would find ourselves calling two events equally likely when we knew so little about either of them as to leave us in a state of complete equivocation; and also if we knew a great deal about them both, but the things which we knew balanced off against one another so nicely as to leave us still in a state of complete equivocation.

To sum up then, the four metaphysical remarks which have seemed to be in order before we can properly appraise the definitions about which I am to speak are these: First, that the definitions refer to the *number* which measures probability and not to *probability* itself. Second, that none of the definitions can be applied to every important situation with which the theory deals. Third, all of them assume some knowledge in advance as to what we mean by the words "equally likely." And fourth, this phrase cannot be defined because the word "likely" refers to the undefinable concept of probability, not to its mathematical twin.

2. *Laplace's Definition.* Now for the definitions. The first one was already in existence, though not very old, when the Century of Progress began. Historically it grew out of the study of a game played with dice, and there is probably no better way than that for us to approach it today.

Suppose then that I ask this question: If I throw a die what is the probability that the ace will turn up? Now I am certainly ignorant as to what the result will be. It may be an ace, or a two, three, four, five or six. I feel equally ignorant about them all. Hence all I can say in advance is that all six possibilities are equally probable—using "probable," you see, in its every-day sense.

Now if there is to be any analogy at all between the every-day idea and the mathematical number it is certainly necessary that equal probabilities be represented by equal numbers, and this of course requires that if we denote the probability of an ace appearing by P the probability of any other face appearing should also be P.

It also seems natural to say that the probability of either an ace or a deuce is twice as great as that of an ace; and that the probability of either an ace, deuce or three is three times as great; and so on. It is not *necessary* to adopt this convention any more than it is *necessary* to say that the length of a stick is twice

as great as the lengths of the two equal pieces into which it can be cut. In the theory of musical sounds, for example, we are accustomed to measure frequencies in octaves, that is, on a logarithmic scale instead of a linear one; and it is also customary to measure the loudness of sounds on a logarithmic scale of decibels. There is nothing to prevent us from measuring lengths the same way, though experience has taught us that it would not be convenient to do so. And probability, too, could be measured by means of some other convention than the one I have stated, though again experience has shown that this one is very satisfactory.

You readily see what I am coming to. We can say immediately that the probability of getting either an ace or a deuce is $2P$; the probability of getting either an ace, deuce or three is $3P$; and so on. In particular, the probability of getting either an ace, deuce, three, four, five or six is $6P$.

But our state of ignorance regarding this last combination of events is in quite a different class from that of any of the rest. There is no longer any element of doubt entering into it; we know absolutely that one or the other of the six faces must appear. Hence the number $6P$ must represent certainty.

Here again we come up against a convention: certainty is represented by the number 1. There is no necessity for this particular convention. There are reasons why it is convenient, perhaps more convenient than any other choice which we could make. But it still remains a convention nevertheless, of exactly the same sort as our conventions regarding the use of millimeters or feet or light-years as units of length. And if we adopt this convention we arrive at the equation $6P=1$, from which we get at once $P=1/6$.

It is by means of this line of argument that we are led to a definition of mathematical probability which reads, "If one or the other of n events must happen and all are equally likely, the probability that a particular one will happen is $1/n$."

This is essentially Laplace's definition and, as I have said, is just a little more than a century old.

Now, what can be said by way of assessing the merits and demerits of this definition?

On the favorable side we must list the fact that it is by all odds the most direct and explicit definition that has ever been invented. When we can use it at all we can write down the value of P at once; not approximately, but exactly and without the least trace of ambiguity.

We must also list in its favor the fact that it follows so closely after the usual pattern of scientific measurement in that it names a unit of measure, certainty, just as all physical measuring methods do, and adopts the "either-or" convention which tells us how to subdivide this unit just as all physical measuring systems do.

Its principal fault lies in the number of problems to which it cannot be applied. You will readily see why if I mention two or three of them.

Suppose, for example, that I have a badly warped die on which the face

marked six is very much larger than the opposite face marked one. I can then say at once that six is much more likely to be down and ace up than ace is to be down and six up. So my six faces are not equally likely to appear and I cannot apply the definition at all.

Or suppose I ask for the probability of a man of 40 dying before he reaches 60. Obviously a man of 40 must die between 40 and 41 or between 41 and 42 or within some other one-year period up to, let us say, age 140. But though I have thus a set of 100 alternatives one or the other of which must occur, I know perfectly well that the likelihood of his dying at ages above 100 is very much smaller than at ages below, and the definition again gives me no direct way of computing the probability.

A second objection is also often made and I must, therefore, mention it in deference to those who believe it valid, though I do not myself subscribe to that view. It is, that in defining probability by means of a set of equally probable events we have proceeded in a vicious circle and defined probability in terms of itself. My own view, of course, is that the probability which we *define* is the number side of our Siamese twins, and the probability that occurs in the "equally probable events" refers to the other side—that is, to the essentially undefinable concept.

But even if my view be not admitted, it still remains true, as I shall show you later on, that this particular objection is not overwhelmingly damning, since the other definitions are all tarred with the same stick.

3. *The Series Definition.* The second definition to which I wish to refer also grew out of the study of dice games. In its rudimentary form it is doubtless much older than the one we have been talking about; for the germ of the idea must have been dimly grasped before any gambling game of the roulette type could be invented, and such games have existed for many centuries. However far into the past its roots may extend, it only blossomed into precise mathematical language in the course of the Century of Progress.

To answer the question "What is the chance of tossing an ace with a die?" this definition requires us to proceed as follows: We first toss the die and note which face appears; then we toss it again and note which face appears; and so on indefinitely. If we keep account of the number of throws and of the number of aces that have appeared we may pause after the first N throws and remark that we have gotten just n aces; or, in other words, that the *proportion* of aces is n/N. We can then go on tossing a while longer, eventually pausing after we have made N' throws in all to note that exactly n' have been aces, so that the proportion is now n'/N'.

The definition then says, that if we keep this process up indefinitely the proportion of aces will approach a limiting value P which is the "probability of throwing an ace."

This is the general picture: a never-ending succession of trials, a computation from time to time of the proportion of trials which have resulted in the

desired way, and the determination of the limit toward which this proportion tends as the number of trials gets larger and larger.

In a way this definition is even simpler than Laplace's. I have, however, left out one very important detail which I must call to your attention before we can properly assess its merits. Perhaps I can best make it clear by means of an illustration.

Suppose we are watching an automatic screw machine turning out screws at the rate of thousands an hour. Most of them will be perfectly good screws. Occasionally a defective one will appear. So I ask, "What is the probability of the machine making a defective screw?" This is an entirely sensible question and in a general way we know that the answer, when we get it, should be quite small. But suppose we attempt to apply the definition as I have just phrased it. We need only come around once each day and count the number of bad screws produced that day. Maybe the machine makes fifty thousand screws a day of which one hundred are found to be bad the first day, one hundred and ten the second, eighty the third, and so on. If we adopt these figures we find the proportion of bad screws to be 2/1000 at the end of the first day, a little more than that for the first two days' output and a little less for the first three. Naturally, we expect the limit, when we get it, to be somewhere around 0.002. But if we keep up this observation day after day and year after year, as the definition requires, what really happens is this: that eventually the machine begins to wear out and the screws become poorer and poorer until they are all quite bad. After this state of affairs has been reached the proportion of bad screws gets bigger and bigger and eventually approaches unity.

Now you could defend this result as being the answer to a certain question, but it is not the question you and I had in mind years before when we asked what the probability was of getting a bad screw. If the series definition were really supposed to give this result, then the series definition would be answering the wrong question. In reality, of course, the inventors of the series definition never intended me to let my screw machine wear out, and so we must go back and patch up our statement of the definition so as to exclude cases of this sort.

This has been done in a number of different ways and it is quite out of the question for me to discuss them all today. I therefore choose the one which is easiest for me to talk about and which fortunately happens to be the one most frequently adopted in textbooks on the theory of probability. According to this scheme the only change which we need make in our definition is to require that all our trials be conducted under the same essential conditions.

But what do these words "the same essential conditions" mean? Obviously what they mean is just this: that the conditions must not be changed in any way which will either increase or decrease our doubts as to the quality of the machine's output, which is merely another way of saying that the probability of getting a bad screw shall be the same today or tomorrow or at any other stage of our infinite succession of trials. Here again we have met the stumbling block of equally probable events right smack in the middle of that definition

which has most frequently been adopted by those mathematicians who have been loudest in their denunciation of Laplace's definition for this very defect.

So far as I am concerned, of course, it is unavoidable that this should be so and is therefore not in any sense a defect of the definition. But it has seemed necessary to make mention of it, nevertheless.

Now that we have a clear picture of just exactly what the definition means we are able to appraise its advantages and disadvantages on much the same basis as was adopted in the case of Laplace's definition.

On the favorable side we have the fact that the definition gives us a practicable method of computing probabilities in a great many cases where the earlier one fell down. We can apply it to our screw machine, for instance, provided we stop our observations soon enough. Or we can use it to find the probability of a badly warped die landing ace up, again provided we do not extend our observations so far as to cause appreciable wear in the die itself. We can even apply it to such a problem as finding the probability of a man aged 40 dying before he reaches 60. All we need to do in this case is to make a record of the ages at death of a very large number of men, each of whom can be regarded as a separate trial in our sequence.

On the unfavorable side we have two kinds of objections; one is practical, the other is theoretical. We shall talk of the practical sort first.

It has already occurred to you, no doubt, that the use of this definition never gives *exact* results, but only *approximate* ones, because it is never possible to extend our sequence of trials to infinity. And of course if we stop short of infinity our final proportion of successes will not be the true limiting value but only a more or less satisfactory approximation to it.

What makes this situation worse is the fact that there are frequently cases where it is inexpedient, if not impossible, to extend our sequence of trials very far. I can give you an illustration. Suppose for some reason an ethnologist wishes to know the probability of a Cro-Magnon man having a skull capacity as great as the average of modern men. All that our ethnologist has to do to answer this question, of course, is to order a carload of Cro-Magnon skulls and measure the volume of each one. But I think he might have some difficulty in locating the wholesaler who would fill his order.

Or again take the case of the life insurance company when it asks "What is the chance that a man aged 40 will die before reaching 60?" It is thinking about the 40 year old man who just applied for a policy and wondering how much it should charge him. But in the very nature of things the only mortality data which it can possibly have refers to men who were 40 years of age a good many years ago. That, you see, is quite a different thing, because the probability in question changes from year to year with the improvement of medical science, just as the product of the screw machine changed as the machine wore out.

All these practical objections are real enough; but they do not destroy the value of the definition since in so many important problems we have no other way of getting an answer at all. There are important practical problems, how-

ever, to which the first definition can be applied directly and when they arise it is a matter of common experience that the first definition is by all odds the more convenient.

I may mention two such examples. The theory of probability has come to play an extremely fundamental part in modern atomic physics and you will find that the physicists who deal with such problems always rely upon the method of measurement suggested by the Laplacian definition no matter what their philosophy may be regarding how probability ought to be measured.

The theory of probability also plays an extremely essential part in the economic design of modern telephone exchanges and here again it is the Laplacian definition rather than the series definition which proves to furnish the best service.

We come now to theoretical objections. These I shall pass over in a word.

You will recall that the series definition really contains two statements instead of one. The first is that the proportion n/N approaches a limit. The other is, that this limit is called the probability. Only the second of these statements is the definition proper. The first is either an established fact (that is, a theorem), or else it is a plain assumption. In reality, as nobody has ever shown it to be true, we must regard it as a pure hypothesis, and when we do this we find that it comes in conflict either with our notion of a limit or with the requirement that the trials occur under the same essential conditions. Thus we meet with a queer sort of theoretical dilemma in which both horns seem to have branches. The subject is too complicated for me to go into in this place, and really is of no practical importance any way since the method of measurement implied by the definition is so indispensable in practice that we shall undoubtedly continue to use it no matter what the philosophers may have to say about it.[1]

4. *The Population.* The third definition to which I wish to refer is a product of this generation and grew, not out of the study of dice games, but out of biological problems of a sort to which I have already rather facetiously referred. As a young man the now famous statistician, Karl Pearson, set out to investigate these problems. For the purpose he needed some new mathematical theory, which he expressed in a language which has been very widely copied since that time.

Perhaps I can explain his point of view most briefly by talking about one of

[1] Just now a favorite form of series definition is that given by Mises. He defines probability in terms of statistical sequences called Kollektiven. The essential departure from the older definitions lies in the fact that not every statistical sequence is a Kollektiv: for example, any sequence for which n/N does not approach a limit is *by definition* not a Kollektiv.

On the practical side this concept has nothing unique to recommend it, in spite of the fact that its inventor is an engineer. To use it, one must go through the same motions as are required by the earlier forms of series definitions, but with the mental reservation that one cannot be sure he is dealing with a Kollektiv until the number of trials becomes infinite—that is, never.

On the theoretical side, in my opinion, it is even worse. If it belongs to the House of Logic at all, it must be by plea of bar sinister.

the questions with which Pearson actually dealt; namely, whether a certain group of prehistoric skulls really represented a single race of men, as certain ethnologists contended, or whether they had come down from two races having well differentiated cranial indices, as other ethnologists believed.

Pearson's picture in dealing with such problems was this. He thought of the vast population of such men that had once existed and imagined that the meager supply of skulls which was available had been selected at random from this population and sent down through the ages for his investigation. And he defined the probability of a skull having a given cephalic index as the proportion of the original population for which this was true.

This idea is so graphic that I need hardly say more by way of explaining it. Indeed its graphic quality is its greatest asset; for as an aid to visualization, particularly in the biological and social sciences, it has every other definition of probability backed off the map.

But it has its defects too. Suppose, for example, that we try to apply it to a dice problem. Before that can be done we must invent an imaginary population in which aces, deuces and so on occur in the proper proportions. Then we call these proportions probabilities. So the proper proportions and the probabilities are in reality the same thing, and the imaginary population cannot be set up at all until the numerical probabilities are known. This is a true vicious circle.

It is only when a true population exists, as in the case of Pearson's prehistoric man, that the definition can lay claim to any semblance of logical validity.

I must also show you that in common with our other definitions it depends upon the notion of equally likely events. In this case the concept does not arise in the definition of the population itself but in the notion of random sampling which is inextricably woven with it. It was only by assuming that the handfull of skulls which had been preserved was a true *random* sample from the original population that Pearson was able to draw any conclusions whatever about the question which he had set out to answer. If he had admitted, for instance, that skulls of high cranial index were much more likely to survive than skulls of low cranial index the whole theory would have gone to pot. And the same situation exists whenever we attempt to make use of this theory at all. Now what do we mean by a *random* sample? Obviously one made in such a way that every individual in the whole population has exactly the same chance of being included as every other, so that so far as the sampling process is concerned the population consists of a set of equally likely individuals, to which the Laplacian definition could be applied without any difficulty whatever.

5. *Conclusion.* We have thus concluded our evaluation of the three most important definitions of probability; and it may not be amiss for me to repeat briefly the points which I have been at greatest pains to leave with you.

One is, that we are really better off with many definitions than with only one; because our world of scientific experience is so broad and complex that a single definition can seldom be good enough to satisfy all our practical needs.

Another is, that there is a connection between our mathematical thought and the world about us—a connection which we cannot shake off by the mere denial of any interest in it, because the very foundations and language of our science have their being in the world of sense, however ethereal the superstructure we may rear upon them.

And finally, that the rejection of any definition because it postulates a concept of equal likelihood is merely a refusal to recognize the true source of our thought processes, which in the last analysis spring, not from a dictionary which defines words by means of words, but from the common experience to which in some mysterious fashion we are all common heirs.

THE RISE AND FALL OF PROJECTIVE GEOMETRY[1]

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1. The Early Period

The subject of projective geometry occupied an important position on the mathematical stage during a large portion of the nineteenth century. In recent years it has moved considerably towards the wings. Why did it appear? Why was it prominent? Why is it now moving aside? These are pertinent questions which perhaps it is worth our while to consider.

First of all, what is projective geometry anyway? It is sometimes defined as that branch of geometrical science which deals with those properties of figures which are unaltered by radial projection from plane to plane or space to space no matter what the number of dimensions involved. This definition is at once too large and too restrictive. The fact that the projective plane has the connectivity "1" or that, looked upon as an assemblage of points, it has the power of the continuum—this fact is invariant under projection, but is not usually looked upon as a projective property. On the other hand, the properties of a figure which are conserved by projection are invariant under the wider group of linear transformations. The proper use of the adjective "projective" would seem to be to describe those properties of figures which are invariant under the general linear transformation but not under all transformations of a wider group. The subject is vast. We can at present deal only with the most significant steps in its development.

At the outset we must make an important distinction. Projective properties were discovered and used long before any one grasped the idea of the projective group. The earliest and most fundamental projective invariant is the cross ratio of four collinear points. We do not know who first discovered this. By a very curious coincidence we have a statement of the corresponding theorem in spherical geometry which is earlier than any we have of the plane theorem. This occurs in Menelaus' "Spherica." The modern texts of Menelaus are based on various Hebrew manuscripts which differ widely one from another. Thus,

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[1] Address delivered before the Mathematical Association of America at the meeting in Cambridge, Mass., December 29, 1933.

Halley's reconstruction does not give this particular theorem at all. The most recent and complete study of Menelaus is by Björnbo, from which it seems safe to assume that Menelaus was cognizant of the theorem in spherical geometry. But who discovered the simpler theorem in the plane? We find it in the second volume of Pappus who wrote long after Menelaus' time, but this portion of Pappus deals with Euclid's lost book of porisms, which suggests that the invariance of the cross ratio was known to Euclid. Chasles has elaborated an ingenious reconstruction of Euclid's work carrying to their logical conclusions the thirty-eight porisms given by Pappus. The reconstruction of this book has been in fact quite a favorite sport with geometers, and there has been not a little speculation as to what a porism is anyway. I am personally somewhat skeptical as to the ultimate importance of such conjectures. One certain thing is that Pappus, and probably Euclid who wrote six hundred years earlier, knew that the cross ratio is invariant.

After this first advance, the subject of projective geometry rested for the substantial period of twelve centuries. The first writer to return thereto was Johannes Werner who wrote in 1522. His rare work has excited a good deal of curiosity among people who have never seen it. He starts out by determining a parabola as a section of a cone as the Greeks did, but keeps his cone by him, instead of discarding it. He treats the hyperbola in the same way. He reaches a few interesting properties of these two curves in this fashion, but really finds nothing which is not much more simply found in the work of Apollonius. There is no sign that he had heard of a cross ratio. For that we must wait another century and turn to the work of a much better known writer, Girard Desargues. When I say that Desargues was better known, I mean he is better known to-day. His work was little esteemed by his contemporaries and immediate followers with the very important exception of Pascal. This is not surprising. His style and nomenclature are weird beyond imagination. Fortunately, his editor Poudra has given the explanation of seventy-one strange, frequently botanical, terms which this writer affects. Who to-day could guess that a *palm* meant a *straight line*? When there are points thereon (a straight line with no points would be more novel), it becomes a *trunk*. A *tree* is a line with three pairs of points of an involution. The *stump* is the point which is the mate of the infinite point. The stump is *engaged* when the involution is elliptic. The figure which Desargues finds most significant is three pairs of an involution which he gives by equations of the form

$$OA \times OA' = OB \times OB' = OC \times OC'.$$

The harmonic set is called an involution of four points. Menelaus' theorem, which is attributed to Ptolemy, comes next and thereby it is proved that an involution is projected into an involution. Next come cones and cylinders, the latter being called cones with an infinite vertex. The polar theory then appears, derived presumably from the special case of the circle. Desargues treats a diameter as the polar of an infinite point. He carries the whole polar theory

much further than did Apollonius, giving the quadrangle construction, the properties of a self-conjugate triangle, and the polar theory with regard to a surface. Moreover, he derives two very famous theorems which bear his name. The first says that the intersections of a line with the pairs of the opposite sides of a complete quadrangle and with any conic circumscribed to that quadrangle, are pairs of an involution. The essentially new element here is including the intersections with the conic. The quadrangle property was known to Pappus. The second theorem is that which tells us that if corresponding pairs of vertices of two triangles be collinear with the fixed point, the intersections of corresponding pairs of sides lie on a line.

The most famous pupil of Desargues was Blaise Pascal who also used the method of projection. There remains very little of what he actually wrote and there is a tendency at present to reduce the previous somewhat exaggerated estimates of the importance of his work. There is, however, no doubt that he discovered the "hexagramma mysticum." There seems no doubt also that Pascal and Desargues were both aware of the invariance of the cross ratio.

Another writer of this period who had a positive passion for conic sections was La Hire. His first venture was in 1673 when he wrote a little book on planoconics. He defined each conic separately by sums or differences of distances and wrote a text which would not be a bad introduction to put into the hands of a student to-day, except for the omission of the focus-directrix property which was known to Pappus. His great work on conics, however, was a splendid folio which saw the light in 1685. It contains over three hundred theorems of a projective sort as well as an appendix showing that all three hundred and sixty-four theorems of Apollonius can be proved by La Hire's method of projection. The book is an attempt to collate all known material connected with conic sections and very nearly succeeds. There is ample evidence that he knew some of Desargues' work, though, curiously enough, he makes no mention of the great theorems which bear that writer's name. I have the impression that he added little to our knowledge of projective geometry as such, that his main object was to prove the superiority of his method over the ancient method of Apollonius or the modern ones of Descartes.

The subject of projective geometry dragged along for about a century after La Hire's time. A few theorems were discovered by Newton, MacLaurin, and Braickenridge. The next writer deserving attention is Carnot. He, like La Hire, was possessed of the desire of overcoming the apparent increase in generality given by the algebraic methods of Descartes. His basic idea, which was much esteemed by subsequent writers—I suppose for its obscurity—seems to have been this. We establish geometrical relations in the simplest case where all quantities involved are positive, but no further restrictions imposed. It is then assumed that these relations are identities which are unaltered when the figure is replaced by what is called a "correlative," that is to say, the figure obtained from the first by continuous deformation. If one is willing to accept such an

axiom, of course it is very convenient. Carnot also wrote about the theory of transversals, giving generalizations of theorems of Menelaus and Ceva.

There is just one other writer who deserves to be in this group—Brianchon. He is especially skillful in handling the theory of polar reciprocation, deducing from Pascal's theorem the hexagon theorem which bears his name.

2. *The Great Period*

A characteristic feature of the work of Brianchon and his predecessors was that they saw isolated theorems and ingenious methods for solving particular problems. The only general conception was that of beating Descartes. They spoke of the geometry of the ruler and sought the solution of geometrical questions which had but one answer without going deeply into the question of what plane figures were unaltered by projection. The great advance marked by putting the question in this latter way was due to Jean Victor Poncelet whose classic "Treaté de Propriétés Projectives des Figures" was commenced in 1813 under the depressing conditions of a Russian military prison at Saratow.

The fundamental task according to Poncelet is to study the graphic properties of figures which he defines as those that do not involve distances or angles. These latter he regards as contingent whereas the graphic ones are unaltered by a central projection. In looking for such invariants, he develops the theory of harmonic separation at great length, though, curiously enough, the invariance of the general cross ratio escapes him.

In his second chapter Poncelet makes a bold attack on the problem of imaginary points in pure geometry with a courage and thoroughness far ahead of anything shown by his predecessors. The need for this first appears in connection with conic sections. If we connect the centre of a central conic with the pole of a given line, we have the diameter conjugate to the line, which passes midway between the intersections of conic and line in case these intersections exist. Suppose that we have an ellipse with a set of parallel chords. The conjugate diameter will bisect these chords and meet the ellipse in two points A and B. There is just one hyperbola having double contact with the ellipses at A and B whose asymptotes are conjugate diameters of the ellipse. The diameter in question bisects not only parallel chords of the ellipse but parallel chords of the hyperbola. Each curve is called a "supplementary" of the other, the chords of one being called "ideal chords" of the other. In this way Poncelet avoids using imaginary points which he defines lamely enough as something which, originally real, becomes impossible or inconstructible when we pass from the first figure to a correlative in the sense of Carnot. He gives a long discussion of supplementaries and ideal chords and closes with a discussion of the line at infinity. He makes quite casually the fundamental remark that two coplanar circles should not be looked upon as completely independent figures, but as having two immovable, infinite points in common. This is one of the basic principles of modern geometry here announced for the first time. Later Poncelet allows imaginary projections throwing real chords into ideal ones. This is essentially Carnot's correlative

idea. Poncelet calls it the principle of continuity. His statement is essentially this. If a theorem be true for an infinite number of real cases in a figure about which we make no particular restriction as to reality, it is true in the complex case also. This amounts analytically to saying that if we have an algebraic identity $f(a_1,a_2, \cdots, a_n) \equiv 0$ which holds for all real values of the variables, it holds equally when the variables are complex, but whereas such an algebraic statement is precise, the geometric one is dangerously vague and may lead to error. Another weakness of Poncelet's theory of ideal chords is that it is not easily carried over to curves of higher order, a difficulty we shall return to later. But when all is said and done, the supplementaries and ideal chords give us something really tangible when dealing with imaginary elements, a great advance over the work of all previous and many subsequent writers.

In the second section Poncelet gives the general projective theory of conics and straight lines, Desargues' involution, harmonic separation, and so on. He notes the analogy between projection from plane to plane and the transformation of a plane into itself by means of a central similitude. This he generalizes into a transformation by homology where corresponding points are collinear with a fixed point and corresponding lines are concurrent on a fixed line. This again he generalizes to three dimensions as he generalizes the idea of harmonic separation.

The last important topic taken up by Poncelet is polar reciprocation. Special applications are made to the study of algebraic curves. He gives in an appendix a long account of a rather sordid controversy between himself and Gergonne on this subject. Poncelet used his methods as a very convenient tool and discovered useful theorems thereby. Gergonne saw that the conic or quadric of reciprocation was of altogether subordinate importance. What was involved was a very deep geometric principle, but he never bothered to deduce many particular results.

I have paid special attention to Poncelet because he saw far deeper into the essential questions than any of his predecessors. He placed the subject in its right aspect and one would expect that numerous followers would follow along his path to splendid results. Such was the case, but, curiously enough, only one of the followers of major importance was a countryman of his, Michel Chasles. This enterprising and ingenious writer enriched geometrical science by several fruitful ideas. On the other hand, he was decidedly uncritical as a historian of mathematics both of his own work and of that of others. In his much vaunted study of the history of geometrical methods, he makes the interesting admission that he has neglected German writers because he did not know their language.

The greatest contribution that Chasles made to projective geometry, and it was certainly very great, was in developing the theory of cross ratio of points, lines, or planes. He discovered the theorem which subsequently played a vital rôle in the work of Steiner, that four points of a conic determine at any fifth point four lines with fixed cross ratio. Another fundamental idea was homography or the general linear transformation of the plane. Central projection and the homology of Poncelet are special cases of this.

If Chasles neglected the work of the German geometers, the reverse was not the case, for, contemporary with him, there sprang up across the Rhine a school of German geometers who closely watched the work of their French neighbors. The first of these in the synthetic field was Jacob Steiner whose "Systematische Entwickelung der Abhängigkeit geometrischer Gestalten von einander" appeared in Berlin in 1832. This is based on three fundamental principles.

(a) Points, lines, and planes are the essential data of geometry. Other figures must be constructed from these in definite fashion.

(b) The principle of duality which appears at the outset and is carried through consistently, much of the work being in double column.

(c) A fundamental concept is that of projective fundamental forms, that is, ranges of points, pencils of lines, and pencils of planes.

Right here is a slip which is hard to account for. I cannot see that Steiner ever really gives a good definition of what he means by projective. A pencil of lines and the range they cut on a transversal are called projective and they are said to remain so when the one or the other is moved about without disturbing distances or angles. This is clumsy and unsatisfactory. Essentially, what he means by projectivity is a one to one relation where corresponding cross ratios and senses of description are the same. He shows that such a relation between two fundamental one-dimensional forms is determined when the fate of three elements is known. The proof is unsatisfactory as there is no proper way to handle questions of limits and continuity. There are no less than eighty pages, mostly in double column, devoted to questions of this sort.

The next form of figure to receive his attention is the cone, that is to say, the cone with circular sections. Construction of a conic by means of projective pencils comes in naturally here. There is one logical slip; at least I cannot see any proof that every conic so constructed can be cut from a circular cone. Except for this defect, we have here the best possible approach to the study of the conics from the point of view of bringing out their projectively invariant properties. No wonder that all the simplest projective properties appear immediately in excellent form. The same methods are applied to the study of the regulus, that is to say, rulings of a surface of the second order. It is rather curious that he does not at once proceed to a study of the cubic space curve which is reached by the intersections of corresponding planes of three pencils.

The most characteristic feature of Steiner's work is given by the adjective in its title, "systematische." He has a consistent and uniform method for treating a variety of figures. He handles it beautifully. He is usually considered the greatest of the German school of projective geometers. It is my own feeling that in originality and power he falls far below his distinguished successor, Johann Karl Christian von Staudt. This deep thinker perceived two essential weaknesses in the synthetic geometry of his predecessors.

(a) The basis of projective relations was the cross ratio. This is projectively

invariant but, as previously given, was based on distances and angles which are not in themselves unalterable.

(b) What are imaginary points anyway? What can be said about them, except that they are imaginary?

Let us give an outline of his method for developing geometry so as to meet these difficulties. In his first book "Geometrie der Lage" published in 1847, he starts with points, lines, and planes as fundamental objects. Desargues' two triangle theorem appears early leading to the configurations of a complete quadrilateral and quadrangle. It never occurred to him that we really need a proof that the diagonals of a complete quadrilateral are not concurrent. Next we have harmonic separation and the fundamental definition that two one-dimensional forms are projective when their members are in one to one correspondence and a harmonic set corresponds to a harmonic set. Here at last we have projectivity defined with no relation to distance. The fact that von Staudt was not able to carry the thing through rigorously does not detract from the originality of the idea. The failure was right here. Sooner or later we have to prove that a projective transformation of a one dimensional form into itself that leaves three elements in place, leaves all elements in place. Von Staudt could not do this as he had no axiom of continuity. Subsequent writers have filled the void and shown how cross ratios can be defined by successive harmonic constructions exactly in the way that distances are defined by successively laying down a fixed length or aliquot part thereof.

Von Staudt defines a collineation of the plane as a transformation of point to point and line to line and shows that this is projective in that it carries harmonic elements into harmonic elements. He goes further, however, and defines a correlation, let us say in the plane, as a correspondence of point to line and points on a line to lines on a point. Poncelet reached such a transformation by polar reciprocation with respect to a conic. Von Staudt exactly reversed the process. Suppose that we have a correlation which is involutory. It is easy to show that such things exist. Then if there be any point lying on the corresponding line, it is easy to show that there is a whole curve of points each of which lies on its line and this is defined as a conic. This definition is clearly self-dual and leads at once to the classic theorems of Pascal, Brianchon, Desargues, and Steiner. The method has the advantage that it is immediately applicable in three or more dimensions. He did not make the mistake of supposing that if we have an involutory correlation in three dimensions, the points lying in the corresponding planes necessarily generate a quadric surface. He was familiar enough with the null-system usually ascribed to Möbius.

Von Staudt was acutely conscious that the treatment of imaginary elements in pure geometry was extremely unsatisfactory. Poncelet's system of ideal chords and supplementaries was the only contribution to the subject that had any real substance. He set to work to remedy this defect in truly heroic fashion. Suppose that on a straight line we have an elliptic involution. A point and its mate in the involution trace the straight line *in the same sense.* "Very well,"

says von Staudt, "We will define an elliptic point involution and a sense of description as an imaginary point. The same involution with the contrary sense shall be defined as the conjugate imaginary point. We can define an imaginary plane as an elliptic involution in a pencil of planes with a sense of description. Reverse the sense and we get the conjugate plane. An imaginary line is defined as the system of points in two planes which are not both real."

These definitions of von Staudt are certainly revolutionary. It was a bold step to define as an imaginary point something that is made up of an infinite number of real points. Von Staudt could not foresee the analogy to Dedekind's definition of an irrational number as a split in the real number system. What he did do was to show by the most careful reasoning that the new elements thus introduced obeyed just the laws of the old ones. Two of his points determine one of his lines which lies completely in any one of his planes through the two points, etc. It is true that his work has the weakness that sense of description is an intuitive concept which he is not able to define and about which he has no definite mathematical axioms. This difficulty can be overcome by methods subsequently invented. His treatment, which is found in his subsequent "Beiträge zur Geometrie der Lage," is a marvellous piece of careful geometry.

But von Staudt did not rest even here. Chasles had based his projective geometry on the idea of the cross ratio which is defined in terms of distances. Von Staudt perceived that this was a blemish which he undertook to correct. Four collinear points, four lines or planes of a pencil are called a "throw." The value of this is defined as unchanged by a double interchange in the pairs of elements, and two throws connected by a train of projective transformations are defined as an equivalent. He proceeds to develop an algebra of throws. If we wish to add two throws, we transform them projectively into two others with three common elements and find the sum by two successive harmonic constructions. In the same way the product of two throws is to be found by elimination between two with three common elements. There follows a very careful demonstration that in this algebra the usual commutative, associative, and distributive laws are obeyed. If a harmonic set be defined as a throw having the value of -1, we can show that any real throw of points is equal to the cross ratio in the usual sense.

But von Staudt does not stop with real throws. He considers complex ones as well. He fixes his attention on an imaginary line of the second sort which has no real points. Let us take four complex points of $ABCD$ on such a line and connect them with their conjugates $A'B'C'D'$ by four real skew lines. Three of these will determine a ruled quadric surface including the given complex line and its conjugate. The necessary and sufficient condition that the throw of the original four points should be real is that the four lines should be generators of one quadric. A set of points on such an imaginary line with the property that the cross ratio of any four is real, was defined as a "chain." Such a set of points can be projectively transformed into the real points of a real line. This theory was further developed by Luröth and later by our own Veblen and Young.

3. The Gradual Decline

The work of von Staudt marks the close of the great period in the history of synthetic geometry. A definite geometrical method had been elaborated. The work of subsequent geometers consisted principally in applying to it such problems as lent themselves readily to treatment. For fifty years the devotees of this branch of science were many and enthusiastic. Progress consisted, however, chiefly in extending the methods already discovered to new problems rather than in finding new methods. An exception to this rule must be made to cover the case of geometrical transformations. Poncelet had a clear grasp of the idea of central projection and the cognate idea of homology. Chasles constructed the general projective transformation. Steiner developed something essentially different in the form of a quadratic transformation by means of a skew projection. Here a point to point transformation is established between two planes by means of projecting lines that do not go through a fixed point but do intersect two skew lines. A straight line in one of the planes will correspond to a conic in the other that passes through three fixed points. Another type of quadratic transformation is the projective generalization of circular inversion where corresponding points are collinear with a fixed point and conjugate with regard to a fixed conic. A better form was developed by Seydewitz where a point in the plane is carried to the single point conjugate to it with regard to a pencil of conics. Many other forms of one to one geometric transformations were discovered. A complete account will be found in the four volumes of Sturm. There are, however, many pitfalls in the path of one who would handle geometrical transformations by means of pure geometry alone, especially when he becomes involved in the complicated singularities of higher curves and surfaces. It is safer to treat the whole question algebraically as a part of the great subject of Cremona transformations.

It was inevitable that students of synthetic geometry should be forced sooner or later to face the question of applying their favorite methods to the study of general algebraic curves and algebraic surfaces and that forces upon us the question of whether synthetic methods really lend themselves well to the study of such curves or surfaces. And at this point I am forced to confess that it seems to me the answer must be frankly negative, though able geometers have striven to prove the contrary. One of these was Cremona whose long memoir "Introduzione alla teoria geometrica della curve piane" was written in 1862. When we examine this work in detail we are forced to acknowledge its defects. The author speaks freely of real and complex points without every saying just what he means by the latter. It is true that we can, if we wish, lay down a system of axioms for the projective geometry of the complex plane, but Cremona never bothered to do anything of the sort. In his time real points were supposed to be given us by nature and no man had a right to speak of complex points, which nature certainly did not provide, without saying what he meant thereby. Cremona's various proofs involve a good deal of credulity on the part of the reader. For instance, he has assumed without a shadow of proof that the

number of intersections of two plane curves is a function of their orders alone and independent of whether they be reducible or irreducible. There is a similar optimism about the number of parameters on which a curve of order n depends. The whole work is shot through with doubtful assumptions. What the writer really does is to use a few essential theorems about the general equation of degree n and then ungratefully discard it.

The two great difficulties have now been mentioned—a sound theory of complex points and something definite to replace the theory of the general equation. Von Staudt, as mentioned above, conquered the first difficulty once for all in most brilliant fashion, but his method was so cumbersome that in practice nobody could be expected to use it. The only way to avoid the second obstacle was by means of mathematical induction. The most ambitious attempt to use this process was made in 1886 by Kötter in his "Theorie der algebraischen ebenen Curven" (Berlin 1887). His favorite implement is the general involution. We may lead up to this as follows. Suppose that we have projective transformation of one of our fundamental forms, let us say the straight line, into itself, whereby the points $ABC \cdots$ correspond to $B'A'C' \cdots$. There will be a pair of self-corresponding points, either a real pair or a conjugate imaginary pair in the von Staudt sense. Now let all of these points remain fixed except C' which traces the whole line. These pairs of self-corresponding points will trace that involution which is determined by the pairs AA' and BB' and the pairs of this are projectively related to the range traced by C'. Suppose now that we know what we mean by a projective relation between a range of points and the groups of an involution of order $n-1$. Suppose that we have such an involution with the group $A_1A_2, \cdots, A_{n-1}$, $B_1B_2, \cdots, B_{n-1}$, $C_1C_2, \cdots, C_{n-1}$ projectively related to the points $B'A'C'$, that everything remain fixed except C' which traces the line. The groups of n, real or imaginary, self-corresponding points will trace the involution of order n determined by the two groups $A'A_1A_2, \cdots, A_{n-1}$, $B'B_1B_2, \cdots, B_{n-1}, \cdots$. In this way we get the general involution. Its establishment involves some delicate considerations of continuity and analysis situs as well as the von Staudt theory. When all is ready, Kötter establishes a projective relation between a pencil of curves or order r and one of order $n-r$ and thus generates a curve of order n.

This is the general idea of Kötter's ambitious attempt. It is a remarkable piece of geometry, also a very difficult one, so that I personally marvel at the thought that I once read it through. I do not wonder that the Berlin Academy awarded it a prize in 1886 or that few have had the temerity to read the thing since. It seems to me that there is really a better approach which consists in generalizing von Staudt's conception of a polar. Suppose that we know a good deal about a thing which we call a curve of order $n-1$ as well as a pencil and a two parameter net of such curves and that we know what we mean by the first polar of a point with regard to such a curve. It may be defined, with Guccia, as the locus of points of contact of tangents from a point P to curves of order $n-1$ linearly dependent on a given curve and $n-1$ arbitrary lines through P.

Suppose further that we have a two parameter set of curves of order $n-1$ which correspond to the points of the plane in such a way that the first polar of P with regard to the curve corresponding to Q is identical with the first polar of Q with regard to the curve corresponding to P. Then, the locus of points which lie on the curves which correspond to them is a curve of order n and the curve of order $n-1$ corresponding to a given point is its first polar with regard to the new curve. I think that this is a better approach than Kötter's. I once carried it out a short distance in an article in the Circolo Matematico di Palermo, Rendiconti (vol. XI. 1915) in ignorance of the fact that the same idea had been previously developed by Thieme ("Die Definition der geometrischen Gebilde." Zeitschrift für Mathematik und Physik. vol. 24. 1879). And yet I now doubt whether it would be worth while to follow such a lead for the algebraic approach is so much easier and more satisfactory. It is for this reason that the latest advances in projective geometry, before it finally flickered out, lay in a closer examination of the fundamental postulates.

We have seen that von Staudt's great work had certain intuitive elements such as the sense of description on a line, continuity, separation, etc. It never occurred to either him or any contemporary that the diagonals of a complete quadrilateral might be concurrent. All of these points were cleaned up by a number of writers who dealt with the postulates of projective geometry. At the risk of creating offense by citing one at the expense of others, I would mention the two-volume work of Veblen and Young. Here have we not only a rigorous postulational basis for projective geometry, but the obvious step is taken of extending it to n dimensions. There is a discussion of nets of rationality which are essentially the same thing as generalizations of the von Staudt chain, of finding projective geometries and other strange systems obtained by varying the axioms. It is not too much to say that this is the last great work dealing with this field.

It is hard to escape the sad conclusion that the field of synthetic projective geometry is pretty much worked out. This gloomy foreboding can only be put forth with all possible caution. There have been so many times when the geometrical field, even the most elementary field, has appeared to be completely exhausted and then surprisingly new and attractive results have appeared. Elementary geometry involving nothing more complicated than a little trigonometry, made far more progress in the nineteenth century than in the sixteen centuries preceding, but the analogy is not perfect. Most of the theorems about the straight line, circle, and sphere which we can demonstrate at all, find their easiest proofs by the most elementary means. Such is not the case with projective geometry. An algebraic handling is almost always the easiest and generally the most adaptable. Until and unless some totally new principle is discovered, the subject of synthetic projective geometry is no longer a very fruitful field for original research. On the other hand, it would be a disaster to the whole geometric fabric if a time ever came when synthetic methods were completely abandoned. Not only have they a permanent beauty which no one who has

ever studied them can forget, but they afford an invaluable insight into the inner significance of geometrical science and an invaluable training for any geometer. If Plato wrote over the gate of the Academy, "Let none ignorant of geometry presume to enter here," surely we may write to-day in the same spirit, "Let none ignorant of the fundamentals of synthetic projective geometry presume to the title of geometer."

THE METHOD OF UNDETERMINED COEFFICIENTS

By N. B. CONKWRIGHT, University of Iowa

In introductory courses in differential equations a certain process, the so-called method of undetermined coefficients, is often employed to find a particular integral of the linear differential equation with constant coefficients

$$(1)\qquad f(D)y = (D^n + a_1D^{n-1} + \cdots + a_{n-1}D + a_n)y = r(x).$$

The rule of procedure is generally given somewhat as follows: If $r(x)$ is of such nature that only a finite number of distinct terms, say $u_1(x), u_2(x), \cdots, u_p(x)$ arise as a result of repeated differentiation, assume that the particular integral is of the form

$$c_1u_1 + c_2u_2 + \cdots + c_pu_p,$$

and determine the constant coefficients $c_1, c_2, \cdots, c_p$ by substitution in the differential equation. As is well known, the process is modified somewhat in case one or more of the u's is identical with a term of the complementary function of the differential equation, or with such a term multiplied by a positive integral power of the independent variable x.

Although the considerations involved are of the most elementary nature there does not seem to be available to the student any discussion of the reasons why one should expect the particular integral to be of the prescribed form, nor a precise statement of the possibilities and limitations of the method of undetermined coefficients. In fact, the procedure is frequently set forth as a sort of trial method and the entire subject treated in a vague and indefinite manner. It would seem worth while from a pedagogical point of view to place the answers to the questions thus raised at the disposal of those who are beginning the study of the subject.

To this end we formally specify that $r(x)$ shall be a function of the type already intimated. That is, we assume the existence of p functions $u_1(x)$, $u_2(x), \cdots, u_p(x)$ of such nature that

$$(2)\qquad D^jr = \sum_{k=1}^{p} b_{jk}u_k(x), \qquad j = 0, 1, 2, \cdots, p,$$

the b's being constants.

A function which satisfies the conditions which have been imposed upon $r(x)$

will be said to be of the type (2). We first inquire what functions satisfy these restrictions. If the u_k are considered as independent variables the $p+1$ polynomials $\sum_{k=1}^{p} b_{jk}u_k$ are linearly dependent. Hence r must satisfy a relation of the form

$$(A_pD^p + A_{p-1}D^{p-1} + \cdots + A_1D + A_0)r = 0,$$

in which the A's are constants and at least one of them different from zero. Since r is an integral of a homogeneous linear differential equation with constant coefficients it appears (from the familiar method of solving such equations) that this function must be expressible as a sum of constant multiples of terms such as

$$x^\lambda e^{\alpha x} \cos \beta x, \qquad x^\lambda e^{\alpha x} \sin \beta x, \tag{3}$$

where λ is a positive integer or zero and α and β are real.

Note that the product of two or more functions of the type (3) is also of this type. Thus the product of $x^5e^{2x} \sin 3x$ and $x^3e^x \cos 2x$ can be written $x^8e^{3x}(\sin 5x + \sin x)$, etc.

We return to the consideration of the linear relation between r and its derivatives. It may happen that $r, Dr, D^2r, \cdots, D^Nr$ are linearly dependent, where $N<p$. Let N_0 be the least value of N for which this is true. Then we can affirm that r is an integral of the differential equation

$$\phi(D)y = (A_0' + A_1'D + \cdots + A_{N_0}'D^{N_0})y = 0.$$

Moreover, $r, Dr, D^2r, \cdots, D^{N_0-1}r$ are independent solutions of this equation. Hence every solution is a sum of constant multiples of the u's since each one is a linear combination of $r, Dr, D^2r, \cdots, D^{N_0-1}r$.

With the facts now in hand it is easy to see how the form of the particular integral of the differential equation (1) is related to the function $r(x)$. Let $y=W(x)$ designate the solution of (1) and $y=w(x)$ the solution of $\phi(D)y=0$. Since the result of operating upon both members of (1) with $\phi(D)$ is

$$\phi(D)f(D)y = 0, \tag{1'}$$

it is evident that every solution of (1) is also a solution of this equation. Consequently the particular integral of (1) can be obtained by subtracting $W(x)$ from the general integral of (1′) and assigning appropriate values to the arbitrary constants in the resulting expression. In particular, if ϕ and f have no common factor, the solution of (1′) is

$$y = W(x) + w(x),$$

and the particular integral of (1) must be an integral of $\phi(D)y=0$ and hence of the form

$$c_1u_1 + c_2u_2 + \cdots + c_pu_p.$$

The more general case in which $f(D)$ and $\phi(D)$ may have a common factor must now be examined. For this purpose it will be convenient and also sufficiently general to consider the case in which r is a single term of the type (3). In this connection we have the rather obvious theorem.

Theorem: *If* $e^{\alpha x}$ *is a term of the complementary function of the differential equation*

$$f(D)y = x^{\lambda}e^{\alpha x}, \tag{4}$$

corresponding to a k-fold root of the auxiliary equation (where k may be zero) then the particular integral is of the form

$$\sum_{j=k}^{k+\lambda} c_j x^j e^{\alpha x}.$$

In proof of this statement let (4) be written

$$(D - \alpha)^k g(D)y = x^{\lambda}e^{\alpha x},$$

and designate the solution of $g(D)y=0$ by $y=v(x)$. Here it is evident that $\phi(D)$ is the operator $(D-\alpha)^{\lambda+1}$. We operate upon both members of (4) with $\phi(D)$. It thus appears that every solution of (4) is also a solution of

$$(D - \alpha)^{k+\lambda+1}g(D)y = 0. \tag{5}$$

The solution of the latter equation is

$$y = v(x) + \sum_{j=0}^{k+\lambda} c_j x^j e^{\alpha x}. \tag{6}$$

Upon subtracting the complementary function of (4) from the second member of (6) it is seen that the particular integral of (4) is of the specified form.

For the more general case in which $r(x) = x^{\lambda}e^{\alpha x} \sin \beta x$ (or $x^{\lambda}e^{\alpha x} \cos \beta x$) where $\beta \neq 0$ we have the theorem

Theorem: *If* $e^{\alpha x} \sin \beta x$ *is a term of the complementary function of the equation*

$$f(D)y = x^{\lambda}e^{\alpha x} \sin \beta x, \tag{7}$$

corresponding to a k-fold root of the auxiliary equation (where k may be zero) the particular integral is of the form

$$\sum_{j=k}^{k+\lambda} x^j(A_j \sin \beta x + B_j \cos \beta x)e^{\alpha x},$$

provided the coefficients in $f(D)$ *are real.*

To establish this statement let (7) be written

$$(D^2 - 2\alpha D + \alpha^2 + \beta^2)^k g(D)y = x^{\lambda}e^{\alpha x} \sin \beta x,$$

and as before designate the solution of $g(D)y=0$ by $y=v(x)$. Here $\phi(D)=(D^2-2\alpha D+\alpha^2+\beta^2)^{\lambda+1}$. Then every solution of (7) is also a solution of

$$(D^2 - 2\alpha D + \alpha^2 + \beta^2)^{k+\lambda+1}g(D)y = 0. \tag{8}$$

The solution of the latter equation is

$$y = v(x) + \sum_{j=0}^{k+\lambda} x^j(A_j \sin \beta x + B_j \cos \beta x)e^{\alpha x}.$$

And upon subtraction the particular integral of (7) is seen to be of the specified form. It is evident that the argument would be unaltered if the second member of (7) were $x^\lambda e^{\alpha x} \cos \beta x$.

If not all the coefficients in $f(D)$ are real, the particular integral of (7) may fail to have the form specified in this theorem. For example, the particular integral of $(D+i)y=\sin x$ is $-e^{ix}/4+ixe^{-ix}/2$, which is not of the form $c_1 \sin x+c_2 \cos x$. Of course, if neither $\alpha \pm i\beta$ is a root of the auxiliary equation the form of the particular integral is correctly given by the theorem regardless of the nature of the coefficients in $f(D)$.

The case will now be considered in which $r(x)$ is a function of the type (2) which is not *explicitly* a sum of constant multiples of terms of type (3) (i.e., $r=\sin ax \cos bx$, $\sin^2 x$, $\cosh x$, $xe^{4x} \sin x \sin 2x \sin 3x$, etc.). It has already been said that if $f(D)$ has no factor in common with the ϕ function associated with $r(x)$ then the method of undetermined coefficients as usually described is adequate to determine the form of the particular integral. That is to say, the particular integral is of the form $\sum_{k=1}^{p} c_k u_k$. However, it may happen that although $r(x)$ *as written* contains no term which is also present in the complementary function of the differential equation (1), the associated ϕ function has a factor in common with $f(D)$. In such cases the particular integral may not be of the form $\sum_{k=1}^{p} c_k u_k$ and the method of undetermined coefficients as ordinarily described is inadequate. Although the method of approach is obvious it does not seem feasible to formulate principles governing the form of the particular integral in such cases. Suffice it to repeat that $r(x)$ can always be written as a sum of terms of type (3) after which the two preceding theorems can be applied.

The discussion of the preceding paragraph is well illustrated by the equation

$$(D^2 + 9)y = \sin 2x \sin x.$$

(Note that r is not a term of the complementary function.) It has the particular integral

$$\frac{\cos x}{16} - \frac{x \sin 3x}{12}$$

which is not a sum of constant multiples of terms of r and terms arising from r by differentiation. The difficulty obviously arises from the fact that the function $\phi(D)$ which one would associate with r in this case, namely $(D^2+1)(D^2+9)$

has a factor in common with $f(D)=(D^2+9)$. In other words, when r is expressed as a sum of terms of the type (3), giving $\frac{1}{2}\cos x-\frac{1}{2}\cos 3x$ then one of these terms is identical with a term of the complementary function of the differential equation. The assumption that this equation has a particular integral of the form $\sum_{k=1}^{p}c_k u_k$ would of course lead to a set of inconsistent equations for the determination of the c's. Analogous remarks might be made relative to the equation

$$(D^2-3D+2)y=\cosh x,$$

which has the particular integral

$$-\frac{1}{2}xe^x-\frac{1}{2}e^x+\frac{1}{12}e^{-x}=-\frac{x}{2}(\cosh x+\sinh x)-\frac{5}{12}\cosh x-\frac{7}{12}\sinh x.$$

It can be readily shown, however, that the usual method of undetermined coefficients can be applied directly to the equations

$$f(D)y=x^\lambda\sinh mx,\qquad f(D)y=x^\lambda\cosh mx,$$

if neither m nor $-m$ is a root of the auxiliary equation. In fact, it can be seen from arguments similar to those already presented that the equation

$$f(D)y=x^\lambda e^{\alpha x}\cosh\beta x$$

has a particular integral of the form

$$\sum_{j=k}^{k+\lambda}x^j(A_j\sinh\beta x+B_j\cosh\beta x)e^{\alpha x},$$

provided $\alpha+\beta$ and $\alpha-\beta$ are k-fold roots, and neither a $(k+1)$-fold root, of the auxiliary equation. Of course, k may be zero.

AN ENVELOPE PROBLEM

By ROBIN ROBINSON and E. F. COOLEY, Dartmouth College

The problems presented to the student in courses in the calculus and related subjects are usually so compounded that straight-forward methods will lead to explicit results. In consequence, many a student reaches an impasse when faced with a problem where conventional methods are impractical or cumbersome, or where they lead to analytic results which cannot readily be interpreted. In such a situation methods must be used which do not necessitate the explicit analytic solution of the problem.

In this paper we shall consider a problem in which parametric methods of attack are the most effective ones. It is hoped that the curves obtained will themselves be of considerable interest to the reader.

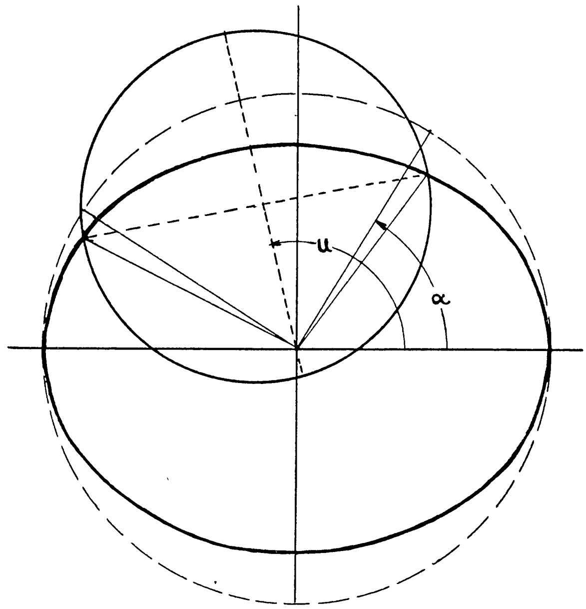
Eugène Fabry has proposed[1] the following problem: Find the envelope of

[1] *Nouveau Traité de Mathématiques Générales*, Paris, 1925, vol. I, p. 309, ex. 2, part 2.

the circles having as diameters the chords joining extremities of conjugate diameters of an ellipse.

If we place our ellipse so that its center is at the origin and its foci are on the x-axis, it may be represented parametrically by the equations

$$x = a \cos \alpha, \qquad y = b \sin \alpha, \qquad a \geqq b.$$



Extremities of conjugate diameters are then given by the coördinates $(a \cos \alpha,\ b \sin \alpha)$, $(-a \sin \alpha,\ b \cos \alpha)$. Making the change of parameter $u=\alpha+\pi/4$, the circle having their chord as diameter has the equation

$$(1) \qquad \phi \equiv x^2 + y^2 - \sqrt{2}a \cos u\, x - \sqrt{2}b \sin u\, y + \frac{c^2}{2} \cos 2u = 0,$$

where $c^2 = a^2 - b^2$.

It will also be convenient to have the equation (1) in polar coördinate form, where $x = r \cos \theta$, $y = r \sin \theta$, as follows:

$$(1a) \qquad \phi \equiv r^2 - \sqrt{2}ar \cos u \cos \theta - \sqrt{2}br \sin u \sin \theta + \frac{c^2}{2} \cos 2u = 0.$$

To determine the equation of the envelope, u must be eliminated between (1) and

$$(2) \qquad \phi_u \equiv \sqrt{2}a \sin u\, x - \sqrt{2}b \cos u\, y - c^2 \sin 2u = 0.$$

This turns out to be a very awkward piece of work, and the result is practically

useless in determining the form of the envelope. It is therefore necessary to determine the nature of the envelope directly from the equations (1) and (2) without elimination. These two equations may be regarded as representing the envelope parametrically, two points of the envelope being determined for each value of the parameter u.

We shall first consider the general procedure for studying curves given parametrically in this manner. To emphasize the importance of this general method of attack is one of the prime purposes of this paper.

For all points of the curve determined by the equations

$$\phi(x, y, u) = \phi_u(x, y, u) = 0, \tag{1), (2}$$

we must have

$$d\phi = \phi_x dx + \phi_y dy = 0, \tag{3}$$

$$d\phi_u = \phi_{ux} dx + \phi_{uy} dy + \phi_{uu} du = 0. \tag{4}$$

Let us determine the condition for a horizontal tangent. At such a point we have $dy=0$, $dx\neq 0$. Equation (1) then tells us that for a *horizontal tangent*

$$\phi_x = 0. \tag{5}$$

Similarly, for a *vertical tangent*

$$\phi_y = 0. \tag{6}$$

If the equations determining the envelope be written in the polar form

$$\phi(r, \theta, u) = \phi_u(r, \theta, u) = 0, \tag{1a), (2a}$$

the equation analogous to (3) is

$$d\phi = \phi_r dr + \phi_\theta d\theta = 0. \tag{3a}$$

Hence when *r is stationary*, $dr=0$, $d\theta\neq 0$, and

$$\phi_\theta = 0. \tag{7}$$

A discussion of nodes will come later. At a cusp, we shall have $dx=dy=0$ as u changes. So from (4), we must have at a *cusp*

$$\phi_{uu} = 0. \tag{8}$$

The conditions (5), (6), (7), and (8) are necessary conditions, which will suffice for this particular problem.

Let us now return to our specific problem. The nature of the envelope depends a great deal upon the shape of the original ellipse. We shall use the notation $k=b/a$, which has greater geometric significance and leads to simpler algebraic work than the more conventional "eccentricity." It will then be necessary to consider the nature of the envelope for all values of k in the interval $0\leqq k\leqq 1$.

Envelope degenerate. A consideration of cases where the envelope is factorable is essential.

If we solve the equation (2) for y, and substitute in (1), we obtain the equation

$$(9)\quad \begin{aligned} 2(b^2\cos^2 u + a^2\sin^2 u)x^2 - 2\sqrt{2}a\cos u(2c^2\sin^2 u + b^2)x \\ + c^2\cos^2 u[(4a^2 - b^2)\sin^2 u + b^2\cos^2 u] = 0. \end{aligned}$$

For the envelope to be factorable, the discriminant

$$(10)\quad B^2 - 4AC = 8a^6\cos^6 u\, k^2[t^2 - (1 - 4k^2 + k^4)t + k^4], \qquad t = \tan^2 u,$$

of (9) must be a perfect square. So either $k=0$, or

$$(1 - 4k^2 + k^4)^2 - 4k^4 = 0.$$

Factoring this, we see that

$$(1 - k^2)^2(1 - 6k^2 + k^4) = 0,$$

i.e., $k=1$, or $k=\sqrt{2}-1$.

In the case $k=0$, the ellipse is considered to have degenerated into a line-segment of length $2a$, extremities of conjugate diameters being determined geometrically, as in the general case, by vertical projection from points at quadrant's distance on the major auxiliary circle. In this case the elimination between (1) and (2) can readily be carried through, and the envelope is determined to consist of the ellipse $x^2+2y^2=a^2$, and its foci $(\pm a\sqrt{2}, 0)$.

In the case $k=1$, the original ellipse is a circle. Here, too, the elimination can be carried through easily, and the envelope is found to consist of the circle $x^2+y^2=2a^2$, and its center $(0, 0)$.

When $k=\sqrt{2}-1$, actual solution of (9) gives as one part of the envelope

$$x = a\cos u, \qquad y = -b\sin u,$$

which is the original ellipse. The other part is given parametrically by

$$\left\{\begin{aligned} x &= -\frac{ak\cos u}{k^2+t}[k^2 - (1+2\sqrt{2})t], \\ y &= -\frac{a\sin u}{k^2+t}[(1+2\sqrt{2})k^2 + t], \end{aligned}\right. \qquad k = \sqrt{2} - 1,\ t = \tan^2 u.$$

Envelope imaginary. The roots of B^2-4AC in (10) are given, aside from $\cos u=0$, which we shall find contributes four distinct points on the y-axis, by

$$(11)\qquad 2t = (1 - 4k^2 + k^4) \pm (1 - k^2)\sqrt{1 - 6k^2 + k^4}.$$

If $0\leqq k<\sqrt{2}-1$, $1-6k^2+k^4>0$, and B^2-4AC has two distinct positive roots t, and between them $B^2-4AC<0$. So for these values of k the envelope is imaginary for an interval, i.e., the circles nest.

We may now drop the cases $k=0$ and $k=1$ from the discussion, and consider the general case $0<k<1$, first applying the conditions (5), (6), (7), and (8). The geometrical nature of the problem indicates that the envelope is symmetric with respect to both axes, so we may restrict u for a time to the interval $0\leqq u\leqq\pi/2$. This does not mean that all the points resulting from these values of u lie in the first quadrant, but that when these points are reflected successively with respect to each axis and the origin, all points of the envelope will have been obtained.

Horizontal tangents. We have seen that the conditions for a point of horizontal tangency are (1) $\phi = 0$, (2) $\phi_u = 0$, and (5) $\phi_x = 0$. Since from (1)

$$\phi_x \equiv 2x - \sqrt{2}a \cos u,$$

equation (5) becomes

$$x = \frac{a}{\sqrt{2}} \cos u. \tag{12}$$

Substituting this value of x in (2) we obtain either

(a) $$\cos u = 0,$$

or

(b) $$y = \frac{2b^2 - a^2}{b\sqrt{2}} \sin u.$$

In case (a), (12) becomes $x=0$. Substituting $x=0$, $\cos u=0$, and $\sin u=1$ in (1), we obtain as points of horizontal tangency on the y-axis

$$\left(0, \frac{b \pm a}{\sqrt{2}}\right).$$

In case (b), we see that the point

$$\left(\frac{a}{\sqrt{2}} \cos u, \frac{2b^2 - a^2}{b\sqrt{2}} \sin u\right) \tag{13}$$

which satisfies (1) is a point of horizontal tangency. Substitution in (1) gives for the proper value of u

$$\tan^2 u = \frac{k^4}{1 - 3k^2 + k^4}\cdot \tag{14}$$

This value of u is real when $0<k\leqq(\sqrt{5}-1)/2$, and as k approaches $(\sqrt{5}-1)/2$, the point moves in toward the y-axis.

Vertical tangents. Exactly similar steps, using (1), (2), and (6), give as points of vertical tangency

$$\left(\frac{a \pm b}{\sqrt{2}}, 0\right)$$

and

$$\left(\frac{2a^2 - b^2}{a\sqrt{2}} \cos u, \frac{b}{\sqrt{2}} \sin u\right),$$

where

$$\tan^2 u = 1 - 3k^2 + k^4.$$

Here, again, the latter point is real when $0 < k \leqq (\sqrt{5}-1)/2$, and moves in toward the x-axis as k approaches $(\sqrt{5}-1)/2$.

Stationary r. The conditions that r be stationary were found to be (1) $\phi = 0$, (2) $\phi_u = 0$, and (7) $\phi_\theta = 0$. Since, from (1a),

$$\begin{aligned} \phi_\theta &\equiv \sqrt{2}ar \cos u \sin \theta - \sqrt{2}br \sin u \cos \theta, \\ &\equiv \sqrt{2}(a \cos u\, y - b \sin u\, x), \end{aligned}$$

equation (7) becomes

$$a \cos u\, y = b \sin u\, x. \tag{15}$$

This is satisfied by $x=0$, $\cos u=0$, so the points of horizontal tangency on the y-axis are also points where r is stationary, as is to be expected. Similarly the points of vertical tangency on the x-axis fall in this category. But there is yet another. Solving (15) for y, and substituting in (2), we obtain $x=\sqrt{2}a \cos u$, and hence from (15) $y=\sqrt{2}b \sin u$. This point satisfies (1) if and only if $\cos 2u=0$, i.e., if $u=\pi/4$. So r is stationary, in fact, is a maximum, at the point (a, b), which lies on the envelope.

Cusps. The conditions for a cusp were found to be (1) $\phi = 0$, (2) $\phi_u = 0$, and (8) $\phi_{uu} = 0$. From (2), we obtain

$$\phi_{uu} \equiv \sqrt{2}a \cos u\, x + \sqrt{2}b \sin u\, y - 2c^2 \cos 2u.$$

Linear combinations of (1), (2), and (8) are actually more useful, viz.,

$$\phi + \phi_{uu} \equiv x^2 + y^2 - \frac{3}{2} c^2 \cos 2u = 0, \tag{16}$$

$$\phi_u \sin u + \phi_{uu} \cos u \equiv \sqrt{2}ax - 2c^2 \cos^3 u = 0, \tag{17}$$

$$- \phi_u \cos u + \phi_{uu} \sin u \equiv \sqrt{2}by + 2c^2 \sin^3 u = 0. \tag{18}$$

Solving (17) for x and (18) for y, we obtain the point

$$\left(\frac{c^2\sqrt{2}}{a} \cos^3 u, \frac{- c^2\sqrt{2}}{b} \sin^3 u\right). \tag{19}$$

This must satisfy (16). It does if and only if

$$4c^2(b^2 \cos^6 u + a^2 \sin^6 u) = 3a^2b^2(\cos^2 u - \sin^2 u).$$

Making the change of parameter $t = \tan^2 u$, this condition becomes

$$\psi(t) \equiv (4 - k^2)t^3 + 3k^2t^2 - 3k^2t + k^2(1 - 4k^2) = 0. \tag{20}$$

To each non-negative root of this cubic equation in t corresponds a cusp. A study of the roots of (20) for values of k in the interval $0 < k < 1$ will then determine the number of cusps on the envelope.

When $0 < k < \frac{1}{2}$, the constant term in (20) is positive, and there are just two changes of sign in the coefficients. So by Descartes' rule, there are either no positive roots or two (which may be coincident). If there are two distinct positive roots, $\psi(t)$ must have a negative minimum between them, for $\psi(0) > 0$, $\psi(\infty) > 0$. We must then seek a positive root of $\psi'(t)$. Since

$$\tfrac{1}{3}\psi'(t) \equiv (4 - k^2)t^2 + 2k^2t - k^2,$$

the desired root of $\psi'(t)$ is

$$t_1 = k/(2 + k).$$

For this value of t we have

$$(2 + k)^2\psi(t_1) = 4k^2(1 + k)^2(\sqrt{2} + 1 + k)(\sqrt{2} - 1 - k),$$

so that $\psi(t_1)$ is positive when $k < \sqrt{2} - 1$, and negative when $k > \sqrt{2} - 1$. We may then state that (a) if $0 < k < \sqrt{2} - 1$, there is no positive root of (20), and hence no cusp; (b) if $k = \sqrt{2} - 1$, there is one double positive root of (20), and hence one double cusp; and (c) if $\sqrt{2} - 1 < k < \frac{1}{2}$, there are two distinct positive roots of (20), and hence two cusps $(0 \leqq u \leqq \pi/2)$. In the case $k = \sqrt{2} - 1$, we find that the double cusp is given by $t_1 = \tan^2 u_1 = (\sqrt{2} - 1)^2$, $u_1 = \pi/8$.

When $k = \frac{1}{2}, t = 0$ is a root of (20), and there is one positive root, $t = (\sqrt{21} - 1)/10$. The latter gives a cusp, but the former is the limit of two cusps and a node, and lies on the x-axis. (See below under *Nodes.*)

When $\frac{1}{2} < k < 1$, there is just one positive root of (20), by Descartes' rule, and hence just one cusp in the interval $0 \leqq u \leqq \pi/2$.

Nodes. Singularities other than simple cusps may arise in one of two ways. In the first place, the two points corresponding to one value of u may be identical. In this case the two tangents at the point must be coincident, for both branches must be tangent there to the generating circle corresponding to the value of u. As a matter of fact, the only values of u for which the two points (x, y) are coincident are given by the roots (11) of the discriminant (10); these are bounding values for the interval in which the envelope is imaginary, except in the case $k = \sqrt{2} - 1$, when a double cusp occurs.

So the only remaining way in which a node may arise is for points (x, y) corresponding to different values of u to coincide. To study this possibility,

let us set up an equation whose roots indicate the values of u giving points of the envelope which lie on a line through the origin. Setting $y=\lambda x$ in (2), and solving for x, we obtain

$$x = \frac{\sqrt{2}c^2 \sin u}{a \tan u - b\lambda} = \frac{\pm \sqrt{2}a(1-k^2)v}{(v-k\lambda)\sqrt{1+v^2}}, \tag{21}$$

where $v=\tan u$. If we then set $y=\lambda x$ in (1), substitute in the resulting equation the value of x given in (21), and then rationalize and simplify, the following relation is obtained:

$$v^4 + 2k\lambda v^3 - [(1-4k^2)+(4-k^2)\lambda^2]v^2 - 2k\lambda v - k^2\lambda^2 = 0. \tag{22}$$

The case $\lambda=0$ is the most vital one and can be disposed of easily. In this case either $v^2=0$, which gives the two distinct points of vertical tangency on the positive x-axis, or

$$v^2 = 1-4k^2, \qquad v = \tan u = \pm\sqrt{1-4k^2}. \tag{23}$$

So if $0<k<\frac{1}{2}$, equations (21) and (23) give us a node at the point

$$\left(\frac{(1-k^2)a}{\sqrt{1-2k^2}},\ 0\right).$$

When $k=\frac{1}{2}$, this node coalesces with two cusps.

The only real points of the envelope on the y-axis are readily shown from (1) and (2) to be points of horizontal tangency already accounted for.

For a node on the line $y=\lambda x$, $\lambda\neq 0$, two *distinct* roots v_1, v_2 of (22) must give the same value of x in (21). That is, we must have

$$\frac{\pm v_1}{(v_1-k\lambda)\sqrt{1+v_1^2}} = \frac{\pm v_2}{(v_2-k\lambda)\sqrt{1+v_2^2}}.$$

Squaring, combining terms, and dividing out the non-vanishing factor (v_1-v_2), this becomes

$$2k\lambda v_1v_2 + v_1^2v_2^2(v_1+v_2) = k^2\lambda^2(v_1+v_2) + 2k\lambda v_1^2v_2^2. \tag{24}$$

On the other hand, if v_3, v_4 are the other two roots of (22), the following relations hold between roots and coefficients:

$$(v_1+v_2)+(v_3+v_4) = -2k\lambda, \tag{25}$$

$$v_1v_2(v_3+v_4)+(v_1+v_2)v_3v_4 = 2k\lambda, \tag{26}$$

$$v_1v_2v_3v_4 = -k^2\lambda^2. \tag{27}$$

Using the expressions for (v_3+v_4) and v_3v_4 obtained from (25) and (27) respectively, (26) reduces at once to

$$(28) \qquad 2k\lambda v_1v_2 + v_1^2v_2^2(v_1 + v_2) = - k^2\lambda^2(v_1 + v_2) - 2k\lambda v_1^2v_2^2 .$$

Comparing (28) with (24), we see that

$$2k\lambda v_1v_2 + v_1^2v_2^2(v_1 + v_2) = 0,$$
$$k^2\lambda^2(v_1 + v_2) + 2k\lambda v_1^2v_2^2 = 0,$$

or, since none of the quantities k, λ, v_1, v_2 vanishes,

$$-\frac{v_1v_2}{k\lambda} = \frac{2}{v_1 + v_2} = \frac{v_1 + v_2}{2v_1v_2},$$
$$(v_1 + v_2)^2 = 4v_1v_2,$$
$$(v_1 - v_2)^2 = 0,$$

which is a contradiction of the assumption that v_1 and v_2 are distinct. Hence no nodes exist other than those on the x-axis.

The singularities may then be summarized from the above and from symmetry as follows:

$k=0$: Two isolated points: $(\pm a/\sqrt{2}, 0)$.
$0<k<\sqrt{2}-1$: Two nodes:

$$\left(\pm\frac{(1 - k^2)a}{\sqrt{1 - 2k^2}}, 0\right).$$

$k=\sqrt{2}-1$: Double cusps (one in each quadrant); two nodes as above.
$\sqrt{2}-1<k<\frac{1}{2}$: Cusps (two in each quadrant) two nodes as above.
$k=\frac{1}{2}$: Cusps (one in each quadrant); limit of two cusps and a node at each of the points $(\pm\frac{3}{4}a\sqrt{2}, 0)$.
$\frac{1}{2}<k<1$: Cusps (one in each quadrant).
$k=1$: Isolated point: (0, 0).

We might ask: May a cuspidal tangent be horizontal? This is readily answered. The coördinates (13)and (19) are the same if and only if

$$\tan^2 u = 1 - 2k^2.$$

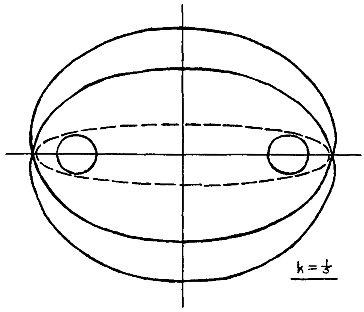
But from (14), we must have for a horizontal tangent

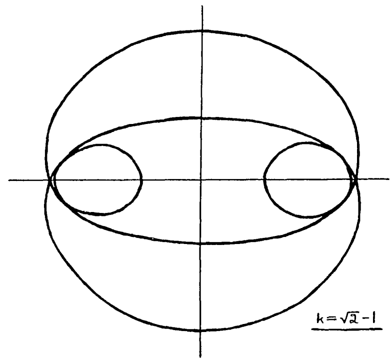
$$\tan^2 u = \frac{k^4}{1 - 3k^2 + k^4}\cdot$$

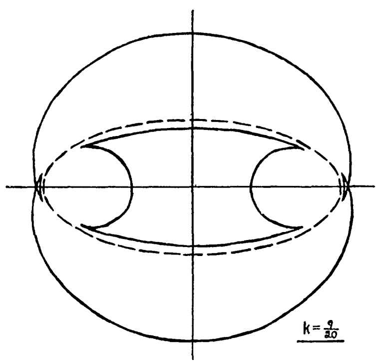
These are the same when $k^2=(2-\sqrt{2})/2$, and then $t=\tan {}^2u=\sqrt{2}-1$. These values for k^2 and t satisfy (20), and hence the cuspidal tangent is horizontal for that value of k.

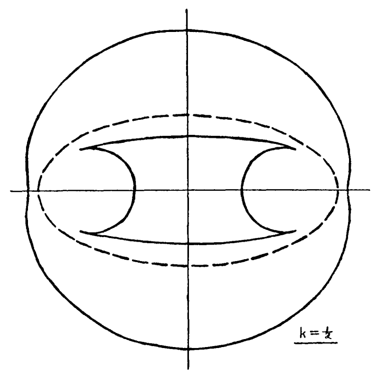
Similar reasoning shows that no real vertical cuspidal tangent exists.

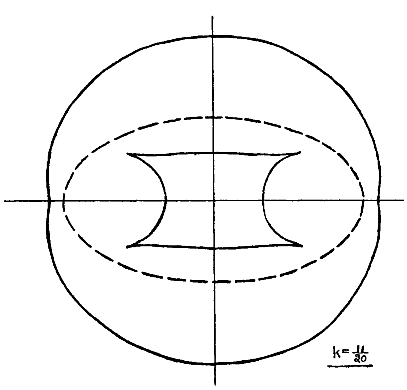
Order of envelope. Since each root v of the equation (22) gives two values of u differing by π, and since the two points corresponding to one value of u do not

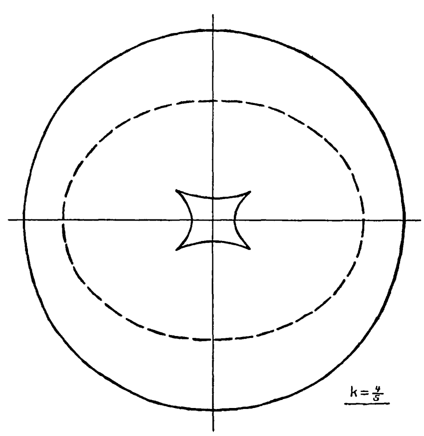

$k = \frac{1}{3}$


$k = \sqrt{2} - 1$


$k = \frac{9}{20}$


$k = \frac{1}{2}$


$k = \frac{11}{20}$


$k = \frac{4}{5}$

in general lie on a line with the origin (cf. equation (2), which represents the line joining them), it appears that the line $y=\lambda x$ cuts the envelope in general in eight points, so that the envelope is an octic curve.

The equation (22) yields another interesting result. If $\lambda=k$, (22) reduces to $(v+k^2)^2(v^2-1)=0$. The root $v=+1$ gives the point (a, b) already found, while the double root $v=-k^2$ indicates that the line $y=kx$ is tangent to the envelope. This is also true for the line $y=-kx$.

The figures show the different stages of the envelope for different values of k. The dashed lines represent the original ellipse, except for $k=\sqrt{2}-1$, where it is part of the envelope. The case $k=11/20$ is nearly the same as the one having horizontal cuspidal tangents. Complete understanding of the arrangement of the cusps and nodes was only brought about by actually drawing in a number of the enveloping circles in each case.

This work has been brought to your attention not with any claim to originality of method, but in the hope that it may bring home the essential power of parametric methods so often slighted in the ordinary formal course.

DISPLACEMENTS OF A RIGID BODY

By C. J. COE, University of Michigan

1. *Introduction.* This paper presents what are believed to be new and simpler proofs in vector form of the basic theorems and formulas concerning the finite displacements of a rigid body. An extension to the case of continuous motion follows. The discussion presupposes only an understanding of the elements of vector algebra, Gibbs's notation being used.[1]

Most modern texts on mechanics express in vector form the basic theorems and formulas on the continuous motion of a rigid body, i.e. Poisson's formula, Mozzi's theorem, the composition of angular velocities, etc., and as so expressed they possess a very striking clearness and simplicity. But a similar treatment of the finite displacements is lacking, the reason being doubtless that the discussions to be found in the literature are in coördinate form or involve concepts much more complex than that of the simple vector. This is peculiarly unfortunate for the two topics are closely associated and both lend themselves very well to vector treatment. The natural extension from the case of finite displacements to that of continuous motion is one of the most satisfying features of the discussion.

2. *Rodrigues' Formula.* Suppose a rigid body to rotate through an angle θ in the positive sense about an axis having the direction and sense of a given unit vector $\boldsymbol{u}$. Let R_1 and R_2 be the initial and terminal positions of a point R

[1] Compare: Hamilton, Wm., *Lectures on Quaternions*, Dublin (1853); Ball, R. S., *Theory of Screws*, Cambridge (1900); Gibbs, J. W., *Vector Analysis*, Yale (1901), p. 334; Study, E., *Geometrie der Dynamen*, Leipzig (1903).

of the rigid body, M the common foot of the perpendiculars let fall from R_1 and R_2 upon the fixed axis. Then from the figure we have

$$\pm (\cos\theta/2)\,|\overline{NR}_2| = (\sin\theta/2)\,|\overline{MN}|$$

where $|\overline{NR}_2|$ means the length of the vector $\overline{NR}_2$, etc. Also from the figure

$$\pm\ |\overline{MN}|\overline{NR}_2 = |\overline{NR}_2|\boldsymbol{u}\times\overline{MN}$$

and consequently

$$(\cos\theta/2)\overline{NR}_2 = (\sin\theta/2)\boldsymbol{u}\times\overline{MN}.$$

If we call

$$\overline{MR}_1 = \boldsymbol{p}_1,\qquad \overline{MR}_2 = \boldsymbol{p}_2,\qquad (\sin\theta/2)\boldsymbol{u} = \boldsymbol{n},\qquad \cos\theta/2 = m$$

the above becomes

$$m(\boldsymbol{p}_2 - \boldsymbol{p}_1) = \boldsymbol{n}\times(\boldsymbol{p}_2+\boldsymbol{p}_1),\qquad m^2+\boldsymbol{n}^2 = 1.$$

If now O be any point of the fixed axis and if we call $\overline{OR}_1=\boldsymbol{r}_1$, $\overline{OR}_2=\boldsymbol{r}_2$, then since

$$\boldsymbol{r}_2-\boldsymbol{r}_1 = \boldsymbol{p}_2-\boldsymbol{p}_1,\qquad \boldsymbol{n}\times(\boldsymbol{r}_2+\boldsymbol{r}_1) = \boldsymbol{n}\times(\boldsymbol{p}_2+\boldsymbol{p}_1)$$

it follows that

$$(1)\qquad m(\boldsymbol{r}_2-\boldsymbol{r}_1) = \boldsymbol{n}\times(\boldsymbol{r}_2+\boldsymbol{r}_1),\qquad m^2+\boldsymbol{n}^2 = 1.$$

We shall call this Rodrigues' formula.[1] For $m\neq 0$ the formula may evidently be written in the simple equivalent form

$$(2)\qquad \boldsymbol{r}_2-\boldsymbol{r}_1 = \boldsymbol{w}\times(\boldsymbol{r}_2+\boldsymbol{r}_1),\qquad \boldsymbol{w} = (\tan\theta/2)\boldsymbol{u}.$$

To solve equations (1) for $\boldsymbol{r}_2$ in terms of $\boldsymbol{r}_1$, m, $\boldsymbol{n}$ we obtain from (1) the equations

$$\begin{aligned} m^3(\boldsymbol{r}_2-\boldsymbol{r}_1) &= m^2\boldsymbol{n}\times(\boldsymbol{r}_2+\boldsymbol{r}_1)\\ m^2\boldsymbol{n}\times(\boldsymbol{r}_2-\boldsymbol{r}_1) &= m\boldsymbol{n}\cdot(\boldsymbol{r}_2+\boldsymbol{r}_1)\boldsymbol{n} - m\boldsymbol{n}^2(\boldsymbol{r}_2+\boldsymbol{r}_1)\\ m\boldsymbol{n}\cdot(\boldsymbol{r}_2-\boldsymbol{r}_1)\boldsymbol{n} &= 0,\end{aligned}$$

and on adding member for member we find

$$m(\boldsymbol{r}_2+\boldsymbol{r}_1) = 2m\{m^2\boldsymbol{r}_1+(\boldsymbol{n}\cdot\boldsymbol{r}_1)\boldsymbol{n}+m\boldsymbol{n}\times\boldsymbol{r}_1\}.$$

For $m\neq 0$ we have therefore

$$(3)\qquad \boldsymbol{r}_2+\boldsymbol{r}_1 = 2\{m^2\boldsymbol{r}_1+(\boldsymbol{n}\cdot\boldsymbol{r}_1)\boldsymbol{n}+m\boldsymbol{n}\times\boldsymbol{r}_1\},\qquad m^2+\boldsymbol{n}^2 = 1$$

or

$$(4)\qquad \boldsymbol{r}_2+\boldsymbol{r}_1 = \frac{2}{1+\boldsymbol{w}^2}\{\boldsymbol{r}_1+(\boldsymbol{w}\cdot\boldsymbol{r}_1)\boldsymbol{w}+\boldsymbol{w}\times\boldsymbol{r}_1\}$$

[1] Journal de Mathématiques, vol. 5 (1840), p. 380.

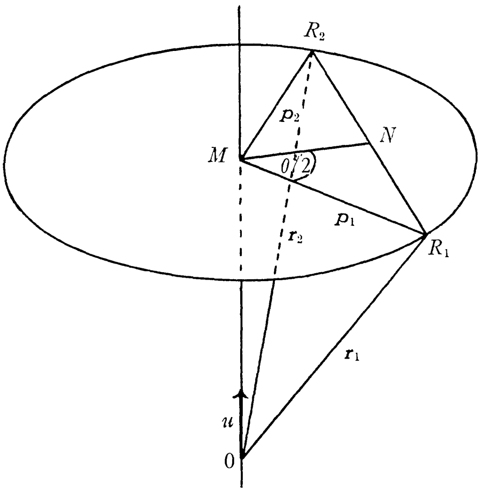
which are immediately solvable for $\boldsymbol{r}_2$. For $m=0$ equations (1) can not be directly solved for $\boldsymbol{r}_2$ but equation (3) must still hold if we make the additional assumption that $\boldsymbol{r}_2$ is a continuous function of $\boldsymbol{r}_1$, m, $\boldsymbol{n}$. The equation then becomes

$$\boldsymbol{r}_2 + \boldsymbol{r}_1 = 2(\boldsymbol{u}\cdot\boldsymbol{r}_1)\boldsymbol{u}. \tag{5}$$

We shall regard this as the form taken by Rodrigues' formula for the case $m=0$, i.e. $\theta=180°$. Since equations (1), (2), (3), (4), (5) involve $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ linearly it follows that they express the relation between the initial and final values of any vector attached to the rigid body.

3. *Euler's Theorem.* We may now readily prove Euler's famous theorem[1] on the displacements of a rigid body with one fixed point:

Any displacement of a rigid body which leaves one of its points fixed may be produced by a rotation of the body through an angle of 180° or less about an axis passing through that point.



Let the position of the body be determined by the position of the fixed point O and two others of its points P and R such that O, P, R are not collinear. Let the initial positions of these points be O, P_1, R_1 and their terminal positions O, P_2, R_2 and let us call $\overline{OP_1}=\boldsymbol{p}_1$, $\overline{OP_2}=\boldsymbol{p}_2$, $\overline{OR_1}=\boldsymbol{r}_1$, $\overline{OR_2}=\boldsymbol{r}_2$. If then we can find a vector $\boldsymbol{w}$ satisfying the equations

$$\boldsymbol{p}_2 - \boldsymbol{p}_1 = \boldsymbol{w} \times (\boldsymbol{p}_2 + \boldsymbol{p}_1), \qquad \boldsymbol{r}_2 - \boldsymbol{r}_1 = \boldsymbol{w} \times (\boldsymbol{r}_2 + \boldsymbol{r}_1) \tag{6}$$

or find a unit vector $\boldsymbol{u}$ satisfying the equations

$$\boldsymbol{p}_2 + \boldsymbol{p}_1 = 2(\boldsymbol{u}\cdot\boldsymbol{p}_1)\boldsymbol{u}, \qquad \boldsymbol{r}_2 + \boldsymbol{r}_1 = 2(\boldsymbol{u}\cdot\boldsymbol{r}_1)\boldsymbol{u} \tag{7}$$

our theorem is proven. This follows from the fact observed above that if $\boldsymbol{w}$ and $\boldsymbol{r}_1$ are given equation (2) determines $\boldsymbol{r}_2$, and if $\boldsymbol{u}$ and $\boldsymbol{r}_1$ are given equation

[1] Novi. Comm. Acad. Petrop. vol. 20 (1775) ed. 1776, p. 202.

(5) determines $\mathbf{r}_2$. For brevity let us call $\mathbf{p}_2-\mathbf{p}_1=\mathbf{b}_1$, $\mathbf{r}_2-\mathbf{r}_1=\mathbf{b}_2$, $\mathbf{p}_2+\mathbf{p}_1=\mathbf{a}_1$, $\mathbf{r}_2+\mathbf{r}_1=\mathbf{a}_2$ and write equations (6) as

$$\mathbf{b}_1 = \mathbf{w} \times \mathbf{a}_1, \qquad \mathbf{b}_2 = \mathbf{w} \times \mathbf{a}_2. \tag{6}$$

Since the points O, P, R are not collinear we have

$$\mathbf{p}_1 \times \mathbf{r}_1 \neq 0 \tag{a}$$

while from the known properties of a rigid body we may write

$$\mathbf{a}_1\cdot\mathbf{b}_1 = \mathbf{p}_2^2 - \mathbf{p}_1^2 = 0, \qquad \mathbf{a}_2\cdot\mathbf{b}_2 = \mathbf{r}_2^2 - \mathbf{r}_1^2 = 0 \tag{b}$$

$$\mathbf{a}_1\cdot\mathbf{b}_2 + \mathbf{a}_2\cdot\mathbf{b}_1 = 2(\mathbf{p}_2\cdot\mathbf{r}_2 - \mathbf{p}_1\cdot\mathbf{r}_1) = 0 \tag{c}$$

$$(\mathbf{a}_2 \times \mathbf{a}_1)\cdot(\mathbf{b}_2 \times \mathbf{b}_1) = (\mathbf{a}_2\cdot\mathbf{b}_1)^2 = (\mathbf{a}_1\cdot\mathbf{b}_2)^2 \tag{d}$$

$$(\mathbf{a}_2 \times \mathbf{a}_1) \times (\mathbf{b}_2 \times \mathbf{b}_1) = (\mathbf{a}_2\cdot\mathbf{b}_1)\mathbf{b}_2 \times \mathbf{a}_1 + (\mathbf{a}_1\cdot\mathbf{b}_2)\mathbf{b}_1 \times \mathbf{a}_2. \tag{e}$$

We consider first the case $\mathbf{a}_2\times\mathbf{a}_1=0$ in which we shall establish the existence of a unit vector $\mathbf{u}$ satisfying equations (7), the rotation being through 180° for this case only. There are essentially two possibilities to be discussed.

$$\text{I},1 \qquad \mathbf{a}_2 \times \mathbf{a}_1 = 0, \qquad \mathbf{p}_2 + \mathbf{p}_1 = \mathbf{r}_2 + \mathbf{r}_1 = 0$$

$$\text{I},2 \qquad \mathbf{a}_2 \times \mathbf{a}_1 = 0, \qquad \mathbf{p}_2 + \mathbf{p}_1 \neq 0.$$

For case I, 1 we choose for $\mathbf{u}$ the unit vector

$$\mathbf{u} = \frac{\mathbf{p}_1 \times \mathbf{r}_1}{|\mathbf{p}_1 \times \mathbf{r}_1|},$$

and have at once

$$\mathbf{p}_2 + \mathbf{p}_1 = 0 = \frac{2[\mathbf{p}_1\mathbf{r}_1\mathbf{p}_1]}{(\mathbf{p}_1 \times \mathbf{r}_1)^2} = \mathbf{p}_1 + \mathbf{r}_1 = 2(\mathbf{u}\cdot\mathbf{p}_1)\mathbf{u}$$

$$\mathbf{r}_2 + \mathbf{r}_1 = 0 = \frac{2[\mathbf{p}_1\mathbf{r}_1\mathbf{r}_1]}{(\mathbf{p}_1 \times \mathbf{r}_1)^2} = \mathbf{p}_1 \times \mathbf{r}_1 = 2(\mathbf{u}\cdot\mathbf{r}_1)\mathbf{u},$$

thus verifying equations (7). For case I, 2 we choose for $\mathbf{u}$ the unit vector

$$\mathbf{u} = \frac{\mathbf{p}_2 + \mathbf{p}_1}{|\mathbf{p}_2 + \mathbf{p}_1|},$$

and remembering that $\mathbf{p}_2^2 = \mathbf{p}_1^2$ we write

$$\mathbf{p}_2 + \mathbf{p}_1 = \frac{\mathbf{p}_2^2 + \mathbf{p}_1^2 + 2\mathbf{p}_2\cdot\mathbf{p}_1}{(\mathbf{p}_2 + \mathbf{p}_1)^2}(\mathbf{p}_2 + \mathbf{p}_1) = \frac{2(\mathbf{p}_2 + \mathbf{p}_1)\cdot\mathbf{p}_1}{(\mathbf{p}_2 + \mathbf{p}_1)^2}(\mathbf{p}_2 + \mathbf{p}_1) = 2(\mathbf{u}\cdot\mathbf{p}_1)\mathbf{u},$$

thus verifying the first of equations (7). By hypothesis

$$0 = \mathbf{a}_2 \times \mathbf{a}_1 = (\mathbf{r}_2 + \mathbf{r}_1) \times (\mathbf{p}_2 + \mathbf{p}_1)$$

while by equation (d)

$$0 = \mathbf{a}_1\cdot\mathbf{b}_2 = (\mathbf{p}_2 + \mathbf{p}_1)\cdot(\mathbf{r}_2 - \mathbf{r}_1)$$

so that we have

$$0 = (\boldsymbol{p}_2 + \boldsymbol{p}_1) \times \{(\boldsymbol{r}_2 + \boldsymbol{r}_1) \times (\boldsymbol{p}_2 + \boldsymbol{p}_1)\}$$
$$= (\boldsymbol{p}_2 + \boldsymbol{p}_1)^2(\boldsymbol{r}_2 + \boldsymbol{r}_1) - (\boldsymbol{p}_2 + \boldsymbol{p}_1)\cdot(\boldsymbol{r}_2 + \boldsymbol{r}_1)(\boldsymbol{p}_2 + \boldsymbol{p}_1).$$

It follows that

$$\boldsymbol{r}_2 + \boldsymbol{r}_1 = \frac{(\boldsymbol{p}_2 + \boldsymbol{p}_1)\cdot(\boldsymbol{r}_2 + \boldsymbol{r}_1)}{(\boldsymbol{p}_2 + \boldsymbol{p}_1)^2}(\boldsymbol{p}_2 + \boldsymbol{p}_1) = \frac{2(\boldsymbol{p}_2 + \boldsymbol{p}_1)\cdot\boldsymbol{r}_1}{(\boldsymbol{p}_2 + \boldsymbol{p}_1)^2}(\boldsymbol{p}_2 + \boldsymbol{p}_1) = 2(\boldsymbol{u}\cdot\boldsymbol{r}_1)\boldsymbol{u}$$

which verifies the second of equations (7).

We next consider the case $\boldsymbol{a}_2 \times \boldsymbol{a}_1 \neq 0$ in which we shall show that there exists a vector $\boldsymbol{w}$ satisfying equations (6). There are here two possibilities to be discussed,

II, 1 $\qquad \boldsymbol{a}_2 \times \boldsymbol{a}_1 \neq 0, \qquad \boldsymbol{b}_2 \times \boldsymbol{b}_1 = 0$

II, 2 $\qquad \boldsymbol{a}_2 \times \boldsymbol{a}_1 \neq 0, \qquad \boldsymbol{b}_2 \times \boldsymbol{b}_1 \neq 0.$

For case II, 1 we choose for $\boldsymbol{w}$ the vector

$$\boldsymbol{w} = \frac{[\boldsymbol{a}_1\boldsymbol{a}_2\boldsymbol{b}_2]\boldsymbol{a}_1 + [\boldsymbol{a}_2\boldsymbol{a}_1\boldsymbol{b}_1]\boldsymbol{a}_2}{(\boldsymbol{a}_2 \times \boldsymbol{a}_1)^2}.$$

To verify the first of equations (6) we observe that under our hypothesis $\boldsymbol{b}_2 \times \boldsymbol{b}_1 = 0$ we have, by equation (d), $\boldsymbol{a}_1\cdot\boldsymbol{b}_2 = \boldsymbol{a}_2\cdot\boldsymbol{b}_1 = 0$ and hence

$$\boldsymbol{b}_1 \times (\boldsymbol{a}_2 \times \boldsymbol{a}_1) = (\boldsymbol{a}_1\cdot\boldsymbol{b}_1)\boldsymbol{a}_2 - (\boldsymbol{a}_2\cdot\boldsymbol{b}_1)\boldsymbol{a}_1 = 0.$$

Consequently we may write

$$0 = (\boldsymbol{a}_2 \times \boldsymbol{a}_1) \times \{\boldsymbol{b}_1 \times (\boldsymbol{a}_2 \times \boldsymbol{a}_1)\} = (\boldsymbol{a}_2 \times \boldsymbol{a}_1)^2\boldsymbol{b}_1 - [\boldsymbol{a}_2\boldsymbol{a}_1\boldsymbol{b}_1]\boldsymbol{a}_2 \times \boldsymbol{a}_1$$

and from this we have at once

$$\boldsymbol{b}_1 = \frac{[\boldsymbol{a}_2\boldsymbol{a}_1\boldsymbol{b}_1]\boldsymbol{a}_2 \times \boldsymbol{a}_1}{(\boldsymbol{a}_2 \times \boldsymbol{a}_1)^2} = \boldsymbol{w} \times \boldsymbol{a}_1$$

thus verifying the first of equations (6). The verification of the second equation follows by a mere interchange of subscripts. In discussing the general case II, 2 we first observe that neither $\boldsymbol{a}_2\cdot\boldsymbol{b}_1$ nor $\boldsymbol{a}_1\cdot\boldsymbol{b}_2$ can be zero, for by equations (d) and (e) we would have either $\boldsymbol{a}_2 \times \boldsymbol{a}_1 = 0$ or $\boldsymbol{b}_2 \times \boldsymbol{b}_1 = 0$ contrary to hypothesis. We choose for $\boldsymbol{w}$ the vector

$$\boldsymbol{w} = \frac{\boldsymbol{b}_2 \times \boldsymbol{b}_1}{\boldsymbol{a}_1\cdot\boldsymbol{b}_2} = \frac{\boldsymbol{b}_1 \times \boldsymbol{b}_2}{\boldsymbol{a}_2\cdot\boldsymbol{b}_1}$$

and equations (6) then become

$$\boldsymbol{w} \times \boldsymbol{a}_1 = \frac{(\boldsymbol{b}_2 \times \boldsymbol{b}_1) \times \boldsymbol{a}_1}{\boldsymbol{a}_1\cdot\boldsymbol{b}_2} = \frac{(\boldsymbol{a}_1\cdot\boldsymbol{b}_2)\boldsymbol{b}_1 - (\boldsymbol{a}_1\cdot\boldsymbol{b}_1)\boldsymbol{b}_2}{\boldsymbol{a}_1\cdot\boldsymbol{b}_2} = \boldsymbol{b}_1$$

$$w \times a_2 = \frac{(b_1 \times b_2) \times a_2}{a_2 \cdot b_1} = \frac{(a_2 \cdot b_1)b_2 - (a_2 \cdot b_2)b_1}{a_2 \cdot b_1} = b_2$$

and are thus verified. This completes the proof of Euler's theorem.

It is evident that the vector w or u here given will satisfy equation (2) or (5) not only for the three points O, P, R here considered but for every other point of the rigid body. The vector w is unique and u is unique except in sense.

4. *Composition of Rotations.* If a rigid body with a fixed point O be given a rotation characterized in our notation by w_1 followed by a rotation characterized by w_2 then by Euler's theorem there must exist a rotation carrying the body from its initial to its final position. Assuming that none of these rotations is through 180° let us seek to determine the vector w_3 characterizing the combined rotation. From Rodrigue's formula (2) we have

$$r_2 - r_1 = w_1 \times (r_2 + r_1) \tag{8}$$

$$r_3 - r_2 = w_2 \times (r_3 + r_2). \tag{9}$$

Multiplying (8) by $w_1 \cdot$ and (9) by $w_2 .$ gives us

$$0 = (w_1 \cdot r_2 - w_1 \cdot r_1)w_2 \tag{10}$$

$$0 = (w_2 \cdot r_3 - w_2 \cdot r_2)w_1. \tag{11}$$

Multiplying (8) by $w_2 \times$ and (9) by $-w_1 \times$ gives us

$$w_2 \times r_2 - w_2 \times r_1 = w_2 \cdot (r_2 + r_1)w_1 - w_2 \cdot w_1(r_2 + r_1) \tag{12}$$

$$w_1 \times r_2 - w_1 \times r_3 = - w_1 \cdot (r_3 + r_2)w_2 + w_2 \cdot w_1(r_3 + r_2). \tag{13}$$

We now add equations (8), (9), (10), (11), (12), (13) member for member obtaining

$$(r_3 - r_1)(1 - w_2 \cdot w_1) = (w_1 + w_2 + w_2 \times w_1) \times (r_3 + r_1).$$

If we call

$$w_3 = \frac{w_1 + w_2 + w_2 \times w_1}{1 - w_2 \cdot w_1}$$

our last equation becomes

$$r_3 - r_1 = w_3 \times (r_3 + r_1). \tag{14}$$

This being Rodrigue's formula for the combined rotation shows that the rotation is characterized by w_3 as here given. The discussion of the cases in which some of these rotations are through 180° is not given here as it presents no points of particular interest.

5. *Chasles's Theorem.* The formal work done in the proof of Euler's theorem makes it easy to establish Chasles's theorem:[1]

Any displacement of a rigid body may be brought about by a translation in a

[1] Bulletin universel des Sciences, vol. 14, (1830).

certain direction combined with a rotation about an axis running in that direction. If A and R be any two points of a rigid body and if the body be given any displacement in which A_1, R_1 and A_2, R_2 are the initial and terminal positions of these points then the identical analytic argument employed in §3 now shows that there exists a vector $\boldsymbol{w}$ or a unit vector $\boldsymbol{u}$ independent of the choice of A and R satisfying the equation

$$(15) \qquad (\boldsymbol{r}_2 - \boldsymbol{a}_2) - (\boldsymbol{r}_1 - \boldsymbol{a}_1) = \boldsymbol{w} \times \{(\boldsymbol{r}_2 - \boldsymbol{a}_2) + (\boldsymbol{r}_1 - \boldsymbol{a}_1)\}$$

or the equation

$$(16) \qquad (\boldsymbol{r}_2 - \boldsymbol{a}_2) + (\boldsymbol{r}_1 - \boldsymbol{a}_1) = 2\boldsymbol{u}\cdot(\boldsymbol{r}_1 - \boldsymbol{a}_1)\boldsymbol{u},$$

where $\boldsymbol{a}_1 = \overline{OA}_1$, $\boldsymbol{a}_2 = \overline{OA}_2$, $\boldsymbol{r}_1 = \overline{OR}_1$, $\boldsymbol{r}_2 = \overline{OR}_2$; O being any fixed point. Let us first discuss the case in which a vector $\boldsymbol{w}$ exists satisfying equation (15). If $\boldsymbol{w} = 0$ it follows that $\boldsymbol{r}_2 - \boldsymbol{r}_1 = \boldsymbol{a}_2 - \boldsymbol{a}_1$ and if the point A be chosen and fixed we have $\boldsymbol{r}_2 - \boldsymbol{r}_1$ a constant independent of the choice of R. The displacements of all points of the body being thus identical the displacement of the body is a translation and Chasles' theorem is proven for this case. If $\boldsymbol{w}$ is not zero let us momentarily fix the point R so that $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ are known quantities and then choose for A that point of the body whose initial position is given by

$$\boldsymbol{a}_1 = \frac{1}{2}\left\{\boldsymbol{r}_2 + \boldsymbol{r}_1 + \frac{\boldsymbol{w} \times (\boldsymbol{r}_2 - \boldsymbol{r}_1)}{w^2}\right\}.$$

Following the method used in deriving equation (4) we obtain from equation (15)

$$(\boldsymbol{r}_2 - \boldsymbol{a}_2) + (\boldsymbol{r}_1 - \boldsymbol{a}_1) = \frac{2}{1 + w^2}\{\boldsymbol{r}_1 - \boldsymbol{a}_1 + \boldsymbol{w}\cdot(\boldsymbol{r}_1 - \boldsymbol{a}_1)\boldsymbol{w} + \boldsymbol{w} \times (\boldsymbol{r}_1 - \boldsymbol{a}_1)\}$$

which for our special choice of $\boldsymbol{a}_1$ yields

$$\boldsymbol{a}_2 - \boldsymbol{a}_1 = \frac{\boldsymbol{w}\cdot(\boldsymbol{r}_2 - \boldsymbol{r}_1)}{w^2}\boldsymbol{w} = \frac{\boldsymbol{w}\cdot(\boldsymbol{a}_2 - \boldsymbol{a}_1)}{w^2}\boldsymbol{w} = k\boldsymbol{w}.$$

The point A being now chosen, the scalar k is independent of the subsequent choice of R and we may write equation (15) in the form

$$\boldsymbol{r}_2 - \boldsymbol{r}_1 = k\boldsymbol{w} + \boldsymbol{w} \times \{(\boldsymbol{r}_2 + \boldsymbol{r}_1) - (\boldsymbol{a}_2 + \boldsymbol{a}_1)\}$$

which must hold for every point R of the rigid body. If we write this in the form

$$(\boldsymbol{r}_2 - \boldsymbol{a}) - (\boldsymbol{r}_1 - \boldsymbol{a} + k\boldsymbol{w}) = \boldsymbol{w} \times \{(\boldsymbol{r}_2 - \boldsymbol{a}) + (\boldsymbol{r}_1 - \boldsymbol{a} + k\boldsymbol{w})\},$$

where $\boldsymbol{a} = \frac{1}{2}(\boldsymbol{a}_2 + \boldsymbol{a}_1)$ and compare with Rodrigues' formula (2) we see that it is a statement that the given displacement may be effected by a translation thru the amount $k\boldsymbol{w}$ followed by a rotation about an axis passing through A_0 in the direction of $\boldsymbol{w}$ ($\overline{OA}_0 = \boldsymbol{a}$). Or again if we write equation (15) in the form

$$(\boldsymbol{r}_2 - \boldsymbol{a} - k\boldsymbol{w}) - (\boldsymbol{r}_1 - \boldsymbol{a}) = \boldsymbol{w} \times \{(\boldsymbol{r}_2 - \boldsymbol{a} - k\boldsymbol{w}) + (\boldsymbol{r}_1 - \boldsymbol{a})\}$$

we see that it is a statement that the given displacement may be effected by the above rotation followed by the above translation.

If a vector $\boldsymbol{w}$ satisfying equation (15) does not exist then there exists a unit vector $\boldsymbol{u}$ satisfying equation (16). Let us momentarily fix the point R so that $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ are known quantities and then choose for A that point of the body whose initial position is given by

$$\boldsymbol{a}_1 = \tfrac{1}{2}(\boldsymbol{r}_2 + \boldsymbol{r}_1).$$

From equation (16) it follows that this is equivalent to choosing A so that

$$\boldsymbol{a}_2 - \boldsymbol{a}_1 = \boldsymbol{u}\cdot(\boldsymbol{r}_2 - \boldsymbol{r}_1)\boldsymbol{u} = \boldsymbol{u}\cdot(\boldsymbol{a}_2 - \boldsymbol{a}_1)\boldsymbol{u} = k\boldsymbol{u}.$$

The point A being now chosen, the scalar k is independent of the subsequent choice of R and we may write equation (16) in the form

$$(\boldsymbol{r}_2 - \boldsymbol{a}) + (\boldsymbol{r}_1 - \boldsymbol{a} + k\boldsymbol{u}) = 2\{\boldsymbol{u}\cdot(\boldsymbol{r}_1 - \boldsymbol{a} + k\boldsymbol{u})\}\boldsymbol{u}, \qquad \boldsymbol{a} = \tfrac{1}{2}(\boldsymbol{a}_2 + \boldsymbol{a}_1)$$

which must hold for every point R of the rigid body. If we compare this with Rodrigues' formula (5) we see that it is a statement that the given displacement may be effected by a translation through the amount $k\boldsymbol{u}$ followed by a rotation of 180° about an axis passing through A_0 in the direction of $\boldsymbol{u}$ ($\overline{OA}_0 = \boldsymbol{a}$). Or again if we write equation (16) in the form

$$(\boldsymbol{r}_2 - \boldsymbol{a} - k\boldsymbol{u}) + (\boldsymbol{r}_1 - \boldsymbol{a}) = 2\{\boldsymbol{u}\cdot(\boldsymbol{r}_1 - \boldsymbol{a})\}\boldsymbol{u}$$

we see that it is a statement that the given displacement may be effected by the above rotation followed by the above translation.

A displacement of the above type, i.e., a rotation about a certain axis combined with a translation along that axis is known as a twist. We have seen that the initial and terminal positions of every point of a rigid body subject to a twist are related by the formula

$$\boldsymbol{r}_2 - \boldsymbol{r}_1 = k\boldsymbol{w} + \boldsymbol{w} \times (\boldsymbol{r}_2 + \boldsymbol{r}_1 - 2\boldsymbol{a})$$

where $k\boldsymbol{w}$ is the translation and $\boldsymbol{w}$ characterizes the rotation about an axis through A_0 ($\overline{OA}_0 = \boldsymbol{a}$). We may also conveniently write this relation in the form

$$\boldsymbol{r}_2 - \boldsymbol{r}_1 = k\boldsymbol{w} + 2\boldsymbol{m} + \boldsymbol{w} \times (\boldsymbol{r}_2 + \boldsymbol{r}_1), \qquad \boldsymbol{w}\cdot\boldsymbol{m} = 0 \tag{17}$$

where $\boldsymbol{m} = \boldsymbol{a} \times \boldsymbol{w}$ is the moment of the rotation vector $\boldsymbol{w}$ with respect to O.

6. *Composition of Twists*. If a rigid body be given a twist characterized as in the previous section by k_1, $\boldsymbol{w}_1$, $\boldsymbol{m}_1$ followed by a twist characterized by k_2, $\boldsymbol{w}_2$, $\boldsymbol{m}_2$ then we know by Chasles' theorem that there exists a twist carrying the body from its initial to its terminal position. If this combined twist be characterized by k_3, $\boldsymbol{w}_3$, $\boldsymbol{m}_3$ let us seek to determine these last three quantities in terms of the first six. We have here by equation (17)

$$(18)\qquad \boldsymbol{r}_2 - \boldsymbol{r}_1 = k_1 w_1 + 2\boldsymbol{m}_1 + w_1 \times (\boldsymbol{r}_2 + \boldsymbol{r}_1), \qquad w_1 \cdot \boldsymbol{m}_1 = 0,$$

$$(19)\qquad \boldsymbol{r}_3 - \boldsymbol{r}_2 = k_2 w_2 + 2\boldsymbol{m}_2 + w_2 \times (\boldsymbol{r}_3 + \boldsymbol{r}_2), \qquad w_2 \cdot \boldsymbol{m}_2 = 0.$$

We apply to these equations the same operations employed in §4 upon equations (8) and (9) and obtain

$$\begin{aligned}(\boldsymbol{r}_3 - \boldsymbol{r}_1)(1 - w_2 \cdot w_1) &= (w_1 + w_2 + w_2 \times w_1) \times (\boldsymbol{r}_3 + \boldsymbol{r}_1) \\ &+ k_1 w_1 + k_2 w_2 - (k_2 w_2^2 w_1 + k_1 w_1^2 w_2) + (k_1 + k_2) w_2 \times w_1 \\ &+ 2(\boldsymbol{m}_1 + \boldsymbol{m}_2 + w_2 \times \boldsymbol{m}_1 - w_1 \times \boldsymbol{m}_2).\end{aligned}$$

If we call $w_3 = (w_1 + w_2 + w_2 \times w_1)/(1 - w_2 \cdot w_1)$ and then resolve the terms in the last two lines of the above equation into components either parallel to w_3 or perpendicular to w_3 our equation takes the form

$$(20)\qquad \boldsymbol{r}_3 - \boldsymbol{r}_1 = k_3 w_3 + 2\boldsymbol{m}_3 + w_3 \times (\boldsymbol{r}_3 + \boldsymbol{r}_1), \qquad w_3 \cdot \boldsymbol{m}_3 = 0,$$

where the desired quantities are as follows:

$$w_3 = \frac{w_1 + w_2 + w_2 \times w_1}{1 - w_2 \cdot w_1}, \qquad k_3 = \frac{(k_1 w_1 + k_2 w_2) \cdot w_3}{w_3^2} + \frac{2(w_1 \cdot \boldsymbol{m}_2 + w_2 \cdot \boldsymbol{m}_1)}{w_3^2 (1 - w_2 \cdot w_1)}$$

$$\begin{aligned}2\boldsymbol{m}_3 = &\left\{ k_1 + k_2 - \frac{(k_1 w_1 + k_2 w_2) \cdot w_3}{w_3^2} - \frac{2(w_1 \cdot \boldsymbol{m}_2 + w_2 \cdot \boldsymbol{m}_1)}{w_3^2 (1 - w_2 \cdot w_1)} \right\} w_3 \\ &- k_2 (1 + w_2 \cdot w_3) w_1 - k_1 (1 + w_1 \cdot w_3) w_2 \\ &+ \frac{2(\boldsymbol{m}_1 + \boldsymbol{m}_2 + w_2 \times \boldsymbol{m}_1 - w_1 \times \boldsymbol{m}_2)}{1 - w_2 \cdot w_1}.\end{aligned}$$

7. *Continuous Motion.* Let a rigid body move with one point O remaining fixed. Let $\boldsymbol{r} = \overline{OR}$ be the radius vector of any point R of the body at the instant t and let $\boldsymbol{r} + \Delta \boldsymbol{r}$ be the radius vector at the subsequent instant $t + \Delta t$. Then as shown in the proof of Euler's theorem there exists a single vector w such that Rodrigues' formula

$$(2)\qquad \boldsymbol{r}_2 - \boldsymbol{r}_1 = w \times (\boldsymbol{r}_2 + \boldsymbol{r}_1) \quad \text{or} \quad \frac{\Delta \boldsymbol{r}}{\Delta t} = \frac{2w}{\Delta t} \times \left(\boldsymbol{r} + \frac{\Delta \boldsymbol{r}}{2} \right)$$

holds for every point R of the body. Furthermore we saw that $2w/\Delta t$ is a continuous function of the values of $\Delta \boldsymbol{r}/\Delta t$ and $\boldsymbol{r} + \Delta \boldsymbol{r}/2$ for any two points of the body and consequently if $\Delta \boldsymbol{r}/\Delta t$ and $\boldsymbol{r} + \Delta \boldsymbol{r}/2$ approach limits, $2w/\Delta t$ must approach a limit also and the limits must satisfy the above equation. If we then allow Δt to approach zero we have in the limit Poisson's formula[1]

$$(21)\qquad v = \omega \times \boldsymbol{r}$$

where

$$v = \lim_{\Delta t \to 0} \frac{\Delta \boldsymbol{r}}{\Delta t}, \qquad \omega = \lim_{\Delta t \to 0} \frac{2w}{\Delta t}.$$

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[1] *Traité de Mécanique*, Paris (1831).

Here v is the velocity of the point R at the instant t and ω is the vector angular velocity of the body at the instant.

If a rigid body with one fixed point be given a rotation w_1 followed by a rotation w_2 then we have seen, §4, that this is equivalent to a single rotation $w_3=(w_1+w_2+w_2\times w_1)/(1-w_2 . w_1)$. If now the rotations w_1 and w_2 take place in two time intervals both of length Δt and if

$$\lim_{\Delta t\to 0}\frac{2w_1}{\Delta t}=\omega_1,\qquad \lim_{\Delta t\to 0}\frac{2w_2}{\Delta t}=\omega_2$$

then we have evidently

$$\omega_3=\lim_{\Delta t\to 0}\frac{2w_3}{\Delta t}=\omega_1+\omega_2. \tag{22}$$

Since ω_3 is thus independent of the order of the two infinitesimal rotations w_1 and w_2 it appears that they might be carried out simultaneously without affecting ω_3. If then we procede to the limit as in deriving Poisson's formula we obtain the law of composition of vector angular velocities:

If a rigid body with one fixed point be given simultaneous angular velocities ω_1 *and* ω_2 *the resulting angular velocity is* $\omega_3=\omega_1+\omega_2$.

If A and R be any two points of a rigid body having radius vectors $\boldsymbol{a}=\overline{OA}$ and $\boldsymbol{r}=\overline{OR}$ at an instant t and radius vectors $\boldsymbol{a}+\Delta\boldsymbol{a}$ and $\boldsymbol{r}+\Delta\boldsymbol{r}$ at a subsequent instant $t+\Delta t$, O being any fixed point, then by equation (15) there exists a unique vector w such that

$$\frac{\Delta \boldsymbol{r}}{\Delta t}-\frac{\Delta a}{\Delta t}=\frac{2w}{\Delta t}\times\left(\boldsymbol{r}-a+\frac{\Delta \boldsymbol{r}}{2}-\frac{\Delta a}{2}\right)$$

for all choices of A and R. If R be given and then A so chosen that

$$\boldsymbol{a}=\boldsymbol{r}+\frac{\dfrac{2w}{\Delta t}\times\dfrac{\Delta \boldsymbol{r}}{\Delta t}}{\left(\dfrac{2w}{\Delta t}\right)^2}+\frac{\Delta \boldsymbol{r}}{2}$$

then we have seen that

$$\frac{\Delta a}{\Delta t}=\frac{\dfrac{2w}{\Delta t}\cdot\dfrac{\Delta a}{\Delta t}}{\left(\dfrac{2w}{\Delta t}\right)^2}\,\frac{2w}{\Delta t}=\frac{k}{2}\,\frac{2w}{\Delta t}\,.$$

With this choice of A we may then write

$$\frac{\Delta \boldsymbol{r}}{\Delta t}=\frac{k}{2}\,\frac{2w}{\Delta t}+\frac{2w}{\Delta t}\times\left(\boldsymbol{r}-a+\frac{\Delta \boldsymbol{r}}{2}-\frac{\Delta a}{2}\right)$$

for all choices of R. Proceeding now to the limit as in deriving Poisson's formula we have finally for all choices of R

$$v = \kappa\omega + \omega \times (r - \alpha), \tag{23}$$

where

$$v = \lim_{\Delta t \to 0} \frac{\Delta r}{\Delta t}, \qquad \omega = \lim_{\Delta t \to 0} \frac{2w}{\Delta t}, \qquad \kappa = \lim_{\Delta t \to 0} \frac{k}{2}$$

and where A is chosen for some given R by the formula $\overline{OA} = \alpha = r + \omega \times v/\omega^2$. Equation (23) enables us to state Mozzi's theorem,[1]

The velocities of the points of a rigid body are at any instant those which they would have if the body were given a certain rotation about a certain axis combined with a certain translation along that axis.

Such a motion is called helicoidal motion. Equation (23) may also be written in the form

$$v = \kappa\omega + \mu + \omega \times r, \qquad \mu = \alpha \times \omega \tag{24}$$

μ being the moment of the angular velocity ω with respect to 0.

If a rigid body be subjected to two successive twists characterized as in §5 by k_1, w_1, m_1 and k_2, w_2, m_2 then the combined twist is characterized by k_3, w_3, m_3 as given in equation (20). If these twists take place in time intervals both of length Δt and if

$$\lim_{\Delta t \to 0} \frac{2w_1}{\Delta t} = \omega_1, \qquad \lim_{\Delta t \to 0} \frac{k_1}{2} = \kappa_1, \qquad \lim_{\Delta t \to 0} \frac{2m_1}{\Delta t} = \mu_1$$

$$\lim_{\Delta t \to 0} \frac{2w_2}{\Delta t} = \omega_2, \qquad \lim_{\Delta t \to 0} \frac{k_2}{2} = \kappa_2, \qquad \lim_{\Delta t \to 0} \frac{2m_2}{\Delta t} = \mu_2$$

then from the definitions of k_3, w_3, m_3 we have

$$\omega_3 = \lim_{\Delta t \to 0} \frac{2w_3}{\Delta t} = \omega_1 + \omega_2$$

$$\kappa_3 = \lim_{\Delta t \to 0} \frac{k_3}{2} = \frac{(\kappa_1\omega_1 + \kappa_2\omega_2 + \mu_1 + \mu_2)\cdot\omega_3}{\omega_3^2}$$

$$\mu_3 = \lim_{\Delta t \to 0} \frac{2m_3}{\Delta t} = \kappa_1\omega_1 + \kappa_2\omega_2 + \mu_1 + \mu_2 - \left\{\frac{(\kappa_1\omega_1 + \kappa_2\omega_2 + \mu_1 + \mu_2)\cdot\omega_3}{\omega_3^2}\right\}\omega_3.$$

Since these quantities are independent of the order of the two infinitesimal twists it appears that they might be carried out simultaneously without af-

[1] *Discorso matematico sopra il rotamento momentaneo dei corpi*, Napoli, (1763).

fecting these limits. If then we procede to the limit as in deriving Poisson's formula we obtain the law of composition of helicoidal motions.

If a rigid body be given simultaneously two helicoidal motions characterized as in equation (24) *by* κ_1, ω_1, $\mathfrak{y}_1$ *and* κ_2, ω_2, $\mathfrak{y}_2$, *then the resulting helicoidal motion is characterized by* κ_3, ω_3, $\mathfrak{y}_3$ *as given above.*

SYSTEMS OF TRIADIC POINTS ON A CUBIC

By H. G. GREEN and L. E. PRIOR, University College, Nottingham, England

In the investigation of the non-singular cubic it is often convenient to use the doubly periodic elliptic functions in which the sum of the parameters of the intersections of the cubic with any straight line or with a conic is congruent to zero (mod ω, ω'). Two points P and Q are defined as conjugates if their parameters differ by one of the three quantities $\omega/2$, $\omega'/2$, $(\omega+\omega')/2$. The conjugacy can also be defined by consideration of the cubic as the Hessian of a second cubic. For any P there are three conjugate Q's and it is easy to show that the chords of contact of two pairs, suitably chosen, of the four tangents from P pass through each Q, and that if a conic is triply tangent to the cubic at the points P, P', P'' the nine conjugate points for the P's can be grouped in three sets of three so that one group is collinear and the others are points of contact of two triply tangent conics. These results can immediately be obtained by parameters or from the Hessian theory.[1] It can be shown also that if a quadrilateral is inscribed in a cubic the pairs of opposite vertices are conjugate points of the same system. Many problems concerned with the three systems of quadrilaterals which thus arise from the three systems of conjugacy can be dealt with by parameters, but the methods of projective geometry show their relationship to more general properties.[2]

We discuss in this paper some properties of triads of three points on a cubic, so placed that a conjugate of one of the points lies on the join of the other two. It can easily be shown, by parameters or otherwise, that the join of two points of the cubic and the join of their conjugates (of the same system) intersect on the cubic. We shall show that the triads of points thus defined are the points of contact of conics with the cubic; in the case of the non-singular cubic there are three systems of triads, but in the case of the nodal cubic, avoiding the node, there is one system only. Using methods of pure geometry we obtain some known results[3] and some which we believe to be new. The proofs are given for the non-singular case only and the conjugate points discussed must be understood to belong to any definite one of the systems, the phrase 'of the same system' being omitted to avoid repetition.

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[1] Durège, *Die Ebenen Curven dritter Ordnung*, §541.

[2] Green and Prior, Journal de L'École Polytechnique, 2nd. Series vol. 30, p. 99, for example, for a treatment of inscribed quadrilaterals.

[3] See Durège, loc. cit.

THEOREM 1: *The conjugate of each point of a triad lies on the join of the other two.*

Let P, Q, R be the triad in which R', the conjugate of R, is the third intersection of PQ with the cubic, and let P', Q' be the conjugates of P, Q. Since the join of two points and the join of their conjugates meet on the cubic, $R'P$ and RP' meet on the cubic at Q. Similarly Q' lies on PR.

COROLLARY: *P', Q', R' are collinear, and with P, Q, R form the vertices of a complete inscribed quadrilateral.*

PQ passes through R', therefore $P'Q'$ passes through R'.

THEOREM 2: *If six points of the cubic lie on a conic, the six conjugate points also lie on a conic.*

Arranging the six points in pairs, the joins of the pairs meet the cubic again in three collinear points. The conjugate points of the six are then collinear in pairs with these three, and therefore lie on a conic.

THEOREM 3: *If three of the six intersections of a conic with the cubic form a triad, so also do the other three intersections.*

The conjugates of a triad are collinear. Hence, by theorem 2, the conjugates of the other three points are collinear and therefore they form a triad.

COROLLARY 1: *The conic through two fixed points of the cubic and any triad passes through a fixed point of the cubic. In particular the circumcircles of triads on a circular cubic pass through a fixed point on the cubic, and the circles of curvature at the conjugates of the inflexions pass through the same point (the conjugate of the real intersection of the cubic with the line at infinity).*

COROLLARY 2: *If the joins of conjugate points P, Q meet the cubic again at V, and L, M, S is any triad, then the conic $LMSVP$ touches the cubic at P.*

THEOREM 4: *Any two triads lie on a conic. (Converse of theorem 3.)*

The conic through one triad and two points of the other meets the cubic in a sixth point which must form a triad with the two. The sixth point must therefore be the third point of the second triad.

COROLLARY 1: *The points of contact of the four tangents from any non-inflexional point O of a non-singular cubic lie on a conic touching the cubic at O. If O is an inflexion the corollary merely reduces to 'the points of contact of tangents from an inflexion are collinear.'*

COROLLARY 2: *A conic can be drawn through the points of a triad to touch the cubic at these points.*

THEOREM 5: *If a non-singular or nodal cubic be transformed into a non-singular or nodal cubic by means of a quadratic transformation in which any triad is taken as base triangle, then conjugate points transform into conjugate points.*

Let X, Y, Z be the chosen triad, and let P, Q, R be any other triad, with P', Q', R' as conjugate points, and forming with P, Q, R a quadrilateral of the system.

The cubic transforms into a cubic through X, Y, Z. In the original figure a conic can be drawn to touch the cubic at X, Y, Z (theorem 4, corollary 2), and hence in the transform the cubic meets the sides of XYZ in three collinear points, and the points X, Y, Z therefore form a triad of the new cubic.

Now in the original figure $P, Q, R; P', Q', R; P', Q, R'; P, Q', R'$ are all triads and therefore lie on conics through XYZ (theorem 4). These points therefore become six points collinear in threes, forming the vertices of a quadrilateral inscribed in the cubic.

Also the four straight lines PQR', etc., of the original figure become conics $XYZPQR'$, etc. But XYZ has become a triad; and hence PQR', etc., in the new figure are triads, and pairs of points P, P', etc., remain conjugate after the transformation.

QUESTIONS, DISCUSSIONS, AND NOTES

EDITED BY R. E. GILMAN, Brown University, Providence, Rhode Island

The Department of Questions and Discussions in the Monthly is open to all forms of activity in collegiate mathematics, including the teaching of mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.

AN ELEMENTARY METHOD FOR CONSTRUCTING A LOGARITHM TABLE

By F. E. WOOD, Northwestern University

To many who use logarithms and who understand the laws which they obey the calculation of the logarithm table is shrouded with mystery. Thus for pedagogical purposes if for no other reason it may well be interesting to observe that logarithms can be obtained with considerable accuracy[1] by a method which uses only the laws of logarithms, interpolation and the solution of two linear equations in two unknowns.

If one knows the logarithms of 2 and 3, the logarithms of 4, 5, 6, 8, 9 and in general of all numbers of the form $2^a3^b5^c$, where a, b, c are positive integers or zero, are easily determined; moreover, then the logarithm of any prime greater than 5 can be found by interpolation between suitable values already known, as will be indicated later for particular cases. The main part of this note is concerned with finding values for log 2 and log 3 to the base 10.

Certain powers of 2 are close to numbers whose logarithms are known, and from any one of them an approximate value of log 2 can be obtained; thus $2^{10}=1024$, and $\log 1000=3$. Moreover certain powers of 3 will be close in value to certain powers of 2, except perhaps for a factor 10 to some integral power,[2] and from any such relation one can obtain a linear equation involving log 2 and log 3.

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[1] The results given here are accurate to four decimal places; the method can be extended to give somewhat more accurate results.

See also the method given by M. Philip in *The Principles of Financial and Statistical Mathematics*, Prentice-Hall, 1932, p. 130. (Editor.)

[2] To some it might seem advisable to examine all numbers of the form $2^a\,3^b\,5^c$ and to pick out pairs which are approximately equal except for a factor 10^k, k an integer or zero. However it can be shown that corresponding to any pair so found will be a pair proportionally close together found by examining the powers of 2, 3, and 5.

Two such relations, found by inspection from the powers of 2 and 3, come from $3^8=6561$, $2^{16}=65536$ and $3^{21}=10460353203$, $2^{20}=1048576$, giving $3^8\times 10=2^{16}\times 1.0011$ and $2^{20}\times 10^4=3^{21}\times 1.0025$, the decimals being accurate to four decimal places only. One obtains, by taking the logarithm of each side,

$$\begin{aligned} 8 \log 3 - 16 \log 2 &= -1 + \log 1.0011 \\ 21 \log 3 - 20 \log 2 &= 4 - \log 1.0025. \end{aligned} \tag{1}$$

If we assume, as a first approximation, that $\log 1.0011=0$, $\log 1.0025=0$ and solve in (1), we find $\log 2=53/176=.3011$, $\log 3=84/176=.4772$. Now $\log 1024=10 \log 2=3.011$ approximately, whence $\log 1.024=.011$ approximately, and by interpolation $\log 1.0011=.00050$, $\log 1.0025=.00115$ approximately. Using these values in (1),

$$\log 2 = \frac{52.9803}{176} = .30102, \quad \log 3 = \frac{83.9716}{176} = .47711$$

which are accurate to four decimal places.[1]

Now log 7 can be obtained from the relation $7^4=2401$, $4 \log 7=2+\log 24+\log(1+1/2400)$ whence $\log 7=.84508$; likewise log 11 from $99^2=9801$, whence $\log 11=1+\log 7-2\log 3+\frac{1}{2}\log 2+\frac{1}{2}\log(1+1/9800)$ giving $\log 11=1.04138$.

This method can be continued to give logarithms of other primes, and from them the logarithms of composite numbers. Moreover when the logarithms of two large numbers which are close together are known, the logarithm of any number between the two can be obtained by interpolation—as from illustration above the numbers between 98,000 and 98,010—with fair accuracy.

RECENT PUBLICATIONS

EDITED BY R. A. JOHNSON, Brooklyn College of the City of New York

All books for review should be sent directly to the editor of this department, at Brooklyn College, 66 Court Street, Brooklyn, N. Y., and not to any of the other editors or officers of the Association.

REVIEWS

Analytical Mechanics for Students of Physics and Engineering. Third Edition—Revised. By H. M. Dadourian. New York, D. Van Nostrand Co., 1931. xiv+428 pages. $4.00.

This new edition of a well-known elementary text on mechanics contains a good deal of material. The book—as the author states in the preface—is espe-

[1] Greater accuracy can be obtained by computing from the value of log 2 more accurate values for logarithms of 1.0011 · · · and 1.0025 · · · each being found more accurately first. But to obtain accuracy to six or more decimal places for log 2 and log 3, the interpolations involved must be over a smaller range than from 1 to 1.024.

cially written to meet the needs of students of physics and of engineering. It should be easy for the instructor to choose the material necessary for a first course.

Mechanics is always a difficult subject for the student of average ability. The full meanings of the principles involved are not easily grasped by the beginner, and when he comes to apply the principles in the solution of problems the difficulties increase immensely.

The author has had much experience in teaching mechanics. Under the table of notations there is given a list of directions for working out problems. This should prove helpful to the student.

The author deviates considerably from the common presentation of the subject in elementary courses. The ordinary procedure is to state Newton's laws of motion, and then take for granted that these laws are the foundation upon which the subject is built. There have been, however, just criticisms of Newton's laws as a foundation of mechanics. For example, the laws are not independent: the first law is a special case of the second. The second is merely a definition of force. The third, though a true principle, is not sufficiently broad for a foundation of mechanics. To get away from these objections the author does not state Newton's laws, but introduces a simplified form of D'Alembert's principle, which may be called the first form of the *action principle.* Stated in words the principle is:

"*The vector sum of all the external actions to which a system of particles or any part of it is subject at any instant vanishes.*"

The action principle thus stated is equivalent to the first three *scalar* equations of D'Alembert's principle. The second form of the action principle applies to rotation about an axis. It may be stated thus:

"*The vector sum relative to any axis of all the angular actions to which a system of particles or any part of it is subject at any instant vanishes.*"

From this form of the statement of the principle the last three *scalar* equations of D'Alembert's principle can be obtained at once.

The object of the statements in the two forms of the action principle is not to derive from them the equations of D'Alembert's principle, but to give to the student in words something which he can apply directly to the solutions of problems. For an elementary course in mechanics this would seem sufficient. The action principle serves to unify the subject. It would seem that it is not too abstruse for the beginner.

Throughout the sixteen chapters of the text there are, in appropriate places, illustrative examples worked out in considerable detail. There is an abundance of well-graded exercises for the student to solve. At the end of each chapter, too, there is a set of general exercises of a somewhat more difficult nature.

It might have added to the many good features of the book if answers had been given to at least some of the exercises which call for numerical results.

H. H. DALAKER

Mathematical Tables (Vol. 3); Minimum Decompositions into Fifth Powers. By L. E. Dickson. London, British Association for the Advancement of Science, 1933.

E. H. Neville remarks in his brief preface—"The opportunity for cooperation in work which would have appealed strongly to the testator (Lt.-Col. A. J. C. Cunningham.) is one upon which the committee congratulates itself." This extensive table of decompositions into fifth powers is indeed a publication which will appeal strongly to those who are interested in the theory of numbers.

For single integers N, decompositions into a minimum number of fifth powers can be read off directly from the table, if $N<150,000$. After 150,000 (the table extends to 300,000) no decompositions are printed, but can readily be found, as is indicated in the author's introduction.

For intervals, the paragraph on the *Distribution of high minima in the table* gives extensive information. For example, all integers between 190,000 and 300,000 are sums of 16 positive or zero fifth powers, the only integer requiring as many as 16 being 191,263. Of the integers which require exactly 15, those after 191,263 are tabulated, and the others can readily be found from the table.

In connection with the Waring problem, information about the distribution of high minima (peaks) in an interval has been exploited extensively in proving new universal theorems. Professor Dickson (Bulletin of the American Mathematical Society, vol. 39, 1933, p. 712) proves that every integer is a sum of 54 fifth powers, the best earlier result being 58. He also proves a number of universal theorems for powers higher than the fifth. For these it is interesting to note that the amount of table which must be constructed is comparatively small, while the results obtained from it are great improvements on the best earlier results.

Later tables show that for the interval from $C=3,470,000$ to 5,489,568 every integer is a sum of at most 12 fifth powers (in fact, from C to 3,600,000 there are only thirteen numbers which require as many as 12). This fact indicates the desirability of considerably lowering the present asymptotic results.

The actual Table of Decompositions has been reproduced photographically from carefully prepared typescript and is remarkably even and uniform. The elaborate checks on the accuracy of the table have thus not been subjected to the hazard of errors introduced in the course of printing.

R. C. Shook

College Algebra. By J. B. Rosenbach and E. A. Whitman. Boston, Ginn and Company, 1933. xi+394 pages. $2.00.

The first ten chapters of this text are devoted primarily to a review of elementary algebra. In these review chapters especially, the authors have included numerous "warnings," "certain incorrect applications of the fundamental prin ciples under discussion." In some instances these warnings are excellent, pointing out algebraic pitfalls into which both the good and the poor students might

easily fall; but in many cases they warn the student against those algebraic atrocities which only the poorest students are prone to commit.

The problems appear to be ample and well arranged; answers are given for the odd-numbered examples. Also, there is an abundance of well chosen illustrative examples. The general presentation of the material is good, and the text should prove very "teachable."

At every opportunity the authors have included interesting historical references.

MALCOLM FOSTER

The Administration of Mathematics in Secondary Schools. By E. R. Breslich. University of Chicago Press, 1933. vi+407 pages. $3.00.

This is the third of a series of three volumes which Professor Breslich has written. All three volumes are devoted to the Teaching of mathematics in secondary schools. The first volume, The *Technique of Teaching Secondary School Mathematics*, deals with problems arising in the choice and use of general teaching procedure and the second, *Problems in Teaching Secondary School Mathematics*, is concerned with specific teaching problems. In his third volume Professor Breslich classifies administrative problems as they relate (1) to the direction and supervision of a department and (2) to the curriculum. The first part is intended to assist the supervisor in organizing the department of mathematics into a unified and cooperative group for the purpose of improving instruction. He discusses such supervisory functions as visitation of teachers, individual and departmental conferences, and the training of teachers. Several chapters are given over to the development of objectives, the formulation of a testing and measurement program, provisions for individual differences in ability, and remedial teaching. The second part relating to curriculum problems gives prominence to how content material should be selected for teaching purposes, the organization and distribution of the various parts of mathematics like arithmetic, geometry, and algebra, how these parts are unified, their organization into teaching units, and the relation of mathematics to modern educational trends.

Professor Breslich's long experience, not only as a teacher but as a supervisor, qualifies him to speak authoritatively upon any subject related to the teaching of mathematics, and this book will be of great value to experienced teachers of mathematics who feel the need of help as heads of departments of mathematics or as supervisors of mathematics in secondary schools.

Not only does Professor Breslich present his own views on many topics, but through a very thorough perusal of the literature of the field and by means of extensive bibliographies he has furnished teachers an opportunity for reading that will help them better to solve their problems and to improve their thinking.

It is to be regretted that a book of this type is necessarily placed at such a high price that in these times it will not be available to a large number of teachers. This is in the last analysis due to the fault of the teachers themselves who

often do not buy such books in large quantities even though the price be fairly low.

W. D. Reeve

MATHEMATICS CLUBS

Edited by F. M. Weida, The George Washington University, Washington, D. C.

All reports of club activities, suggestions and topics for club programs, and material of interest to clubs should be sent to F. M. Weida, The George Washington University, Washington, D. C. All manuscript should be typewritten, with double spacing, and with margins at least one inch wide. All club activity manuscripts for the academic year 1933–1934 should be submitted for publication not later than June 1, 1934.

Club Topics

1934 as a Centennial Year in the History of Mathematics

By W. C. Eells, Stanford University

In continuation of previously published lists (See this Monthly, Vol. 40, (1933), pp. 359–360, for a list of 1933 centennial events, and for references to previous volumes for corresponding lists from 1925 to 1932) of centennial dates in the history of mathematics, the following group of important 1934 centennial dates is presented.

A.D. 1534. Publication at Venice of Giovanni Sfortunati's work on commercial arithmetic which went through several editions.

A.D. 1634. Publication at Paris of the first volume of Herigone's five volume *Cursus Mathematicus.*

A.D. 1634. Herigone uses the notation *sin* for the sine of an angle. (*Si*, *S*, etc. used by later writers)

A.D. 1734. Birth of Edward Waring, English mathematician who contributed important theorems on number theory, first gave the so-called Cauchy's test of convergence of series, and devised a process for the approximation of the values of imaginary roots of equations.

A.D. 1734. Birth of Honda Teiken, better known by his later name Fujita Sadasuke, Japanese author of a notable work on algebra.

A.D. 1734. Introduction by Euler of the notation $f(x)$ to indicate a function of x.

A.D. 1734. Publication of Bishop Berkeley's *The Analyst*: or *a Discourse addressed to an Infidel Mathematician, wherein it is examined whether the Object, Principles, and Inferences of the Modern Analysis are more distinctly conceived, or more evidently deduced than religious Mysteries and Points of Faith*, a bold attack on the principles of fluxions which, according to Cajori, "was the most spectacular mathematical event of the eighteenth century in England."

A.D. 1734. Death of De Lagny, French algebraist who devised the method of differences for the solution of numerical equations of higher degree.

A.D. 1834. Birth of Laguerre, French mathematician who made contributions to the theory of equations based on Descartes' rule of signs and its application to infinite series.

A.D. 1834. First use of the designation "Rolle's Theorem" by Drobisch of Leipsig.

A.D. 1834. Foundation of the chair of geometry at the University of Berlin for Steiner, Swiss mathematician, sometimes characterized as "the greatest geometrician since the time of Euclid." (Quoted by Cajori.)

A.D. 1834. Death of Hachette, French mathematician who wrote a treatise on descriptive geometry.

Club Activities

1932–1933

LOCAL MATHEMATICS CLUBS

The Mathematics Club of Milwaukee-Downer College

The club exists for the purpose of discussing mathematical subjects not ordinarily included in the curriculum courses. Membership is open to all who are interested in the subject and are willing to contribute to the value of the programs.

The meetings for the academic year 1932–1933 were as follows:

November 1932: Supper meeting and geometrical treasure hunt.
December 1932: "Mathematical symbolism." Selection of the "Magic square" as club symbol.
January 1933: "Mathematical tricks and puzzles."
February 1933: "George Washington as an engineer"; "Map projection"; "Four color problem."
March 1933: "The vogue for numerology."
April 1933: "Mathematics applied to physics."
May 1933: Closing dinner.

Ethel Schoenbaum, *Secretary*

The Mathematics Club of Washington Square College

The officers for the year 1932–1933 were: Ely Wagner, President; Selma Karlin, Vice President; Pearl Brecher, Secretary; Jacob Schneider, Treasurer; Zvi Feinstein, Editor of "X"; Dr. C. K. Payne, Faculty Adviser. The election of officers is held the last Friday of the school year and is decided by a majority vote of the members present.

Our club was organized in 1928 by a small group of students interested in research in mathematics. The present number of active members exceeds fifty. There is no special eligibility requirement for membership. It is only necessary for the student to maintain his standing in the university.

The club carries on a variety of activities. Coaching classes for freshmen are held throughout the year. A magazine, "X," of about thirty pages is published semi-annually. In addition to two socials, the club gives a reception of welcome to the freshmen entering the University in February. After the final examinations of each school year, a boat ride up the Hudson is held.

Meetings of the Club are held every Friday that school is in session. Papers by Club members and also by members of the faculty are read and discussed. The programs for the school year 1932–1933 were as follows:

October 7, 1932: "Matrices and determinants" by Mr. H. R. Cooley.
October 14, 1932: "Calculus of variations" by Professor Fay Farnum.
October 21, 1932: "Elements of differential equations" by Mr. Zvi Feinstein.

October 25, 1932: "Symbolic logic" by Mr. J. C. C. McKinsey.
November 11, 1932: "Measurement of Stellar distances" by Dean P. H. Graham.
November 25, 1932: "Derivatives and differentials" by Mr. M. Kline.
December 2, 1932: Discussion of the previous lecture.
December 9, 1932: "Mathematical statistics" by Professor W. S. Schlauch.
February 10, 1933: "Vector analysis" by Mr. Al Miller.
March 3, 1933: "The elements of relativity" by Dr. H. H. Sheldon.
March 10, 1933: A review of "The case against Einstein" by Mr. E. N. Grisewood and Dr. C. K. Payne.
March 24, 1933: "The standardization of error" by Mr. V. Stefanssan.
April 7, 1933: "The philosophy of relativity" by Mr. A. I. Katsh.
April 21, 1933: Debate on "The obvious in mathematics."

Pearl Brecher, *Secretary*

The Mathematics Club of the New York State College for Teachers

The purpose of the organization is to further the interests and broaden the perspective of students in mathematics. There are sixty-five active members. Faculty members of the mathematics department and Dean William H. Metzler are also members. All students who have successfully completed the first semester of analytical geometry are eligible for membership.

The officers for the year 1932–1933 were: George Hisert, President; Louise Wells, Vice President; Mariam Wood, Secretary; Myrtle Peck, Treasurer; Ellen C. Stokes, Faculty Adviser. The club elects its officers in May.

The meetings were held ordinarily once every two weeks of the academic year in the College Lounge. The meetings and programs were as follows:

October 19, 1932: Initiation.
November 3, 1932: "Number systems" by Alice McEwan.
November 17, 1932: "History of mathematics up to 1500" by Babette Hutzenlaub.
December 5, 1932: "History of mathematics from 1500 to 1800" by Myrtle Peck; "Sugar-coated pills of mathematics" by Annette Lewis.
January 5, 1933: "The nature of graduate courses in mathematics" by Dr. R. Beaver.
February 2, 1933: "Ciphers and cryptograms" by Charles Kissam.
February 16, 1933: "Problems in the construction of the new Albany-Rennselaer bridge" by O. H. Schermerhorn, Assistant Chief Engineer of the New York State Department of Public Works.
March 9, 1933: Initiation.
March 23, 1933: "Some aspects of finite groups" by Instructor Carolyn A. Lester.
April 6, 1933: "The development of algebra" by Loraine Loder.
May 4, 1933: "Various methods of trisecting an angle" by Laurence Rupert.
May 25, 1933: Picnic.

Eunice E. Sisbower, *Chairman*

The Mathematics Club of Oberlin College

The Mathematics Club of Oberlin College endeavors to promote interest in the study and in the clear exposition of mathematics. The membership is limited to those who have had at least one year of college mathematics. For the year 1932–1933 the number of active members was thirty-five.

Officers were elected on October 7, 1932. A nominating committee submitted names, and the officers chosen were: Philip Severance, President; Sarah Metcalf, Vice President; Doris Bottom, Secretary-Treasurer; Mary Louise Matteson, Zoe Schnabel and Robert Cornelius, Social Committee; Edward Tenney, Clara Coates and Harriet Bayle, Program Committee. Professor Marie M. Johnson acted as faculty adviser for the year.

The meetings and programs were as follows:

October 7, 1932: The club members were the guests of Professor and Mrs. W. D. Cairns at their home. Election of officers.

October 21, 1932: "Object and elements of mathematics" by Harriet Bayle; "Recent work at Mount Wilson Observatory" by Professor F. E Carr.

November 4, 1932: "The use of symbols in mathematics" by Neil Gilbert; "The postulational method" by Robert Bolbach.

November 18, 1932: "The postulates of common algebra" by Doris Bottom; "Non-Euclidean geometry" by Eva Mae Parker.

December 2, 1932: "Mathematics in architecture" by Willard Pye; "Mathematics in radio" by Paul Brown.

December 16, 1932: The annual Christmas party.

January 6, 1933: "Transformations" by William Correll; "Matrices" by Harold Rivkind.

January 20, 1933: "Mathematics and philosophy" by Clara Coates; "Mathematics in chemistry" by Charles Krister. The picture of the club group was taken for the college yearbook.

February 10, 1933: "Oscillographs" by Richard Nuckolls; "Calculating machines" by Dana Whitmer.

February 20, 1933: "Aspects of unity and reality in elementary mathematics" by Professor W. G. Simon of Western Reserve University.

March 10, 1933: "The calculus" by Dorothy Waterbury; "Counting the infinite" by Mary Louise Matteson.

March 24, 1933: "Mathematics in economics" by Paul Kennedy; "Mathematics in Music" by Robert Cornelius.

April 6, 1933: Eight members of the club attended the meeting of the Ohio Section of the Mathematical Association of America held at Columbus.

April 21, 1933: "Algebraic equations solvable by radicals" by Florence Oberlin; "Non-harmonic vibrations" by Kenneth Hubbard.

May 5, 1933: "The solution of integrals by known functions" by Philip Severance; "A problem in mathematical astronomy" by Neil Gilbert.

May 12, 1933: Annual banquet.

DORIS BOTTOM, *Secretary*

The Mathematics Club of Northeastern University

The officers for the year 1932–1933 were: Howard C. Cookingham, President; Charles Davis and Amos H. Fenlason, Vice Presidents; Charles W. Perry, Secretary; James C. Soutter, Treasurer.

There were thirty-two active members and five faculty members. One faculty member, Mr. Elmer E. Haskins of the mathematics department, was adviser to the club.

The aim of the club is to encourage among the students the study of topics of mathematical interest which are beyond the scope of the regular mathematics courses which they attend.

Membership eligibility is restricted to those engineering students who have completed at least one and one-half years of school work and have obtained an average of not less than *C* in all their mathematics courses. Meetings are open to members only.

The meetings and programs were as follows:

November 29, 1932: "Theory of equations" by Elmer H. Haskins of the faculty.

January 3, 1933: "Sturm's theorem" by Charles Perry; "Number systems" by Amos H. Fenlason; "Some geometric problems in the construction of the triangle" by Leroy Kelley.

March 14, 1933: "The problem of damped vibration" by Professor J. Spear.

February 7, 1933: "Operational calculus" by Professor C. F. Muckenhoupt.

April 25, 1933: "Steiner's and Mascheroni's geometrical constructions" by Professor W. R. Ransom of Tufts College.

May 25, 1933: Annual banquet and election of officers for 1933–1934.

A tentative program for the year 1933–34 was discussed.

CHARLES W. PERRY, *Secretary*

The Mathematics Club of the Oshkosh State Teachers College

The purpose of the club is to promote interest in the subject of mathematics, and to afford an opportunity to study interesting matters connected with mathematics that do not find a place in the usual class discussion. Membership is open to all students who have completed at least one year of college mathematics.

The officers for the year 1932–1933 were: Louis Gardipee, President; Walter Bohman, Vice President; Gertrude Kushman, Secretary; Gordon Kester, Treasurer.

The meetings and programs were as follows:

October 4, 1932: "Review of the book *Flatland*" by Irene Timm.

November 1, 1932: "Trisection of the angle" by Harry Heinrich; "Mathematical card tricks" by Roger Sloan.

December 6, 1932: Illustrated lecture on "Famous mathematicians" by Vialor Dumdie.

January 3, 1933: "Mathematical wrinkles" by Rose Schlegel.

February 7, 1933: "Calculating machines" by Clarence Discher.

February 20, 1933: An enjoyable evening was spent at a mathematics party given by Dr. May M. Beenken and Dr. Irene Price of the faculty.

March 14, 1933: "The scope of mathematics and its applications in modern life" by Dr. May M. Beenken.

April 4, 1933: "Mathematics in field artillery" by John Adams; "Life and works of Charles L. Dodgson" by Dorothy Mortson.

May 16, 1933: "Mathematics in music" by Alton Davis; "The mathematics exhibit at the Century of Progress Exposition" by Loretta Golz.

May 29, 1933: The annual picnic was held at the home of Bertram Lyngaas on the shores of Lake Winneconne.

A prize of $5.00 was offered by the faculty of the mathematics department for the best paper presented during the year. This prize was awarded to Mr. Clarence Discher.

GERTRUDE KUSHMAN, *Secretary*

PROBLEMS AND SOLUTIONS

EDITED BY OTTO DUNKEL, H. L. OLSON AND W. F. CHENEY, JR.

ELEMENTARY PROBLEMS

Send all communications about Elementary Problems and Solutions to Wm. Fitch Cheney, Jr., Dept. Box 35, Storrs, Connecticut.

The Department of Elementary Problems welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. Problems may be submitted unaccompanied by their solutions.

PROBLEMS FOR SOLUTION

E 87. *Proposed by William Douglas, Courtenay, British Columbia.*

Find the radius of the circle in which a thirty foot chord subtends a thirty-two foot arc.

E 88. *Proposed by R. K. Morley, Worcester Polytechnic Institute.*

Show that m arctan x is greater or less than arctan mx according as m is greater or less than unity, providing m and x are positive and the angles are taken in the first quadrant.

E 89. *Proposed by E. P. Starke, Rutgers University.*

It is required to find two integers, one the square of the other, which together contain each of the nine digits from one to nine just once. Show that there are just two solutions.

E 90. *Proposed by W. B. Campbell, Rangoon, Burma.*

Show that all the planes which cut the tetrahedron $ABCD$ and are parallel to AB and CD, cut it in parallelograms which are equiangular to each other. In case $AB=CD=r$, these parallelograms are of constant perimeter $2r$.

E 91. *Proposed by Morgan Ward, California Institute of Technology.*

Let d be the greatest common divisor of the two positive integers, a and b, with $a=a'd$ and $b=b'd$. Now if n is any integer greater than unity, show that (n^a+1) and (n^b-1) can not have any common factor greater than 2 as long as b' is odd.

E 92. *Proposed by L. S. Johnston, University of Detroit.*

It is desired to make a rectangular box by cutting square corners out of a rectangular sheet of cardboard and turning up the sides and ends. If the depth of the box is to be a inches, find the dimensions of the sheet of cardboard such that the box made in the manner described is the largest rectangular box which can be made from the sheet. This is the converse of an elementary problem which appears in most calculus texts.

SOLUTIONS

E 13. [1932, 606]. *Proposed by W. F. Cheney, Jr., Connecticut State College.*

It is required to find two factors which, together with their product, contain each of the nine digits from one to nine just once. Each of the several solutions should be found. (Zero must not appear in any solution.)

Solution by V. Thébault, Le Mans, France

1. The numbers of digits in the two factors and the product must be either one, four and four, or else two, three and four.

2. If the digit sums of the factors and product are a, b and c, we have first that

$$(A) \qquad ab \equiv c \pmod{9},$$

and secondly, since $a+b+c=45$, $a+b+ab\equiv 0 \pmod{9}$; or, adding 1 and factoring,

$$(B) \qquad (a+1)(b+1) \equiv 1 \pmod{9}.$$

3. If the first factor is a two-digit number, a must be at least 3 and at most 17. The second factor is then a three-digit number and b must be at least 6 and at most 24. There must thus be a finite number of solutions to (B) for a and b compatible with these conditions, and each must be examined to see if it furnishes solutions to the original problem.

4. In illustration of the procedure involved, let us consider the case of $a=12$, $b=15$. Here $(a+1)\equiv 4$, $(b+1)\equiv 7$, and $4\times 7\equiv 1$ (mod. 9). Then the first factor may be 39, 48, 57, 75, 84 or 93. Let us try 39. Then the first digit of the second factor may only be 1 or 2, lest their product have five digits. If this digit be 1, the second factor can only be 168 or 186, since its digit sum is 15 and no digit may repeat. When we multiply these two out, we get $39\times 168=6552$ and $39\times 186=7254$, and the first is ruled out by the repeated 5, but the second is a valid solution.

5. When all possibilities have been investigated by these methods, there result the seven following solutions which have two, three and four digits in the factors and product respectively: $12\times 483=5796$, $18\times 297=5346$, $27\times 198=5346$, $28\times 157=4396$, $39\times 186=7254$, $42\times 138=5796$ and $48\times 159=7632$.

6. A similar investigation of the case in which the factors and product contain respectively one, four and four digits, yields just the two solutions, $4\times 1738=6952$ and $4\times 1963=7852$. This confirms the findings of W. E. Buker which appeared in the MONTHLY, [1933, 559].

7. This problem may be generalized[1] by considering the case in which all ten of the number symbols must appear once each in the problem, thus adding the zero in with the nine digits. In this case the investigation is still more complicated, and the following solutions are submitted: $4\times 3907=15628$, $4\times 7039=28156$, $7\times 4093=28651$, $7\times 9304=65128$, $27\times 594=16038$, $39\times 402=15678$, $54\times 297=16038$.

E 60. *Proposed by M. W. Aylor, University of Virginia.*

A horse is tied by a rope to one corner of a barn which is thirty feet square. Find the length of the rope such that the horse can graze over one acre of ground without grazing over any part twice.

Solution by W. R. Ransom, Tufts College

One acre is 43,560 square feet, and if the barn were not there, the rope necessary for grazing over one acre plus the area of the barn would be 119.0 feet. With the barn in the way, this rope would not allow the horse so much grazing area, so we take a better approximation of 120.00 feet for the length of the rope. With this rope, the area grazed over is composed of five pieces. The first is three-quarters of a circle of radius 120 feet, or 33,929 square feet. The next two are two equal triangles with sides of thirty and ninety feet, with an angle of 135° opposite the ninety-foot side. By the sine law, the smallest angle in each triangle is 13°37′59″ and their combined area is 1405 square feet. Finally there are two equal sectors whose central angles are 58°37′59″ and their radii are ninety feet. Their combined area is 8287 square feet, making a total area for the 120 foot rope of 43621 square feet. This is 61 square feet over the desired amount.

[1] V. Thébault, Mathesis, 1933, 327.

Now a small change ϵ in the length of the rope will alter the area by an amount approximately equal to ϵ times the perimeter of the grazed area. This perimeter is about 750 feet, so $\epsilon \doteq 61/750 \doteq .08$ feet, which must be deducted from the length of the rope, leaving it 119.92 feet long.

Solved also by James M. Schenkel and the proposer.

E 61. *Proposed by Raphael Robinson, University of California at Berkeley.*

Given the common logarithms of the integers from 1 to 10 correct to fifteen decimal places (as in Table Ia of the MacMillan Tables), find log 2 correct to sixteen decimal places.

Solution by the proposer

The table gives log 8 = .903,089,986,991,944, correct to fifteen places, which means that the true logarithm of 8 lies between .903,089,986,991,943,5 and .903,089,986,991,944,5. Dividing by 3, we find that the logarithm of 2 lies between .301,029,995,663,981,16 and .301,029,995,663,981,50.

Similarly, since log 4 = .602,059,991,327,962 and hence lies between .602,059,991,327,961,5 and .602,059,991,327,962,5, division by 2 shows that log 2 lies between .301,029,995,663,980,75 and .301,029,995,663,981,25.

Since one-third log 8 gives a lower limit of 1.6 and one-half log 4 gives an upper limit of 2.5 for the sixteenth digit of log 2, the correct value of log 2, to sixteen digits, must be .301,029,995,663,981,2.

E 62. *Proposed by W. R. Ransom, Tufts College.*

Defining a "C-angle" as the figure formed by two internally tangent circles, and its magnitude as the difference of the curvatures of those circles, show how to bisect a C-angle geometrically. (If the circles are tangent externally, the magnitude of the C-angle is the sum of their curvatures.) If the circles are tangent to the X-axis at the origin, O, and cut the circle $x^2+y^2=2x$ also at P and Q, show that the magnitude of the C-angle equals the difference between the slopes of the chords OP and OQ.

Solution by Simon Vatriquant, Athênée Royale d'Ixelles, Brussels, Belgium.

Let O be the contact-point of the circles (O_1, R_1) and (O_2, R_2), and let (O_3, R_3) be the bisecting circle. Assume $R_1<R_2$. We obtain $2/R_3=1/R_1+1/R_2$. The point O_3 is thus the harmonic conjugate of O with respect to the segment O_1O_2. If the circles are externally tangent, we take R_2 negative and the result is the same.

The chords OP and OQ are radical axes of the circles (1) $x^2+y^2=2R_1y$ and (2) $x^2+y^2=2R_2y$ respectively with the given circle (3) $x^2+y^2=2x$. Their equations are consequently obtained by subtracting (1) or (2) from (3), which gives $y=x/R_1$ and $y=x/R_2$. Consequently the difference of the slopes of these chords is $1/R_1-1/R_2$, which is defined as the magnitude of the C-angle.

Solved also by Roy MacKay, C. W. Trigg, and the proposer.

E 63. *Proposed by W. B. Carver, Cornell University.*

A number of less than thirty digits begins with the two digits 15 on the left, 15 ———; and when it is multiplied by 5, the result is merely to move these two digits to the right-hand end, thus, ——— 15. Find the number, and show that the solution is unique.

Solution by H. T. R. Aude, Colgate University.

Let f denote the proper fraction which, when written as a decimal, consists of the repetend 15 ———, one cycle of which is the number sought. According to the conditions of the problem, $5f=0.$ ——— 15 ——— 15 $\cdots$ and $100f=15.$ ——— 15 ——— 15 $\cdots$ so that $95f=15$, and $f=3/19$. By performing the division, the decimal repetend of f is found to consist of the eighteen digit sequence, 157,894,736,842,105,263, and this is the unique number of less than thirty digits which satisfies the conditions of the problem. If the thirty digit limit were raised to fifty, there would be just one more solution admitted, consisting of two cycles of f, or a number of thirty-six digits.

It is interesting to note that if k is any integer prime to 19, then the decimal expansion of k/f contains exactly the same repetend, but starting at a different part of the cycle, although the integer portion of k/f does not appear in the repetend. This is because 10 is a primitive root of 19, and would be equally true if f were any other fraction whose denominator had 10 for a primitive root.

Also solved by L. M. Bauer, W. E. Buker, M. L. Constable, Arnold Court, Bernard Friedman, J. O. Garcia, E. L. Harp, Jr., T. C. Horton, C. F. Lewis, Roy MacKay, F. L. Manning, C. W. Munshower, W. R. Ransom, B. D. Roberts, E. P. Starke, C. W. Trigg, Simon Vatriquant, Maud Willey and the proposer.

E 64. *Proposed by J. Rosenbaum, the Milford School, Milford, Connecticut.*

The bisectors of the interior angles of the triangle ABC meet the sides in the points P, Q and R. Prove that the ratio of the area of the triangle PQR to the area of the triangle ABC is $2abc/[(a+b)\ (b+c)\ (c+a)]$.

Solution by Olga Larson, State College for Women, Tallahassee, Florida.

The bisector of an angle of a triangle divides the opposite side in the ratio of the adjacent sides. Therefore $AR/RB=b/a$, $BP/PC=c/b$, $CQ/QA=a/c$.

The areas of two triangles with a common angle are in the ratio of the products of the including sides. If we denote the area of a triangle by the letters at its vertices, $AQR/ABC=(AQ\cdot AR)/(AB\cdot AC)=(AQ/AC)\ (AR/AB)$ $=[c/(a+c)]\ [b/(a+b)]=bc/[(a+b)\ (a+c)]$. Similarly,

$$BPR/ABC=ac/[(a+b)\ (b+c)] \text{ and } CPQ/ABC=ab/[(a+c)\ (b+c)].$$

Now $PQR=ABC-AQR-BPR-CPQ$, so that

$$\frac{PQR}{ABC}=1-\frac{bc}{(a+b)\ (a+c)}-\frac{ac}{(a+b)\ (b+c)}-\frac{ab}{(a+c)\ (b+c)}$$

$$= \frac{2abc}{(a+b)(b+c)(c+a)}.$$

Also solved by M. W. Aylor, W. E. Buker, Bernard Friedman, L. M. Kelly, Roy MacKay, E. P. Starke, C. W. Trigg, Simon Vatriquant and the proposer.

Advanced Problems

Send all communications about Advanced Problems and Solutions to Otto Dunkel, Washington University, St. Louis, Mo. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide.

Problems containing results believed to be new, or extensions of old results are especially sought. The editorial work would be greatly facilitated, if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known textbooks or results found in readily accessible sources, will not be proposed as problems for solution in this department. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

PROBLEMS FOR SOLUTION

3673. *Proposed by E. B. Escott, Oak Park, Ill.*

Factor $x^8+98x^4y^4+y^8$ into two polynomial factors with integral coefficients.

3674. *Proposed by Garrett Birkhoff, Harvard University.*

For any positive integer k, show that

$$\phi_k = \frac{(2k-2)!}{k!(k-1)!} = \binom{2k-1}{k}\Big/(2k-1)$$

is an integer; and prove the recurrence formula

$$\phi_n = \sum_{i=1}^{n-1} \phi_i \phi_{n-i}.$$

3675. *Proposed by H. Grossman, New York.*

Denote by $P^{-1}(ABC)$ the triangle which contains the given triangle ABC in its interior as its pedal triangle, by $P^{-2}(ABC)$ the triangle determined by $P^{-1}(ABC)$ in the same manner, and so on. Show that as n becomes infinite the angles of $P^{-n}(ABC)$ approach 60°.

3676. *Proposed by R. E. Gaines, University of Richmond.*

Let $P_1, P_2, P_3, \cdots$ be a series of points on a parabola such that the successive chords $P_1P_2, P_2P_3, \cdots$ are the bases of segments of the parabola whose areas are in decreasing geometric progression. Then P_n approaches a limit P as n becomes infinite; and the series of areas of the segments on the chords $P_1P, P_2P, P_3P, \cdots$ are in geometric progression with the same ratio as the former. Also any set of P_s whose subscripts are in arithmetic progression determines a series of areas in geometric progression, and the areas of the triangles formed by successive sets of three such points are also in geometric progression.

3677. *Proposed by Otto Dunkel, Washington University.*

Given the two equations

$$y^2 = (a^2 \pm 1)x^2 + 1,$$

in which a is a given positive integer greater than unity in the second equation; deduce for each equation, without use of the theory of continued fractions, iterative formulae which give all the positive integral solutions in x and y.

3678. *Proposed by B. W. Jones, Cornell University.*

Prove the following theorem:

If $f(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0 = 0$, where n is a positive integer, $a_n \neq 0$, and all coefficients are real, has only real roots, then

1. Descartes' Rule of Signs gives exactly the number of positive and the number of negative roots of the equation.

2. $a_r = 0$ $(n > r > 0)$ implies $a_{r+1} \cdot a_{r-1} < 0$.

The characteristic equation of a real symmetrical matrix is such an equation.

Note by the Editors. This is an old theorem which is not often mentioned. It may be deduced from Descartes' rule of signs or from the Budan-Fourier theorem. A simple proof is desired.

SOLUTIONS

3541 [1932, 175]. *Proposed by J. B. Reynolds, Lehigh University.*

A rod of length $2a$ standing on a rough level plane of coefficient of friction μ falls from a vertical position. At what angle with the vertical will the rod begin to slip at the bottom?

II. *Solution by G. E. Raynor, Lehigh University.*

In the solution of this problem [1933, 305] it is stated that no slipping occurs if $\theta > \cos^{-1} 2/3$; in the present solution it is shown that, for certain values of μ, the slipping takes place for a value of θ in this range.[1]

Let θ be the angle between the rod and the vertical, and ω and α its angular velocity and angular acceleration, respectively, before it has begun to slip. Also let R_h and R_v be the horizontal and vertical reactions, respectively, of the horizontal plane. Then the equations of motion are

$$Mg - R_v = M(a\alpha \sin\theta + a\omega^2 \cos\theta), \tag{1}$$

$$R_h = M(a\alpha \cos\theta - a\omega^2 \sin\theta), \tag{2}$$

$$Mga \sin\theta = I\alpha, \tag{3}$$

where M is the mass of the rod and $I = 4Ma^2/3$ is its moment of inertia about one end.

If in (3) we replace α by $\omega d\omega/d\theta$ and integrate we find, since $\omega = 0$ for $\theta = 0$, that

[1] The error in V. F. Ivanoff's solution occurs at the top of page 306 where an incorrect denominator makes it appear (erroneously) that μ would be negative. (EDITOR.)

$$(4) \qquad \omega^2 = 3g(1 - \cos\theta)/2a.$$

Substituting the values of α and ω^2 given by (3) and (4) in (1) and (2), we obtain

$$(5) \qquad R_v = \tfrac{1}{4}Mg(3\cos\theta - 1)^2,$$

$$(6) \qquad R_h = \tfrac{3}{4}Mg\sin\theta(3\cos\theta - 2).$$

For $\theta < \cos^{-1} 2/3$, R_h is positive, and if the rod be thought of as falling toward the right, if it slips at all it will slip toward the left. On the other hand for $\theta > \cos^{-1} 2/3$, R_h will be negative and the rod will tend to slip toward the right.

For actual slipping to occur $|R_h|$ must become greater than μR_v. To see what may happen consider the curves V and H given by the following equations, in rectangular coordinates with θ as abscissa.

$$(V) \qquad y_v = \mu(3\cos\theta - 1)^2,$$

$$(H) \qquad y_h = 3\sin\theta(3\cos\theta - 2).$$

y_v and y_h are merely the functions μR_v and R_h with the common factor $Mg/4$ removed. If we consider the derivatives $dy_v/d\theta$ and $dy_h/d\theta$ it is easy to see that y_v decreases steadily from its value 4μ for $\theta = 0$ to zero at $\cos\theta = 1/3$, while y_h is zero for $\theta = 0$, increases to a maximum at $\cos\theta = (1+\sqrt{19})/6$, then falls steadily to zero at $\cos\theta = 2/3$, and continues to decrease to $-2\sqrt{2}$ at $\cos\theta = 1/3$. Now if there is to be slipping toward the left the curve H must rise above the curve V in the interval $0 < \theta < \cos^{-1} 2/3$. Three cases are possible: (1) V crosses H at two points, (2) V and H are tangent and have no other points in common, (3) V lies entirely above H. That these are the only possibilities between $\cos\theta = 1$ and $\cos\theta = 2/3$ will become apparent below. In the last two cases no slipping to the left will occur while in the first case the rod will slip as soon as θ becomes greater than the value corresponding to the left hand point of intersection.

These points of intersection are given by the equation

$$y_v = y_h$$

or

$$(7) \qquad \mu(3\cos\theta - 1)^2 = 3\sin\theta(3\cos\theta - 2).$$

Between $\cos\theta = 2/3$ and $\cos\theta = 1/3$, since R_h changes sign, we must take the equation

$$y_v = -y_h$$

or

$$(8) \qquad \mu(3\cos\theta - 1)^2 = -3\sin\theta(3\cos\theta - 2),$$

to obtain the critical angle at which slipping may take place. Between $\cos\theta = 1$ and $\cos\theta = 2/3$, (8) can have no roots since y_v is positive and $-y_h$ is negative, while between $\cos\theta = 2/3$ and $\cos\theta = 1/3$, (7) can have no roots since for this interval y_v is positive and y_h negative. Hence, we will obtain precisely all the crit-

ical angles, between $\cos\theta=1$ and $\cos\theta=1/3$, at which slipping may occur from the equation

$$y_v^2 = y_h^2,$$

which reduces to

$$\begin{aligned}(9)\qquad & 81(\mu^2+1)\cos^4\theta - 108(\mu^2+1)\cos^3\theta \\ & + 9(6\mu^2-5)\cos^2\theta - 12(\mu^2-9)\cos\theta + \mu^2 - 36 = 0.\end{aligned}$$

This equation is algebraic in $\cos\theta$ and if we compute the Sturm functions corresponding to it we obtain, after dropping certain positive factors, the following expressions

$$81(\mu^2+1)\cos^4\theta-108(\mu^2+1)\cos^3\theta+9(6\mu^2-5)\cos^2\theta-12(\mu^2-9)\cos\theta+\mu^2-36,$$

$$54(\mu^2+1)\cos^3\theta-54(\mu^2+1)\cos^2\theta+3(6\mu^2-5)\cos\theta-2(\mu^2-9),$$

$$33\cos^2\theta-49\cos\theta+18=33\left(\cos\theta-\frac{9}{11}\right)\left(\cos\theta-\frac{2}{3}\right),$$

$$-(3318\mu^2-675)\cos\theta+1970\mu^2-450,$$

$$1125-8192\mu^2.$$

From these functions we find that for μ less than the critical value

$$\mu_1 = \left(\frac{1125}{8192}\right)^{1/2} = \frac{15\sqrt{10}}{128} = 0.37058$$

equation (9) will have exactly two real simple roots between $\cos\theta=1$ and $\cos\theta=2/3$, while for $\mu=\mu_1$ there will be a double root which occurs for

$$\cos\theta_1 = \frac{9}{11}$$

or

$$\theta_1 = 35°5'.8,$$

and for $\mu>\mu_1$ no root. These give the three cases noted above. On the other hand, for *any* finite μ there will be one and only one root between $\cos\theta=2/3$ and $\cos\theta=1/3$, and hence $|R_h|$ will become greater than μR_v in this interval. For $\mu=\mu_1$ this root will occur at

$$\cos\theta_2 = \frac{-35+48\sqrt{14}}{231}$$

or

$$\theta_2 = 51°14'.8.$$

We may sum up these results as follows. For $\mu<\mu_1$, the rod will slip to the left before it reaches an inclination with the vertical equal to θ_1, while for $\mu\geqq\mu_1$ it will slip to the right at an inclination with the vertical greater than or equal to θ_2.

Note by the Editors. The discussion of the roots of (9) in the above solution can be put in a simpler form which gives a more comprehensive view of the variations of μ and the roots. Set $x=\cos\theta$ and $z=\mu^2/9$, then

$$(1)\quad z = (1-x^2)(3x-2)^2(3x-1)^{-4},\quad \frac{dz}{dx} = 2(3x-2)(9-11x)(3x-1)^{-5};$$

and we see that z increases from 4 at $x=0$ to $+\infty$ at $x=1/3$; it then decreases from $+\infty$ and the curve becomes tangent to the x-axis at $x=2/3$; z then increases to the maximum $z_1=125/8192$ at $x_1=9/11$; from this maximum it decreases to zero at $x=1$. For a given value of z, $0<z<z_1$, the parallel to the x-axis cuts the curve for three positive values of x, $x'>x''>x'''$, where this order is selected since, as x increases, θ decreases, and the corresponding values of θ are $\theta'<\theta''<\theta'''$. The root θ'' is rejected in the problem; for, as the root θ'' increases, x'' decreases and the corresponding z decreases. This shows that the left side of (9) becomes positive when θ increases from the root θ'' and z remains fixed; hence $\mu R_v-|R_h|$ becomes positive. On the other hand, as θ', θ''' increase, x', x''' decrease and the corresponding z's increase; hence for z fixed the left side of (9) becomes negative when θ increases from the root values, and $\mu R_v-|R_h|$ becomes negative. When $z=z_1$ there are two equal roots $\theta_1=\theta'=\theta''$ and a simple root $\theta'''=\theta_2$. When $4>z>z_1$, there is only one real positive root θ'''.

A method for the determination of the nature of the roots of a numerical biquadratic equation, which is often very convenient, may be used. Change the variable x to y so that the coefficient of the term of the third degree is zero; replacing the variable y by $1/t$, we obtain the equation of reciprocal roots $1/y$. The first derivative equation in t has the root $t=0$ and the roots of a quadratic. These three roots enable us to determine precisely the nature of all four roots of the t equation and hence the nature of the roots of the original equation. The method is very convenient if the roots of the quadratic are imaginary. For the equation (1) the change is made by setting $1/t=3x-1$. We can see in advance that one root of the quadratic is $t=1$; since, if $z=0$, $x=2/3$ is a double root of the original equation. After a change of sign, the resulting equations are

$$f(t) = 8t^4 - 18t^3 + 11t^2 - (1+9z) = 0,$$

$$f'(t) = 2t(t-1)(16t-11).$$

The signs of $f(-\infty)$, $f(0)$, $f(11/16)$, $f(1)$, $f(+\infty)$ determine completely the nature of the roots for a given value of z.

3608 [1933, 243]. *Proposed by N. A. Court, University of Oklahoma.*

The vertex of a positive angle fixed in magnitude describes a given straight line, while the initial line constantly passes through a fixed point. Find the envelope of the second side.

Solution by H. W. Bailey, University of Illinois.

Let the fixed point be $(0, b)$; the fixed line, $y=0$; the fixed angle arc tan k; and let the vertex of the angle be at a variable point $(a, 0)$. Then the slope of the initial side is $-b/a$; call the slope of the terminal side m. Setting up the equation for the tangent of the angle of intersection of the initial and terminal sides and solving for m, we get

$$m = (ak - b)/(a + bk).$$

The equation of the terminal side is then

$$(ak - b)x - (a + bk)y = a(ak - b),$$

where a is a parameter. On eliminating a from this equation and its derivative with respect to a, the equation of the envelope is found to be

$$k^2x^2 - 2kxy + y^2 - 2bkx - 2b(1 + 2k^2)y + b^2 = 0.$$

The discriminant of this parabola is

$$-4b^2k^2(1 + k^2)^2;$$

it vanishes only when $b=0$ or when $k=0$, so that the parabola is degenerate if, and only if, the fixed point is on the fixed line or the angle is 0° or 180°. The parabola has the focus $(0, b)$, directrix $x+ky+bk=0$, vertex $(-bk/(1+k^2), b/(1+k^2))$, and latus rectum $4bk/(1+k^2)^{1/2}$.

Solved also by J. H. Butchart, J. W. Clawson, A. S. Householder, Ethel I. Moody, O. J. Ramler, B. D. Roberts, Ruth B. Smyth, F. Underwood, S. Vatriquant, R. C. Yates, and Margaret M. Young.

Note by the Editors. By simple geometric considerations this problem reduces to a part of problem 3535 [1932, 174], solution [1933, 56]. Let F be the given point and l the given line, not containing F, and let N be the foot of the perpendicular from F on l. Let Q be the variable vertex of the angle on l, and let P be the foot of the perpendicular from F on the side QP of the angle. Since F, Q, N, P lie on a circle $\angle FNP = \angle FQP$, and hence straight line $m=NP$ is fixed in position. Thus the problem reduces to finding the inverse pedal curve of m with respect to F. The envelope, as shown in the above reference, is a parabola with F as focus and m the tangent at the vertex V, the foot of the perpendicular from F on m. Let FV meet l in U, and let T be the point on l so that $NT=UN$. It is then clear that the angles of the isosceles triangle FUT at U and T are each equal to $\angle FNV$, and thus T is the position of Q when the terminal line QP falls along l. It then follows that T is the point of contact of the parabola with l.

NEWS AND NOTICES

Readers are invited to contribute to the general interest of this department by sending items to Professor J. H. Weaver, Ohio State University, Columbus, Ohio.

The Mathematical Association has received the first volume of the "Mathematics Student," a quarterly dedicated to the service of students and teachers of mathematics in India. This journal is edited by A. Narasinga Rao, M.A., L.T., and is published by the Indian Mathematical Society. This is in addition to the Journal of the Indian Mathematical Society, which hereafter will print only advanced matter. The editor states that, "The Mathematics Student is the outcome of the realization of the urgent need for better methods of teaching mathematics in this country, and dedicates its services to the students and teachers of mathematics and young research workers. . . . It will seek to stimulate interest, encourage wide reading and a critical appreciation of results. It will be the official organ for all announcements of the Society and will attempt to keep its readers in touch with the mathematical world of India and abroad. Finally, it will devote itself, wholeheartedly, to raise the standard of mathematical instruction in the country by providing a medium for the exchange of ideas, and for drawing attention to defects in syllabuses and methods on instructions and suggestions for their improvement."

In memory of its late Director, the review "Scientia" founded in 1930 the Eugenio Rignano Prize of the value of 10,000 lire to be conferred by international competition upon the author of the best essay on "The Evolution of the Notion of Time." The prize has recently been divided between Professor Sigismond Zawirski of the University of Poznàn and Professor Giovanni Giorgi of the University of Palermo.

The following courses in Mathematics are announced for the Summer, 1934.

University of Chicago, First term, June 20 to July 20; Second term, July 23 to August 24. In addition to Calculus II and Elementary differential equations, the following advanced courses will be offered: By Professor L. E. Dickson: Topics in the theory of numbers. By Professor E. P. Lane: Metric differential geometry; Projective differential geometry. By Professor R. W. Barnard: Algebraic invariants; Postulational methods. By Professor L. M. Graves: Introduction to the theory of functions; Theory of functions of real variables. By By Professor A. A. Albert: Theory of fields. By Dr. W. T. Reid: Boundary value problems. By Professor Walter Bartky: Theoretical mechanics; Theory of the potential. By Dr. Max Coral: Solid analytic geometry.

Columbia University, July 9 to August 17. In addition to courses in trigonometry, solid geometry, analytic geometry, calculus, and methods of teaching secondary mathematics, the following advanced courses are offered: By Professor E. Kasner: Survey of modern mathematics; Differential geometry. By

Professor J. F. Ritt: Theory of functions of a complex variable. By Dr. A. C. Berry: Differential equations. By Dr. A. B. Brown: Groups of finite order.

Cornell University, July 9 to August 18. In addition to the usual elementary work, the following advanced courses will be offered: By Professor J. I. Hutchinson: Modern algebra. By Professor Virgil Snyder: Advanced analytic geometry. By Professor W. A. Hurwitz: Functions of a complex variable. By Professor W. B. Carver: Projective geometry. By Professor D. C. Gillespie: Advanced calculus. Reading and research work will be directed by Professors J. I. Hutchinson, Virgil Snyder, F. R. Sharpe, W. A. Hurwitz, W. B. Carver, D. C. Gillespie, and R. P. Agnew.

The George Washington University, nine weeks term, June 11 to August 11. The regular courses in college algebra, trigonometry and analytic geometry will be offered.

University of Illinois, June 18 to August 11. In addition to the usual courses in college algebra, trigonometry, analytic geometry, and calculus, the following advanced courses are offered: By Professor A. B. Coble: Introduction to higher algebra; Algebra. By Professor Arnold Emch: Introduction to higher geometry; Geometry. By Associate Professor A. R. Crathorne: Statistics; Analysis. By Assistant Professor H. W. Bailey: Advanced aspects of euclidean geometry; Introduction to higher analysis.

University of Iowa, First term, June 11 to July 19. In addition to courses in College Algebra, Trigonometry, analytic geometry and calculus, the following courses are offered: By Miss Ruth Lane: Subject matter and teaching of mathematics. By Dr. Conkwright: Differential equations, theory of numbers; theory of equations. By Assistant Professor Ward: Second course in differential equations; Modern geometry; Infinite series. By Associate Professor Woods: Constructive geometry. By Professor Reilly: Numerical solution of equations; Summation; Seminar in topics in finite differences. By the staff: Reading and research.

Second term, July 23 to August 23. By Professor Chittenden: Calculus; Advanced calculus; Seminar in analysis. By Associate Professor Wylie: Advanced algebra; Astronomy; Meteors. By Dr. Craig: Differential equations; Matrices and determinants; Theory of statistics. By the staff: Reading and research.

Johns Hopkins University, June 25 to August 4. By Professor F. D. Murnaghan: College algebra; Analytic geometry; Real variables.

University of Kansas, June 13 to August 8. In addition to the usual elementary courses, the following advanced courses are offered: By Professor Mitchell: Theory of numbers; Higher algebra; Teachers course; Seminar. By Professor Smith: Advanced calculus; Modern analytic geometry; Series; Seminar.

University of Kentucky, First term. By Dean P. P. Boyd: Modern geometry. By Professor J. M. Davis: Differential equations. By Professor C. G. Latimer: Solid analytics. Second term. By Professor H. H. Downing: Infinite series. By Professor F. E. LeStourgeon: Theory of functions of a complex variable.

University of Maine, July 2 to August 10. In addition to the usual elementary work, the following advanced courses are offered: By Associate Professor Bryan: History of mathematics; Teachers course. By Associate Professor Jordan: Practical astronomy. By Professor Willard: Differential equations, or other graduate courses by arrangement.

Massachusetts Institute of Technology, First period, June 12 to July 24. Calculus and differential equations, covering the prescribed work of the first two years; Course in theoretical aeronautics; Advanced calculus; Theory of functions with applications to engineering. Second period, July 26 to September 6. Courses in the first period repeated; Vector analysis. August 7 to September 9. Courses in algebra, solid geometry and trigonometry, in preparation for fall entrance examinations in these subjects.

University of Michigan, June 25 to August 17. In addition to courses in college algebra, trigonometry, plane and solid analytic geometry, calculus, statistics, and finance, the following advanced courses will be offered: By Professor W. L. Ayres: Advanced calculus. By Professor H. C. Carver: Finite differences. By Professor R. V. Churchill: Differential geometry. By Professor C. C. Craig: Theory of probability; Empirical formulas; Mathematical theory of statistics, By Professor Peter Field: Analytic projective geometry; Vector analysis. By Professor W. B. Ford: Advanced calculus; Infinite series, with special reference to Fourier series. By Professor T. H. Hildebrandt: Theory of functions of a complex variable; Integral equations. By Professor L. C. Karpinski: Teaching of algebra; History of geometry. By Professor W. O. Menge: Advanced solid analytic geometry. By Professor J. A. Nyswander: Algebraic theory. By Professor V. C. Poor: Differential equations; Applied mathematics —Engineering problems. By Professor R. L. Wilder: Higher algebra; Introduction to the foundations of mathematics. By Professor T. H. Hildebrandt and others: Seminar in pure mathematics.

University of Minnesota, First term, June 18 to July 28. In addition to the usual elementary work, the following courses will be offered: By Professor Dunham Jackson: History of mathematics; The Gamma function and related topics. By Associate Professor Anthony L. Underhill: Differential equations. By Professors Jackson and Underhill: Reading in advanced mathematics. Second term, July 30 to September 1. No courses will be offered beyond the sophomore level.

Northwestern University, June 25 to August 18. In addition to the usual elementary work, the following advanced courses will be offered: By Professor

D. R. Curtiss: Theory of functions. By Professor H. A. Simmons: Advanced calculus. By Mr. J. F. Kenney: Theory of statistics.

Ohio State University, June 18 to August 31. In addition to the usual elementary work, the following advanced courses will be offered: By Professor H. Blumberg: Point sets and real functions; Introduction to the theory of relativity. By Associate Professor C. C. MacDuffee: Solid analytic geometry; Theory of matrices. By Assistant Professor F. R. Bamforth: Advanced calculus; Partial differential equations. By Assistant Professor Beatty: Advanced Euclidean geometry. Reading and research work will be directed by Professor Blumberg, Associate Professor MacDuffee, and Assistant Professor Bamforth.

University of Pennsylvania, July 2 to August 11. In addition to the usual courses in elementary subjects, the following will be offered: By Professor J. A. Shohat: Differential equations; Theory of probability. By Professor H. H. Mitchell: Theory of equations. By Professor J. R. Kline: Theories of integration; Non-Euclidean geometry. By Professor G. G. Chambers: Functions of a complex variable.

University of Pittsburgh, July 2 to August 10. In addition to the undergraduate courses the following more advanced courses will be offered: By Professor F. A. Foraker: Synthetic geometry; Mathematical statistics. By Professor J. S. Taylor: Functions of a complex variable; Functions of a real variable. By Professor Culver: Modern algebraic theories.

Syracuse University, July 2 to August 10. In addition to the usual elementary work, the following advanced courses will be offered: By Professor I. S. Carroll: Teaching of mathematics. By Professor F. F. Decker: History of mathematics or Fundamental concepts of mathematics. By Professor A. D. Campbell: College plane geometry or Elementary solid analytic geometry.

University of Vermont, By Professor Bullard: Differential calculus; Analytical geometry. By Professor Millington: Elementary algebra; Plane trigonometry; Integral calculus.

University of Wisconsin, six weeks session, June 25 to August 3. By Professor T. Bennett: Modern analytic geometry; Differential geometry; College geometry. By Dr. M. L. Hartung: The content of secondary mathematics; The teaching of mathematics. By Professor E. B. Skinner: Theory of equations; Finite groups; Theory of numbers. By Professor I. S. Sokolnikoff: Analytic geometry; Advanced calculus; Conformal representation. Special nine weeks session for graduates, June 25 to August 24. These courses may be taken for six weeks. By Professor R. E. Langer: Infinite series; Calculus of variations.

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EDITORIAL CORRESPONDENCE should be addressed to the EDITOR-IN-CHIEF, W. B. CARVER, White Hall, Cornell University, Ithaca, N.Y.

BOOKS FOR REVIEW should be addressed to R. A. JOHNSON, Brooklyn College, 66 Court Street, Brooklyn, N.Y.

BUSINESS CORRESPONDENCE should be addressed to the SECRETARY-TREASURER of the Association, W. D. CAIRNS, 33 Peters Hall, Oberlin, Ohio.

CHANGE OF ADDRESS: Members should send notice of any change of address to the SECRETARY-TREASURER, W. D. CAIRNS, Oberlin, Ohio, before the 10th of each month.

MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Eighteenth Summer Meeting of the Association, Williamstown, Mass., Sept. 3-4, 1934.

The following is a list of the sections of the Association, with dates of those section meetings which have been scheduled for 1934 and reported to the Secretary.

ALLEGHENY MOUNTAIN, Pittsburgh, Pa., Feb. 10; Washington, Pa., May 5.
ILLINOIS, Jacksonville, May 4-5.
INDIANA, La Fayette, May 11-12.
IOWA, Des Moines, April 20-21.
KANSAS, Topeka, Mar. 17.
KENTUCKY, May.
LOUISIANA-MISSISSIPPI, Jackson, Miss., Mar. 23-24.
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Williamsburg, Va., May.
MICHIGAN, Ann Arbor, Mar. 17.
MINNESOTA, Northfield, May 12.
MISSOURI.
NEBRASKA.
OHIO, Columbus, Apr. 5.
OKLAHOMA, Oklahoma City, Feb. 9.
PHILADELPHIA, Philadelphia, Dec. 1.
ROCKY MOUNTAIN.
SOUTHEASTERN, University, Ala., Mar. 30-31.
SOUTHERN CALIFORNIA, Riverside, Mar. 3.
TEXAS.
WISCONSIN, Oshkosh, May 5.

AFFILIATED ORGANIZATIONS: THE NEW ENGLAND ASSOCIATION OF TEACHERS OF MATHEMATICS. THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS.

GEORGE BANTA PUBLISHING COMPANY, MENASHA, WISCONSIN

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THE SECULAR EFFECT OF TIDES ON THE MOTION OF PLANETARY SYSTEMS

By TULLIO LEVI-CIVITA, University of Rome

The late Sir G. H. Darwin devoted a great part of his busy life to the study of tides, not only in their terrestrial aspects but also in connection with their slow but important astronomical consequences. He thoroughly investigated the effects of tidal friction on the motion of two viscous spheroids,[1] say the Earth and Moon, recognizing in particular the asymptotic effect of this dissipative force on the revolution and rotation of two bodies.

Darwin's analysis considers in detail the essential mechanical features of both local and general motions. This treatment requires, in concept and calculation, a formidable amount of work before yielding rigorous results concerning the final evolution of a planetary system. On the other hand it must be acknowledged that this method has certain intuitive advantages, for one gets an insight into the small and slow, but increasing and accumulating mechanical action of tidal friction.

With special reference to the couple, Earth and Moon, the following results are to be noted:

(1) Retardation of the period of rotation of each spheroid around its axis, which implies a lengthening of the day.

(2) Retardation (to a smaller extent) of the period of revolution, that is a lengthening of the month.

(3) Tendency (many thousand times stronger for the Moon than for the Earth) to equalize the periods of rotation and revolution.

(4) Tendency of both axes to arrange themselves perpendicularly to the orital plane.

When Lord Kelvin (at that time Sir William Thomson) became acquainted with Darwin's work he suggested that a more general theory might be obtained by means of arguments concerning the energy and the conservation of angular momentum. The remarkable possibilities of this method were shown soon after by Darwin himself, who studied the behaviour of the energy of two gravitating bodies, each moving in a circular orbit and rotating about its own axis perpendicular to the plane of the orbits. If this general motion is accompanied by frictional tides of some sort, the system must be losing energy and this energy must tend to a minimum compatible with the supposed conditions, among which the conservation of angular momentum is the most important. The simplicity with which the effects of complicated mechanical interactions may then be inferred is a tribute to Kelvin's genius.

The treatment of Darwin[2] is confined to the case in which, after friction,

[1] G. H. Darwin Scientific papers Vol. II, *Tidal Friction and Cosmogony*, Cambridge University Press, 1908.

[2] L.c. pp. 195–207; 374–382; or more briefly in Kelvin and Tait, *Treatise on Natural Philosophy*, appendix G., vol. II.

viscosity and other dissipative forces have been neglected, the residual conservative system belongs to the limited class of integrable cases.

A few years ago I made an attempt to apply adiabatic invariants to this type of problem.[3] The construction of adiabatic invariants does not depend on the previous integration of the unperturbed conservative problem, and from their expression it was possible to draw some interesting conclusions concerning cosmic evolution. However it would be very tedious, and perhaps impossible, to detect by this method the final state to which a general planetary system tends on account of tides and other small but continual dissipative forces.

A way by which one can easily derive the most fruitful conclusions from Kelvin's suggestion has recently been discovered by G. Krall who has devoted many papers to this fascinating subject and has exhaustively investigated the final effects caused by tidal friction and cognate phenomena. He starts from Darwin's model, Earth and Moon, and proceeds to the more general case of planetary systems of $n+1$ gravitating bodies, each of which may be rotating about its center of mass.[4]

I desire here to sketch the general outlines and some straightforward applications of Krall's method.

1. *Introductory remarks on canonical systems*

As you know, a differential system is said to be *canonical* if the $2n$ dependent variables, when paired in *conjugate* couples p_h, $q_h (h=1, \cdots, n)$, satisfy the relations,

$$\frac{dp_h}{dt} = -\frac{\partial H}{\partial q_h}, \qquad \frac{dq_h}{dt} = \frac{\partial H}{\partial p_h} \qquad (h = 1, 2, \cdots, n), \tag{1.1}$$

where H is any function whatever of the dependent variables, and perhaps even of the independent variable t. We shall suppose, however, that H, the *characteristic function* or simply the *Hamiltonian*, depends only on $p_1, \cdots, p_n, q_1, \cdots, q_n$, which we shall refer to collectively as (p, q). We have, regarding (p, q) as functions of t,

$$\frac{dH}{dt} = \sum_{h=1}^{n}\left(\frac{\partial H}{\partial p_h}\frac{dp_h}{dt} + \frac{\partial H}{\partial q_h}\frac{dq_h}{dt}\right).$$

The right hand side vanishes on account of (1.1), and a necessary consequence of our canonical system is,

$$H(p, q) = E, \tag{1.2}$$

where E is an arbitrary constant.

The fundamental importance of this class of systems is that the *conservative*

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[3] *Applicazioni astronomiche degli invarianti adiabatici*, Atti del Congresso Int. dei Mat. (Bologna, 1928), T.V. pp. 17–28.

[4] Rend. Acc. Lincei, vol. 15 (1932), pp. 219–225, 371–376, 664–669.

problems of mechanics, and of other branches of physics admitting of mechanical models, may all be summarized in canonical form. The Hamiltonian represents the energy of the moving system, so the general integral $H=E$ expresses the *conservation of energy*. For a general material system, S, the q's usually denote general or Lagrangian coordinates, capable of fixing its position; their number, n, is called the *degree of freedom* of the system. The conjugate p's are called *momenta* and define the kinetic state of the system (i.e. the velocities of the constituent particles). The variable (p, q) may be interpreted as the Cartesian coordinates of a point in a Euclidean manifold, Φ_{2n}, which is called, after Gibbs, the phase space.

Obvious instances of the existence of integrals of canonical systems (1.1) occur when some of the variables do not appear explicitly in the Hamiltonian. Should this be the case for a particular variable q_μ, then $\partial H/\partial q_\mu = 0$ and (1.1) gives for the conjugate variable p_μ

$$p_\mu = c_\mu = \text{constant}. \tag{1.3}$$

In a similar way, if $\partial H/\partial p_\mu = 0$, we get

$$q_\mu = d_\mu = \text{constant}. \tag{1.4}$$

2. *Minimizing and stationary solutions in strict sense. Necessary shrinkage to equilibrium.*

A generic solution γ,

$$p_h = p_h(t), \qquad q_h = q_h(t) \qquad (h = 1, 2, \cdots, n), \tag{2.1}$$

of the differential system (1.1) may be interpreted as a parametric representation of a curve in the phase space, Φ_{2n}. Along every path the condition (1.2) is satisfied, that is the energy E has a constant value. This value will in general undergo an infinitesimal variation δE in passing from γ to some neighbouring path.

In 1887 Routh called attention to the exceptional solution (or solutions) γ_0 for which this δE is zero. Such solutions he called *stationary*. *Solutions minimizing* energy must naturally be stationary and may conveniently be singled out from among the stationary solutions.

Starting from a stationary solution γ_0 we must have $\delta E = 0$. Now, by (1.2), $H=E$ and δE is simply the increment δH which H assumes as function of (p, q) in passing from a point P_0 of γ_0 to some point of a neighbouring solution γ, as a matter of fact to any point P in the neighbourhood of P_0, for through every point of the space Φ_{2n} there passes one and only one solution of (1.1). A necessary condition that γ_0 be stationary is therefore,

$$\delta H = \sum_{h=1}^{n}\left(\frac{\partial H}{\partial p_h}\delta p_h + \frac{\partial H}{\partial q_h}\delta q_h\right) = 0, \tag{2.2}$$

where δp_h and δq_h are components of the displacement P_0P and hence com-

pletely arbitrary. Therefore (2.2) requires that along γ_0,

$$\frac{\partial H}{\partial p_h} = 0, \qquad \frac{\partial H}{\partial q_h} = 0 \qquad (h = 1, 2, \cdots, n). \tag{2.3}$$

Conversely, let us suppose that these $2n$ equations in $2n$ unknowns are satisfied by a set of numerical values p_h^0, q_h^0, which correspond to a point P_0. In general the $2n$ equations (2.3) will be compatible and there will only be a finite number of solutions P_0, but it may happen that the points P_0 will form a continuum (line, or more generally, subspace in Φ_{2n}). At any rate at each of these points P_0 the right hand side of (1.1) will vanish and therefore our canonical system admits of the particular solution

$$p_h = p_h^0, \qquad q_h = q_h^0 \qquad (h = 1, 2, \cdots, n),$$

that is the solution reduces to the point P_0 itself. In mechanical problems this obviously means a state of equilibrium.

At P_0 the conditions (2.3) are satisfied by construction. Hence necessary and sufficient conditions that γ_0 be stationary are that the system be in a state of equilibrium (degenerate to a point).

3. *Relatively stationary solutions in a first particular case.*

As remarked in section 1, if some of the q's, say $q_1, q_2, \cdots, q_m$, are cyclic variables, that is

$$\frac{\partial H}{\partial q_\mu} = 0 \qquad (\mu = 1, 2, \cdots, m), \tag{3.1}$$

then (1.1) admits the integrals

$$p_\mu = c_\mu \qquad (\mu = 1, 2, \cdots, m). \tag{3.2}$$

The next question is to determine more general solutions γ_0 of (1.1) for which (or better from which) $\delta H = 0$, not with respect to all solutions γ, but only with respect to those along which the p_μ's have the same values c_μ as along γ_0, (for $\mu = 1, \cdots, m$).

Since $\delta p_\mu = 0$, by (3.1) δH reduces to

$$\sum_{h=m}^{n} \left(\frac{\partial H}{\partial p_h} \delta p_h + \frac{\partial H}{\partial q_h} \delta q_h \right); \tag{3.3}$$

its vanishing imposes therefore the $2(n-m)$ conditions

$$\frac{\partial H}{\partial p_h} = 0, \qquad \frac{\partial H}{\partial q_h} = 0 \qquad (h = m + 1, \cdots, n).$$

These equations, for $p_\mu = c_\mu$ $(\mu = 1, \cdots, m)$, will furnish constant values p_h^0, q_h^0 for p_h, q_h for which the equations,

$$\frac{dp_h}{dt} = -\frac{\partial H}{\partial q_h}, \qquad \frac{dq_h}{dt} = \frac{\partial H}{\partial p_h} \qquad (h = m + 1, \cdots, n),$$

will be automatically satisfied.

Of the residual $2m$ equations of (1.1)

$$\left.\begin{aligned} &(3.4) \qquad \frac{dp_\mu}{dt} = -\frac{\partial H}{\partial q_\mu} \\ &(3.5) \qquad \frac{dq_\mu}{dt} = \frac{\partial H}{\partial p_\mu} \end{aligned}\right\} \qquad (\mu = 1, 2, \cdots, m),$$

the first group (3.4) is satisfied because of (3.2), while in the second group, (3.5), the right hand sides, since they involve only $q_h(h>m)$ and the p's are constants,

$$\omega_\mu = \frac{\partial H}{\partial p_\mu} \qquad (\mu = 1, 2, \cdots, m). \tag{3.6}$$

Therefore

$$q_\mu = \omega_\mu t + q_\mu^0 . \tag{3.7}$$

The q_μ^0 are m new arbitrary constants. The ω_μ are clearly constant velocities corresponding to the cyclic variables. Every such solution is stationary in the restricted sense (that is after imposing (3.2)), for $\delta H=0$ as soon as $\delta p_\mu=0$, $(\mu=1, \cdots, m)$.

These solutions are stationary with respect to the $2n-m$ variables, p_h $(h>m)$, and q_h (for all h). We may say then that there are ∞^{2n-m} solutions. On the other hand, since the ω's are in general not zero, not all coordinates q_μ are constant along γ_0 and some of them will vary linearly with t. This indicates an effective motion of the corresponding mechanical system and not a pure equilibrium. For a given ∞ set of conditions (3.2) there are ∞^m of these relatively stationary solutions, for they involve m arbitrary constants q_μ^0 (initial values of $q_1, q_2, \cdots, q_m$).

4. *Second particular case.*

We shall suppose now that H, in addition to being independent of $q_h(h<m)$, does not depend on certain of the corresponding momenta, says $p_1, p_2, \cdots,$ $p_{m'}(m'\leqq m)$. Then

$$\frac{\partial H}{\partial q_\mu} = 0 \qquad (\mu = 1, 2, \cdots, m), \tag{4.1}$$

$$\frac{\partial H}{\partial p_{\mu'}} = 0 \qquad (\mu' = 1, 2, \cdots, m'), \tag{4.2}$$

and

$$(4.3) \qquad p_\mu = c_\mu \qquad (\mu = 1, 2, \cdots, m),$$

$$(4.4) \qquad q_{\mu'} = d_{\mu'} \qquad (\mu' = 1, 2, \cdots, m').$$

We wish to investigate the stationary solutions γ_0 under these $m'+m$ conditions. Remembering that the solutions are to be stationary only with respect to the $2n-m-m'$ variables $p_h(h>m)$, $q_{h'}(h'>m')$, we get, in order that $\delta H=0$,

$$(4.5) \qquad \frac{\partial H}{\partial p_h} = 0 \qquad (h > m), \qquad \frac{\partial H}{\partial q_{h'}} \qquad (h' > m').$$

Some of these conditions are automatically satisfied because of (4.2), since we have already supposed that $m' \leqq m$. Therefore we need only consider

$$(4.6) \qquad \frac{\partial H}{\partial p_h} = 0, \qquad \frac{\partial H}{\partial q_h} = 0 \qquad (h > m),$$

which is the same as (3.3). If we proceed to the characterization of possible stationary solutions γ_0, we recognize that they may be represented by exactly the same formulas as were used in the preceding paragraph. However, on account of (4.2), the velocities $\omega_\mu(\mu=1, 2, \cdots, m')$ vanish and therefore the corresponding cyclic variables are constant on γ_0.

If $m=m'$ we are again reduced to a state of equilibrium. This particularly unfavorable case can not occur when the total number, $m+m'$, of integrals playing the role of restraints is odd, for then $m-m'$ is also odd. In any event *the number of constants involved*, for a given set of conditions (4.3), (4.4), is just $m-m'$.

While the stationarity is by hypothesis confined to solutions verifying (4.3) *and* (4.4), it is only necessary to consider the latter conditions in the final step.

5. *Changes of variables. Invariant form of the condition that a solution be stationary.*

It is familiar from the theory of maxima and minima that the property of a function to assume extreme values, or simply stationary values, is not affected by a change of variables. Let us perform a general transformation in the case just considered by substituting for the canonical variables (p, q) $2n$ linearly independent variables

$$x_\nu = x_\nu(p, q) \qquad (\nu = 1, 2, \cdots, 2n).$$

The system (1.1) will, in general, lose its canonical form and will appear as the generic normal system,

$$(5.1) \qquad \frac{dx_\nu}{dt} = X_\nu(x) \qquad (\nu = 1, 2, \cdots, 2n)$$

where x stands for the set $x_1, x_2, \cdots, x_{2n}$. The right hand sides preserve the property of not involving t.

Under the hypothesis of section 4, that H does not involve certain of the p's and q's, the canonical system admits the integrals (4.3), (4.4). If we set $p_\mu = F_\mu(x)$, $q_\mu = G_\mu(x)$, the integrals (4.3), (4.4) become

$$F_\mu = c_\mu \qquad (\mu = 1, 2, \cdots, m), \tag{5.2}$$

$$G_{\mu'} = d_{\mu'} \qquad (\mu' = 1, 2, \cdots, m'), \tag{5.3}$$

or

$$\phi_\rho(x) = k_\rho \qquad (\rho = 1, 2, \cdots, m + m'), \tag{5.4}$$

where the k's are constants and

$$\phi_\rho = F_\rho \quad (\rho = 1, 2, \cdots, m); \quad \phi_\rho = G_{\rho-m} \quad (\rho = m + 1, \cdots, m + m'). \tag{5.5}$$

In the discussion of the previous paragraph it was seen that the conditions that H be relatively stationary led to a class of $\infty^{m-m'}$ solutions γ_0 of (1.1). If we express the integrals, under which H is to be rendered stationary, in terms of the x's and use Lagrange's method of indeterminate multipliers, we are led to the equation

$$\delta\left(H + \sum_{\rho=1}^{m+m'} \lambda_\rho \phi_\rho\right) = 0. \tag{5.6}$$

This is equivalent to the following $2n$ equations in finite form relating the x's and the λ's,

$$\frac{\partial H}{\partial x_\nu} + \sum_{\rho=1}^{m+m'} \lambda_\rho \frac{\partial \phi_\rho}{\partial x_\nu} = 0 \qquad (\nu = 1, 2, \cdots, 2n). \tag{5.7}$$

Of course it will be granted, from the results obtained with reference to the original (p, q), that the association of (5.7) to the differential system (5.1) is in general compatible and gives rise to $\infty^{m-m'}$ stationary solutions γ_0 whose determination may require, in addition to operations in finite terms, at most the integration of a differential system of order $m - m'$.

6. *Canonical systems with known integrals of any form.*

The method of constructing particular integrals of a canonical system used in section 4 supposes that the known integrals, relative to which the value of H is to be made stationary, have the form $p = \text{const}$, $q = \text{const}$. This restriction is not essential as may be seen by combining the general theory of canonical transformations with the remarks of the preceding paragraph. Any set of integrals,

$$\phi_\rho = \text{const.} \qquad (\rho = 1, 2, \cdots, r),$$

may be supposed to form a Lie group. This means that any Poisson bracket

$$(\phi_\rho, \phi_\sigma) = \sum_{h=1}^{n} \left(\frac{\partial \phi_\rho}{\partial p_h} \frac{\partial \phi_\sigma}{\partial q_h} - \frac{\partial \phi_\rho}{\partial q_h} \frac{\partial \phi_\sigma}{\partial p_h}\right),$$

is expressible in terms of the ϕ's alone. Then the general Lie theory states:

1) There exists a canonical form for the group in which, if the ϕ's are replaced by r independent combinations $\Phi_\rho(\phi_1, \phi_2, \cdots, \phi_r)$, the Poisson's brackets are either zero or unity.

2) There are some canonical transformations between (p, q) and the new variables (P, Q), r of which are just the functions Φ.

If the transformation between (p, q) and (P, Q) is canonical, then our system (1.1) preserves the canonical form, the Hamiltonian being the same H expressed in the new variables.

If H is stationary when the Φ's are constant, then (by section 4) by means of at most $m-m'$ quadratures, and some finite operations, we are led to $\infty^{m-m'}$ particular solutions. What may be inferred from this if we return to the primitive variables (p, q)? In the first place, as the Φ's are independent combinations of the ϕ's, the condition that the Φ's be constant is identical with the condition that the ϕ's be constant. Also the existence of maxima, minima and stationary values is not affected by changes of variables; we may therefore conclude that, in searching for stationary solutions of (1.1) under the constraints of r given integrals,

$$\phi_\rho = K_\rho \qquad (\rho = 1, 2, \cdots, r),$$

the conditions (5.6) [or (5.7)] are still to be associated with the differential system (1.1).

The transformation to the special canonical variables, which we have used only to simplify the demonstration and have not actually performed, leads to one more inference (see the end of the preceding paragraph): that is the conclusion that, even in the general case of subordination to some integrals $\phi_\rho = k_\rho$, the variational condition (5.6) is normally compatible with the differential system (1.1) and the determination of the corresponding solution requires, in addition to operations in finite terms, at most the integration of a differential system of order $m-m'$.

The avoidance of the use of the special canonical coordinates (P, Q) in the actual construction of stationary solutions γ_0 is not a matter of convenience only, but an absolute necessity from the methodological standpoint, for the determination of the canonical transformation between the generic variables (p, q) and the special (P, Q) is in general more difficult than the integration of (1. 1), since it depends upon differential operations of rank much higher than $m-m'$.

7. *Teleological remarks. Systems involving parameters.*

If we know that the energy of a natural process tends to decrease because of a dissipative force of any kind in such a way that it is either not possible or at least extremely difficult to follow the whole evolution of the process, it becomes highly desirable to determine the ultimate effects of these losses of energy. For mechanical phenomena these ultimate effects are much easier to investigate than the sequence of intermediary phases.

We shall consider a material system which, except for very small influences which are negligible at first, is conservative and determined by the Hamiltonian $H(p, q)$, i.e. by the canonical system

$$(7.1)\qquad \frac{dp_h}{dt} = -\frac{\partial H}{\partial q_h}, \qquad \frac{dq_h}{dt} = \frac{\partial H}{\partial p_h} \qquad (h = 1, 2, \cdots, n).$$

We assume that H, besides being a function of (p, q) depends on some parameters a which vary so slowly that the integration of (7.1) is not sensibly affected by their variation. We shall call such a variation *adiabatic*. The existence of such a variation will eventually be noticed in the integral expressions for the solutions of the canonical system. We lastly assume that, due to the diminution of energy, H tends asymptotically to a limit compatible with certain preassigned conditions. These conditions are expressed by the r integrals,

$$(7.2)\qquad \phi_\rho(p, q, a) = k_\rho \qquad (\rho = 1, 2, \cdots, r),$$

where the k's are unaffected by the adiabatic variation of the a's.

The problem now is to determine, among the solutions γ of (7.1), those solutions γ_0 which correspond to the minimum of H. Of course it will be necessary that H have a stationary value in passing from a point of γ_0 to a point of any other solution satisfying the same constraints; i.e. admitting for the integrals (7.2) the same values of the k's.

For the final state only stationary solutions are allowable along which (5.6) holds. Since H involves, besides (p, q), the parameters a, its minimum must be taken not only as a function of (p, q) but also as a function of a. For the sake of clearness we split up the variation of a function $F(p, q, a)$ into two parts, the first δF, arising from (p, q) and the second $\delta_a F$, arising from a. In this notation, for the final state we must have $(\delta+\delta_a)H=0$ under the conditions (7.2). This implies separately

$$(7.3)\qquad \delta H = 0, \qquad \delta_a H = 0$$

under the same conditions.

The first group implies that (5.7) must be satisfied and also that the asymptotic solution is to be found only among the stationary solutions γ_0. The second group gives the additional conditions for the existence of a final state

$$(7.4)\qquad \frac{\partial H}{\partial a} + \sum_{\rho=1}^{r} \lambda_\rho \frac{\partial \phi_\rho}{\partial a}\cdot$$

This condition (parametric stationarity) adjoined to the motional stationarity just emphasized, is precisely Krall's rule, which adequately expresses Kelvin's intuitions.

Of course as in the ordinary problems of maxima and minima the conditions are necessary but not sufficient. To fully characterize the final state, if any, a

fuller discussion is needed in order to distinguish those solutions which actually render H a minimum.

8. *Problem of two bodies. Preparatory formulas for energy and angular momentum.*

Let us consider two rigid bodies B_0 and B_1 (planet and satellite), concentrating our attention on the ideal case in which the external forces may be neglected. This is the case in actual instances of planetary systems. We call the centers of mass of B_0 and B_1, P_0 and P_1 respectively and we call their masses m_0 and m_1. Let r be the distance P_0P_1. We denote by x, y, z, the coordinates of P_1 with respect to a fixed system of axes through P_0 and by p_x, p_y, p_z, the components of the momentum of B_1. We shall suppose that B_0, B_1 are oblate spheroids, or more generally gyroscopes, having $C_i(i=0, 1)$ as polar and $A_i(<C_i)$ as equatorial moments of inertia. If we assume, as we may, that the center of mass of the system is fixed, the problem of revolution and rotation of the two bodies under mutual gravitation is one of 9 degrees of freedom, three corresponding to revolution and three for each body corresponding to rotation about the respective centers of mass. We shall confine ourselves to the case where the bodies, though having a mass which contributes noticeably to their energy, are far enough apart that the ratio D/r of their greatest dimensions to the distance between them is always small. This is always the case in actual planetary systems.

The Newtonian potential of mutual attraction has the general form

$$\Omega = f\int_{B_0} dm_0 \int_{B_1} \frac{dm_1}{\Delta},$$

where f denotes the gravitational constant and Δ the distance between two volume elements of the bodies B_0, B_1. If D/r is negligible, then we may simply replace Δ by the constant r and obtain

$$\Omega = fm_0m_1/r.$$

We wish to go a step further, however, and procure a more accurate expression for Ω in terms of the relative positions of the two bodies, neglecting only terms of the order of $(D/r)^3$.

At this juncture it is convenient to fix the geometrical parameters upon which Ω may finally depend. These parameters for any rigid body are usually the Eulerian angles, θ_i, f_i, ϕ_i, which define the orientation of B_i about its center of mass P_i with respect to some frame of reference. According to Kirchhoff, we may assume that θ_i and ϕ_i are the polar angles of the figure axis of B_i, which has the direction cosines

$$\alpha_i = \sin\theta_i \cos\phi_i, \qquad \beta_i = \sin\theta_i \sin\phi_i, \qquad \gamma_i = \cos\theta_i.$$

The remaining angle f_i has the function of fixing the orientation of B_i about its figure axis. Since the mass of each body is distributed symmetrically about its

axis, an axial rotation does not alter the gravitating matter. This means that Ω does not involve f_i and depends at most on

$$r, \ \theta_i, \ \phi_i.$$

Furthermore it is obvious that a rigid rotation of the whole system does not affect Ω. In particular we may rotate the system about the axis, say the ζ axis, to which the angles ϕ_i are referred as azimuths without changing Ω. This means that Ω is not changed if we add equal increments to ϕ_0 and ϕ_1 and hence Ω depends not on ϕ_0 and ϕ_1 but on their difference $\phi_0 - \phi_1$.

It can be shown that, to the degree of accuracy to which we confine ourselves, assuming as ζ axis the line of centers P_0P_1 renders Ω independent of ϕ and reduces it to

$$\Omega = U' + U_0 + U_1, \tag{8.1}$$

where,[5]

$$\begin{aligned} U' &= \frac{fm_0m_1}{r} \\ U_i &= \frac{fm_0m_1}{2m_ir^3}(C_i - A_i)(1 - 3\cos^2\theta_i) \qquad (i = 0, 1). \end{aligned} \tag{8.2}$$

Note that in this formula it is indifferent how we take the positive sense, either on P_0P_1 or on the axes of symmetry of the two bodies. The kinetic energy T' of the translatory motion reduces to

$$2T' = \left(\frac{1}{m_0} + \frac{1}{m_1}\right)(p_x^2 + p_y^2 + p_z^2). \tag{8.3}$$

The kinetic energy T_i of the rotation of the rigid body B_i about its center of mass P_i has the familiar expression,

$$2T_i = A_i(p_i^2 + q_i^2) + C_ir_i^2,$$

p_i, q_i, r_i being the components of angular velocity. But A_ip_i, A_iq_i, C_i, r_i are the corresponding components of the angular momentum $\mathbf{K}_i$ arising solely from the rotation.[6] We may then write

$$2T_i = \frac{1}{A_i}\{(A_ip_i)^2 + (A_iq_i)^2\} + \frac{1}{C_i}(C_ir_i)^2 \qquad (i = 0, 1),$$

and therefore, if K_i denotes the absolute value of $\mathbf{K}_i$ and ψ_i its angular deviation from the polar axis,

[5] Levi-Civita and Amaldi, *Lezioni di Meccanica Razionale*, vol. 2 (second part), Bologna, Zanichelli, 1927, p. 391.

[6] The pole (or center of reduction) with respect to which this moment $\mathbf{K}_i$ is taken may be chosen at will, since the resultant vector of the momenta of a rigid body corresponding to an instantaneous rotation about its center of mass is zero.

$$(8.4) \qquad 2T_i = K_i^2\left(\frac{\sin^2\psi_i}{A_i} + \frac{\cos^2\psi_i}{C_i}\right) = K_i^2\left\{\frac{1}{C_i} + \left(\frac{1}{A_i} - \frac{1}{C_i}\right)\sin^2\psi_i\right\} \quad (i = 0, 1).$$

The total kinetic energy of the system is simply

$$T = T' + T_0 + T_1$$

and the Hamiltonian is

$$(8.6) \qquad H = T - U.$$

The angular momentum about the center of mass is the vector

$$(8.7) \qquad \boldsymbol{K} = \boldsymbol{K}' + \boldsymbol{K}_0 + \boldsymbol{K}_1,$$

where $\boldsymbol{K}'$ is the angular momentum of revolution about the center of mass and has the components

$$(8.8) \qquad K_x' = yp_z - zp_y, \qquad K_y' = zp_y - xp_z, \qquad K_z' = xp_y - yp_x.$$

This quantity may be regarded as *rigorously constant*, not only for the conservative problem just stated, but even in the more general case in which there exist internal forces of any kind.

In the form here adopted the Hamiltonian depends only on 16 variables; namely: (x, y, z, p_x, p_y, p_z) corresponding to the revolution of the system, the four which define the orientation of the vectors $\boldsymbol{K}_0$, $\boldsymbol{K}_1$, the absolute values of $\boldsymbol{K}_0$ and $\boldsymbol{K}_1$, and the inclinations θ_i, ψ_i $(i=0, 1)$ of the symmetry axes to P_0P_1 and $\boldsymbol{K}_i$. In general in canonical coordinates, the H corresponding to a system with nine degrees of freedom would depend on 18 variables, but in this case, even if referred to canonical coordinates, H would only involve 16 for two, the azimuthal angles, do not affect the problem.

This shows that for our problem two integrals must exist corresponding to the two cyclic variables. These integrals simply state that the axial component of the angular momentum $\boldsymbol{K}_i$, for each body B_i, is a constant, that is

$$(8.9) \qquad K_i \cos\psi_i = \text{const.} \qquad (i = 0, 1).$$

These integrals might be inferred directly from the Eulerian equations of motion of the bodies B_i. The axial component of the acting couple arising from the Newtonian attraction of the other body is always zero and the third Euler equation gives (8.9). This dynamical origin shows that the constant values in (8.9) would not in general be unaffected by the superposition of tidal actions.

It is now time to return to H and to search for its minimum subject to the constraints (8.8).

9. *Necessary conditions.*

Before using variational methods we will first investigate the dependence of

H on the four angular variables

$$\theta_i, \psi_i \qquad (i = 0, 1).$$

These variables appear only in the terms U_i, T_i and represent the inclination of the gyroscopic axis of B_i with respect to P_0P_1 and $\boldsymbol{K}_i$. The instantaneous orientation of these axes has obviously nothing to do with the vectors $\boldsymbol{K}_0$, $\boldsymbol{K}_1$, nor with the canonical variables (p_x, x), (p_y, y), (p_z, z). We may write, after applying (8.1)–(8.6),

$$H = \mathfrak{H} + H^*, \tag{9.1}$$

where

$$\mathfrak{H} = \sum_{i=0}^{1}\left\{\frac{1}{2}K_i^2\left(\frac{1}{A_i} - \frac{1}{C_i}\right)\sin^2\psi_i + \frac{3}{2}\,\frac{fm_0m_1}{m_ir}\,(C_i - A_i)\cos^2\theta_i\right\}, \tag{9.2}$$

and

$$\begin{aligned} H^* = {} & \frac{1}{2}\left(\frac{1}{m_0} - \frac{1}{m_1}\right)(p_x^2 + p_y^2 + p_z^2) + \frac{1}{2}\sum_{i=0}^{1}\frac{K_i^2}{C_i} \\ & - \frac{fm_0m_1}{r}\left\{1 + \frac{1}{2}\sum_{i=0}^{1}\frac{C_i - A_i}{m_ir^2}\right\}. \end{aligned} \tag{9.3}$$

The latter quantity is obviously independent of θ_i, ψ_i. Therefore, insofar as the variability of θ_i, ψ_i is concerned, since $C_i > A_i$, $\mathfrak{H}$ reaches its absolute minimum, zero, for

$$\sin\psi_i = 0, \qquad \cos\theta_i = 0 \qquad (i = 0, 1). \tag{9.4}$$

These values, if compatible with the constraints (8.7), must be taken on in the final state. Sin $\psi_i = 0$, implies that the moment $\boldsymbol{K}_i$ has the direction of the symmetry axis of B_i and therefore the rotation of each body takes place about its figure axis. Without fixing a positive sense we may say that the angular velocity ω_i has the absolute value

$$|\omega_i| = \frac{K_i}{C_i} \qquad (i = 0, 1). \tag{9.5}$$

Putting $\sin\psi_i = 0, \cos^2\psi_i = 1$ in (8.9) we see that the absolute values of $\boldsymbol{K}_0$, $\boldsymbol{K}_1$, and therefore those of the gyroscopic rotations, are certainly constant. Moreover, from the well known kinematical remark that the axis about which a body rotates is fixed in space, if it is fixed in a body, we conclude that *the two vectors* $\boldsymbol{K}_0$ *and* $\boldsymbol{K}_1$ *have not only constant length but also constant directions.*

Now let us pass to the equations of the second group of (9.4),

$$\cos\theta_i = 0 \qquad (i = 0, 1).$$

They tell us that both gyroscopic axes, and with them $\boldsymbol{K}_0$, $\boldsymbol{K}_1$ must be directed

perpendicularly to the line P_0P_1. As an immediate consequence we get, by projecting K_0, K_1 on P_0P_1 and making use of (8.8),

$$K_x x + K_y y + K_z z = 0. \tag{9.6}$$

If the vector K is not zero, (9.6) shows that the motion of revolution takes place, as in the ordinary problem of two variables, in one fixed plane. We may take this plane as the coordinate plane xy, whence the first two of the given constants K_x, K_y, K_z, assume the special values

$$K_x = K_y = 0, \tag{9.7}$$

while the third may be identified with the length of the vector K, if the direction Oz is that of K. Also for the motion in question it is necessary that

$$z = p_z = 0. \tag{9.8}$$

To shorten the discussion it will be useful to remember that our aim is to determine the final state in the case of planetary *revolutions*. Thus we may exclude the case where P_0P_1 is fixed and therefore the *constant* vectors K_0, K_1 must be normal to the orbital plane.

The projection of K on the orbital plane is zero and the projection of Oz gives

$$K = c + c_0 + c_1, \tag{9.9}$$

where $K>0$,

$$c = xp_y - yp_x \tag{9.10}$$

and c_0, c_1 are the components of K_0, K_1 along the z axis. We now adopt the direction Oz as the positive sense along the symmetry axes. We obtain from (9.5) the following form for the angular velocities,

$$\omega_i = \frac{c_i}{C_i} \qquad (i = 0, 1). \tag{9.11}$$

10. *Plane problem. Upright gyroscopes.*

The conditions for a minimum already derived allow us to give a mechanical interpretation to the remaining steps. In the expression $\mathfrak{H} + H^*$, for H, the term $\mathfrak{H}$ disappears and z, p_z are set equal to zero.

We write for brevity,

$$R(r) = \frac{fm_0m_1}{r}\left\{1 + \frac{1}{2}\sum_{i=0}^{1}\frac{C_i - A_i}{m_i r^2}\right\} = \frac{fm_0m_1}{r} + \mathrm{X}(r) \tag{10.1}$$

The remainder term,

$$\mathrm{X}(r) = \frac{1}{2}\,\frac{fm_0m_1}{r}\sum_{i=0}^{1}\frac{C_i - A_i}{m_i r^2}, \tag{10.2}$$

may be regarded as the potential of the perturbing forces acting between the two particles P_0 and P_1. Introducing instead of K_0, K_1 their z components $c_0 = \pm K_0$, $c_1 = \pm K_1$, we have

$$H = \frac{1}{2}\left(\frac{1}{m_0} + \frac{1}{m_1}\right)(p_x^2 + p_y^2) + \frac{1}{2}\sum_{i=0}^{1}\frac{c_i^2}{C_i} - R(r). \tag{10.3}$$

This is obviously the Hamiltonian of the plane problem of two gyroscopic bodies gravitating and rotating about their axes, which are supposed to be perpendicular to the xy plane in which the motion of the centers takes place. In this problem, considered as a conservative one, c_0, c_1 are regarded as constants and, since the potential $R(r)$ depends only on r, the integral

$$xp_y - yp_x = C$$

is admitted. If we adopt polar coordinates r, θ and complete the canonical transformation by the introduction of the corresponding momenta p_r, p_θ, we obtain the identities

$$p_x^2 + p_y^2 = p_r^2 + \frac{1}{r^2}p_\theta^2, \qquad xp_y - yp_x = C.$$

The first suffices for the transformation of H into polar coordinates. As θ does not appear explicitly, we have $p_\theta = \text{const}$. This in virtue of the second identity is nothing but the integral $xp_y - yp_x = c$ in the new variables; so

$$p_\theta = C. \tag{10.4}$$

On the other hand the equation,

$$\frac{d\theta}{dt} = \frac{\partial H}{\partial p_\theta},$$

determines the angular velocity $n = d\theta/dt$ in terms of r and $p_\theta = c$. This has the value

$$n = \frac{\partial H}{\partial p_\theta} = \left(\frac{1}{m_0} + \frac{1}{m_1}\right)\frac{c}{r^2}\cdot \tag{10.5}$$

The final expression for H is therefore,

$$H = \frac{1}{2}\left(\frac{1}{m_0} + \frac{1}{m_1}\right)\left(p_r^2 + \frac{c^2}{r^2}\right) + \frac{1}{2}\sum_{i=0}^{1}\frac{c_i}{C_i} - R(r), \tag{10.6}$$

in which only two variable arguments are involved, namely r and p_r.

The quantities c, c_0, and c_1 (constants in the conservative problem) are connected by the equation (9.9), that is

$$c + c_0 + c_1 = K > 0. \tag{10.7}$$

If adiabatic influences arising from interior forces are noticeable, then the c's are not in general constant but assume the character of adiabatic parameters. On the contrary, K is rigorously constant and so the c's are still subject to the condition (10.7). For a stationary solution of the present problem we must minimize H as a function of p_r and r. This requires

$$\frac{\partial H}{\partial p_r} = \left(\frac{1}{m_0} + \frac{1}{m_1}\right) p_r = 0, \qquad \frac{\partial H}{\partial r} = 0.$$

The first equation gives $p_r = 0$, or $dr/dt = 0$ (as $dr/dt = \partial H/\partial p_r$), that is

$$r = \text{const.} = r_0, \tag{10.8}$$

which implies that the orbits are circular.

The radius $r = r_0$ is connected with the other parameters of the problem by the equation $\partial H/\partial r = 0$, or, in view of (10.6),

$$\frac{\partial H}{\partial r} = -\left(\frac{1}{m_0} + \frac{1}{m_1}\right)\frac{c^2}{r^3} - R'(r) = 0. \tag{10.9}$$

Using (10.1) this gives us

$$\left(\frac{1}{m_0} + \frac{1}{m_1}\right) c^2 = f m_0 m_1 r - r^3 \mathrm{X}'(r), \tag{10.10}$$

where the term $r^3\mathrm{X}'(r)$ is very small in comparison with the first term. From (10.9) and (10.1) we have

$$\frac{\partial^2 H}{\partial r^2} = \left(\frac{1}{m_0} + \frac{1}{m_1}\right)\frac{3c^2}{r^4} - 2\,\frac{f m_0 m_1}{r^3} - \mathrm{X}''(r),$$

and in particular, for stationary solutions γ_0 along which $(1/m_0 + 1/m_1)c^2$ has the value (10.10)

$$\frac{\partial^2 H}{\partial r^2} = \frac{f m_0 m_1}{r^3} - \frac{3}{r}\,\mathrm{X}'(r) - \mathrm{X}''(r).$$

Here again the terms in X are very small (at most of order D^2/r^2) so that undoubtedly $\partial^2 H/\partial r^2 > 0$. This inequality together with

$$\frac{\partial^2 H}{\partial p_r^2} = \frac{1}{m_0} + \frac{1}{m_1} > 0, \qquad \frac{\partial H}{\partial p_r \partial r} = 0$$

assures us that the couple $p_r = 0$, $r = r_0$ corresponds to a minimum value for H.

Squaring (10.5) and eliminating c^2 by means of (10.10), we get

$$n^2 = f\,\frac{m_0 + m_1}{r^3} - \left(\frac{1}{m_0} + \frac{1}{m_1}\right)\frac{1}{r}\,\mathrm{X}'(r). \tag{10.11}$$

This, if we neglect the term in X$'$, is the usual expression for n^2 in Keplerian motion.

We conclude that for the Hamiltonian (10.6) in which the c's are treated as constants the possible stationary solutions correspond to circular revolution of the centers of mass P_0, P_1 with angular velocity given by (10.5) and (10.11). Also each body rotates gyroscopically about an axis perpendicular to the orbital plane, with angular velocities (9.11), that is

$$\omega_i = \frac{c_i}{C_i} \qquad (i = 0, 1). \tag{10.12}$$

These angular velocities might of course be inferred from the reduced Hamiltonian (10.6) as derivatives of H with respect to the angular momenta c_i. In fact (10.12) is identical with

$$\omega_i = \frac{\partial H}{\partial c_i} \qquad (i = 0, 1). \tag{10.13}$$

The three angular velocities n, ω_0, ω_1 are connected by a relation in which also occur the material and mechanical characteristics of the system (masses, moments of inertia, angular momentum). This relation may be deduced by eliminating c, c_0, and c_1 from (10.5), (10.7) and (10.12).

The final step in selecting the possible final state is to minimize H as a function of the adiabatic parameters c, c_0, c_1 which are connected by (10.7). Using the expression (10.6) and a Lagrangian multiplier λ for (10.7), we must have,

$$\frac{\partial H}{\partial c} = \left(\frac{1}{m_0} + \frac{1}{m_1}\right)\frac{c}{r^2} = \lambda, \qquad \frac{\partial H}{\partial c_0} = \frac{c_0}{C_0} = \lambda, \qquad \frac{\partial H}{\partial c_1} = \frac{c_1}{C_1} = \lambda. \tag{10.14}$$

This shows in the first place that all three derivatives have the same sign as λ. Since the sum of c, c_0, and c_1 is positive, they and the above three derivatives must all be positive. Therefore c_0 and c_1 coincide with the absolute values K_0, K_1 of the angular momentum. In addition, by (10.4), (10.5) and (10.13), the equalities $\partial H/\partial c = \partial H/\partial c_0 = \partial H/\partial c_1$ give

$$n = \omega_0 = \omega_1$$

which expresses the double fact that *revolution and rotation take place uniformly in the same senses and with the same angular velocities*. In other words B_0 and B_1 behave as though they were rigidly connected.

11. *The problem of three or more bodies.*

Professor Krall, by a skilful application of the methods used in paragraph 7, has treated the case of a planetary system of any number, say $n+1$, bodies. The conclusion is still that the bodies all describe circular orbits about their common center of mass with equal angular velocities. Also each of them rotates about an axis perpendicular to the orbital plane and they all have the same period of rotation.

In Krall's paper the $n+1$ bodies are supposed to have kinetic energy arising from rotation comparable with the translatory energy. The problem is further simplified by the hypothesis that the finite size may be disregarded in evaluating the Newtonian potential. Its expression is therefore the same as in the case of particles.

I am certain that one would reach the same conclusions in the more general case but it seems desirable, since we are in possession of a very straightforward method to complete the problem in its utmost generality. It is also desirable to study the case, till now neglected, in which the resultant angular momentum $\boldsymbol{K}$ is zero. Then the final issue is probably catastrophic the tendency being toward a general collision of all bodies. This may be handled in the simple case of two bodies. A general attack however seems to require a previous regularization of the differential equations.

I may perhaps end with a general remark. It is true that only asymptotic aspects, without intermediary stages, may be caught in this way, but in my opinion, the rigor and simplicity in so high and delicate a matter make the determination of final states of paramount importance among the soluble problems of celestial mechanics.

A METHOD OF SOLVING THE LINEAR DIFFERENTIAL EQUATION WITH CONSTANT COEFFICIENTS

By L. J. PARADISO and R. H. CAMERON, Cornell University

The usual procedure given in standard text-books for solving the differential equa tion

$$a_n \frac{d^n y}{dx^n} + a_{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1 \frac{dy}{dx} + a_0 y = f(x) \tag{1}$$

(where the coefficients are constants) is to obtain the general solution of the associated homogeneous equation, and to combine this with a particular solution of the non-homogeneous equation (1) which may be obtained by any one of a number of methods. The method of undetermined coefficients given for the determination of the particular solution is applicable only to a certain class of functions $f(x)$—roughly those which have a finite number of linearly independent derivatives. For functions of other types the method of variation of parameters is often given. This method however has the disadvantage of being laborious for differential equations of the third or higher orders. Moreover, the theory of this method is involved if it is to be justified rigorously. Another method sometimes given is that of using an integrating factor. Again this method is applied to special differential equations and when complex quantities result the justification of the procedure leads to the theory of complex variables.

In this note the method of integrating factors is used to treat all of the cases usually given in the treatment of the differential equation (1). Complex quanti-

ties are avoided entirely and the whole treatment is made to depend on the solution of the linear differential equation of the first order.

Let us use the symbol D for the operation of differentiation. Then we may write the differential equation (1) symbolically as

$$F(D)y = (a_nD^n + a_{n-1}D^{n-1} + \cdots + a_1D + a_0)y = f(x). \tag{2}$$

It is easy to verify that when the coefficients a_i are constants this operator on y behaves as a polynomial in D and hence may be factored in the ordinary way. Furthermore the factors may be applied in any order.

THEOREM 1. *Let $D-A$ be a factor of $F(D)$, where A is a real constant. Let the other factor which is a polynomial in D of the $(n-1)$th degree be $F_1(D)$. Then e^{-Ax} is an integrating factor of equation* (2) *and its integral is given by*

$$F_1(D)y = e^{Ax}\int e^{-Ax}f(x)dx + Ce^{Ax}.$$

The above formula may be derived and the theorem proved by writing $U=F_1(D)y$ and solving the resulting first order linear differential equation $(D-A)U=f(x)$ by the ordinary method, using the integrating factor e^{-Ax}.

COROLLARY. *Let $(D-A)^r$ be an r-fold factor of $F(D)$, where A is a real constant. Let $F_r(D)$, a polynomial of degree $n-r$ in D, be the other factor. Then*

$$F_r(D)y = f_r(x) + (c_1 + c_2x + \cdots + c_rx^{r-1})e^{Ax}$$

where

$$f_r(x) = e^{Ax}\int^r \cdots \left[\int e^{-Ax}f(x)dx\right]dx \cdots dx.$$

This corollary follows immediately by applying r times successively the theorem above.

We notice that if $f(x)$ is zero then this corollary gives us the rule for obtaining for the homogeneous equation $F(D)y=0$ the r-parameter family of solutions which is associated with the r-fold root A of $F(D)=0$.

THEOREM 2. *If $(D-\alpha)^2+\beta^2$ is a factor of $F(D)$, where α and β are real constants, and if $F_2(D)$ is the other factor, then*

$$(D - \alpha - \beta\tan\beta x)[(D - \alpha + \beta\tan\beta x)\{F_2(D)y\}] = F(D)y. \tag{3}$$

This theorem can easily be verified by carrying out the indicated differentiation. It may be noted in passing that the verification of this formula is a good exercise for students. It illustrates very clearly that the operators do not obey the ordinary laws of algebra when the coefficients are not constants. We observe that equation (3) does not hold if we interchange the order of the factors as given.

THEOREM 3. *If* $(D-\alpha)^2+\beta^2$ *is a factor of* $F(D)$, *where* α *and* β *are real constants, and if* $F_2(D)$ *is the other factor, then* $e^{-\alpha x}\cos\beta x$ *is an integrating factor of* $F(D)y=f(x)$ *and its integral is*

$$(D-\alpha+\beta\tan\beta x)F_2(D)y = e^{\alpha x}\sec\beta x\int e^{-\alpha x}\cos\beta x\, f(x)dx + c_1e^{\alpha x}\sec\beta x.$$

PROOF: Let

$$(D-\alpha+\beta\tan\beta x)F_2(D)y = V.$$

Then the differential equation (1) may be written using Theorem 2

$$F(D)y = (D-\alpha-\beta\tan\beta x)V = f(x).$$

This is a linear equation of the first order in V and x. Its integrating factor is

$$e^{\int(-\alpha-\beta\tan\beta x)dx} = e^{-\alpha x}\cos\beta x.$$

From this the theorem follows at once. We notice that the resulting differential equation is of order $n-1$.

THEOREM 4. *If* $(D-\alpha)^2+\beta^2$ *is a factor of* $F(D)$ *and* α *and* β *are real constants and if* $F_2(D)$ *is the other factor then*

$$F_2(D)y = e^{\alpha x}(c_1\cos\beta x + c_2\sin\beta x) + f_2(x), \tag{4}$$

where

$$f_2(x) = e^{\alpha x}\cos\beta x\int\sec^2\beta x\left\{\int e^{-\alpha x}\cos\beta x\, f(x)dx\right\}dx.$$

If we let $F_2(D)y=U$, then from Theorem 3 we have

$$(D-\alpha+\tan\beta)U = e^{\alpha x}\sec\beta x\int e^{-\alpha x}\cos\beta x\, f(x)dx + c_1e^{\alpha x}\sec\beta x.$$

This is a linear equation of the first order in U and x and its integrating factor is

$$e^{\int(-\alpha+\beta\tan\beta x)dx} = e^{-\alpha x}\sec\beta x.$$

Applying this integrating factor we get the theorem at once. The resulting differential equation (4) is now of order $n-2$.

By using Theorems 3 and 4 we can remove all of the quadratic factors of $F(D)$ and each time by two integrations reduce the order of the differential equation by two. If the quadratic factor $(D-\alpha)^2+\beta^2$ occurs r times we can reduce the order of the differential equation and remove this factor by repeated

applications of the integrating factors $e^{-\alpha x}\cos\beta x$ and $e^{-\alpha x}\sec\beta x$ in this order successively r times.

We thus find the

COROLLARY. *If $f(x)=0$ and $F(D)$ has the quadratic factor $[(D-\alpha)^2+\beta^2]^r$, where α and β are real constants, and if $F_{2r}(D)$ is the other factor, then*

$$F_{2r}(D)y = e^{\alpha x}\{(c_1 + c_2x + \cdots + c_rx^{r-1})\cos\beta x + (k_1 + k_2x + \cdots K_rx^{r-1})\sin\beta x\}.$$

By applying these theorems and corollaries we thus obtain all of the rules which are frequently given for solving the differential equations of type (1) treated in elementary courses. It is not intended that the formulas given in these theorems should be applied directly, but that in each case the procedure of using the integrating factor is to be applied. The proofs of the rules are simple; complex numbers have been avoided. The function $f(x)$ may or may not have a finite number of linearly independent derivatives; the method applies in either case.

We illustrate the use of these theorems by the following example.
Example: Find the general solution of

$$(D - 1)(D^2 - 2D + 2)y = e^x\sec^2 x.$$

An integrating factor is e^{-x}. Applying this we obtain

$$e^{-x}(D - 1)(D^2 - 2D + 2)y = \sec^2 x.$$

Integrating we have

$$e^{-x}(D^2 - 2D + 2)y = \tan x + c_1.$$

Hence

$$(D^2 - 2D + 2)y = [(D - 1)^2 + 1]y = e^x(\tan x + c_1).$$

An integrating factor of this is $e^{-x}\cos x$. Applying this factor, we have

$$e^{-x}\cos x[(D - 1)^2 + 1]y = e^{-x}\cos x[D - 1 - \tan x][D - 1 + \tan x]y = \sin x + c_1\cos x.$$

Integrating, we obtain

$$e^{-x}\cos x(D - 1 + \tan x)y = -\cos x + c_1\sin x + c_2.$$

Applying the integrating factor $e^{-x}\sec x$, we obtain

$$e^{-x}\sec x\,(D - 1 + \tan x)y = -\sec x + c_1\sec x\tan x + c_2\sec^2 x;$$

or, integrating, we have

$$e^{-x}\sec x\cdot y = -\log(\sec x + \tan x) + c_1\sec x + c_2\tan x + c_3.$$

Hence

$$y = -e^x\cos x\log(\sec x + \tan x) + e^x(c_1 + c_2\sin x + c_3\cos x).$$

SOME INTERPOLATION SERIES[1]

By J. L. WALSH, Harvard University

1. *Introduction: polynomial series of interpolation.* A series of the form

$$f(z) = a_0 + a_1(z - \beta_1) + a_2(z - \beta_1)(z - \beta_2) + \cdots \tag{1}$$

is called an *interpolation series* or *series of interpolation.* It is to be noted that if the points β_k are all distinct and if a function $f(z)$ is defined in each β_k, then a *formal* expansion of $f(z)$ of type (1) exists—this in the sense that the coefficients a_n can be determined in terms of the β_k and the values $f(\beta_k)$. We need merely set $z=\beta_1$: $a_0=f(\beta_1)$; $z=\beta_2$: $a_1=[f(\beta_2)-f(\beta_1)]/(\beta_2-\beta_1)$; $\cdots$. Each a_n can be expressed as a function[2] of $\beta_1, \beta_2, \cdots, \beta_{n+1}, f(\beta_1), f(\beta_2), \cdots, f(\beta_{n+1})$.

Even if the points β_k are not all distinct, the formal expansion exists provided the function $f(z)$ is analytic in each of the points β_k. Let us proceed to prove this statement by induction. We always have $a_0=f(\beta_1)$. Suppose now that $a_0, a_1, \cdots, a_{n-1}$ have been determined and that precisely m of the points $\beta_1, \beta_2, \cdots, \beta_n$ are equal to β_{n+1}. Then the m-th derivative of the factor of a_n:

$$(z - \beta_1)(z - \beta_2) \cdots (z - \beta_n)$$

is different from zero at the point $z=\beta_{n+1}$, whereas the m-th derivatives of the factors of $a_{n+1}, a_{n+2}, \cdots$ all vanish at this point $z=\beta_{n+1}$. Thus the coefficient a_n can be computed in terms of the coefficients $a_0, a_1, \cdots, a_{n-1}$, hence in terms of the β_k and the values and derivatives of $f(z)$ at the points β_k. In particular we may have all the β_k equal to β; then (1) reduces to the Taylor development of $f(z)$ about the point $z=\beta$.

An expansion (1) valid in the distinct points β_k must be the unique formal expansion of $f(z)$. An expansion (1) valid together with the corresponding derived equations in the points β_k not necessarily distinct, must also be this unique formal expansion of $f(z)$.

2. *Polynomial sequences of interpolation.* Series (1) is closely related to a more general expansion problem: to expand a given function $f(z)$ analytic in a region in a sequence of polynomials found by interpolation to that function at points of that region. A polynomial[3] $p_n(z)$ of degree n is uniquely determined by the requirement of coinciding with $f(z)$ (assumed analytic at the points considered) in $n+1$ distinct points $z_1, z_2, \cdots, z_{n+1}$. Indeed, the determination of $p_n(z)$ depends on the solution of $n+1$ linear equations for the $n+1$ coefficients a_μ in the polynomial $p_n(z)$:

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[1] Address delivered by invitation before the Association at Atlantic City, December 28, 1932.

[2] The actual formulas for the a_k are well known. For these formulas, for further details on particular series and sequences of interpolation in the complex domain, and for further references to the literature, the reader may consult Nörlund, *Differenzenrechnung*, Berlin 1924, Chapter 8.

[3] Any expression of the form $a_0z^n+a_1z^{n-1}+\cdots+a_n$ is called a *polynomial of degree n.* We do not assume $a_0\neq 0$.

$$\begin{aligned} a_0 z_1^n + a_1 z_1^{n-1} + \cdots + a_n &= f(z_1), \\ a_0 z_2^n + a_1 z_2^{n-1} + \cdots + a_n &= f(z_2), \\ \cdots\cdots\cdots&\cdots\cdots, \\ a_0 z_{n+1}^n + a_1 z_{n+1}^{n-1} + \cdots + a_n &= f(z_{n+1}). \end{aligned}$$

The vanishing of the determinant Δ (Vandermonde determinant) of this system is a necessary and sufficient condition for the existence of a polynomial of degree n not identically zero which vanishes in the points $z_1, z_2, \cdots, z_{n+1}$. Such a polynomial is known not to exist. Hence Δ is different from zero and $p_n(z)$ is uniquely determined.

If the points z_k are not all distinct, we introduce the convention that at a point z_k of multiplicity $\nu+1$, the equations

$$p_n(z_k) = f(z_k), \; p_n'(z_k) = f'(z_k), \cdots, p_n^{(\nu)}(z_k) = f^{(\nu)}(z_k)$$

are all required to hold. Under this convention, it is still true and can be proved by an immediate extension of the reasoning just given, that the polynomial $p_n(z)$ of degree n is uniquely determined by the requirement of coinciding with $f(z)$ in the $n+1$ points $z_1, z_2, \cdots, z_{n+1}$, not necessarily all distinct.

Suppose now that $f(z)$ is analytic in each of the points of the array

$$\begin{array}{l} z_1^{(0)}, \\ z_1^{(1)}, z_2^{(1)}, \\ z_1^{(2)}, z_2^{(2)}, z_3^{(2)}, \\ \cdots\cdots\cdots, \\ z_1^{(n)}, z_2^{(n)}, \cdots, z_{n+1}^{(n)}, \\ \cdots\cdots\cdots. \end{array} \tag{2}$$

There exists a unique polynomial $p_n(z)$ of degree n which coincides with $f(z)$ in the $n+1$ points $z_1^{(n)}, z_2^{(n)}, \cdots, z_{n+1}^{(n)}$. It is of interest to study the convergence of this sequence $p_n(z)$.

If the points (2) are chosen interior to a region R within which $f(z)$ is analytic, it does not necessarily follow that we have

$$\lim_{n\to\infty} p_n(z) = f(z)$$

for every z interior to R. For instance, we may choose $z_k^{(n)}=0$; the polynomial $p_n(z)$ is the sum of the first $n+1$ terms of the Maclaurin development of $f(z)$. If we set $f(z)=1/(1-z)$, then $f(z)$ is analytic at every point of the plane except $z=1$, yet the sequence $p_n(z)$ converges only for $|z|<1$.

Another example is due to Méray.[4] Choose $f(z)=1/z$, and choose the set

[4] Annales de l'Ecole Normale (3) vol. 1 (1884), pp. 165–176.

$z_1^{(n)}, z_2^{(n)}, \cdots, z_{n+1}^{(n)}$ as the $(n+1)$st roots of unity.[5] We verify directly the equation $p_n(z) \equiv z^n$. The sequence $p_n(z)$ converges to the value zero for $|z| < 1$, actually diverges for $|z| > 1$, and converges to the value $f(z)$ only in the point $z = 1$, a point of interpolation for all the polynomials $p_n(z)$.

Whenever the point $z_k^{(n)} = z_k$ of the array (2) is independent of n (for $n \geqq k-1$), the polynomial $p_n(z)$ defined by interpolation to $f(z)$ in the points $z_1, z_2, \cdots, z_{n+1}$ is the sum of the first $n+1$ terms of a series of form (1). Indeed, the polynomials $p_n(z)$ and $p_{n-1}(z)$ both coincide with $f(z)$ in the points $z_1, z_2, \cdots, z_n$, so the difference $p_n(z) - p_{n-1}(z)$ is a polynomial of degree n which vanishes in those points, hence a constant multiple of $(z-z_1)(z-z_2)\cdots(z-z_n)$. This reasoning is valid whether the points $z_1, z_2, \cdots, z_n$ are distinct or not.

3. *Interpolation series of rational functions.* Much of our discussion of section 1 is clearly applicable to interpolation series of the form

$$a_0\phi_0(z) + a_1(z-\beta_1)\phi_1(z) + a_2(z-\beta_1)(z-\beta_2)\phi_2(z) + \cdots, \quad \phi_k(\beta_{k+1}) \neq 0.$$

We shall discuss in some detail the particular case

$$f(z) = a_0 + a_1\frac{z-\beta_1}{z-\alpha_1} + a_2\frac{(z-\beta_1)(z-\beta_2)}{(z-\alpha_1)(z-\alpha_2)} + \cdots, \quad \alpha_i \neq \beta_j. \tag{3}$$

The points α_k or β_k are not necessarily all distinct. The values α_k or $\beta_k = \infty$ are also not excluded; we consider the factor of a_n in (3) as a rational function of z with the zeros $\beta_1, \beta_2, \cdots, \beta_n$ and the poles $\alpha_1, \alpha_2, \cdots, \alpha_n$, whether these values are finite or infinite. Thus, if all the α_k are infinite, series (3) takes precisely the form (1).

It is clear from the discussion of section 1 that an arbitrary function $f(z)$ analytic in the points β_k (distinct or not) has a formal development (3) found by interpolation to $f(z)$ in the points β_k.

We remark that any rational function $r(z)$ of the form

$$r(z) = \frac{b_0z^n + b_1z^{n-1} + \cdots + b_n}{(z-\alpha_1)(z-\alpha_2)\cdots(z-\alpha_n)} \tag{4}$$

can be expressed (the β_k being given) in a unique way as

$$a_0 + a_1\frac{z-\beta_1}{z-\alpha_1} + \cdots + a_n\frac{(z-\beta_1)\cdots(z-\beta_n)}{(z-\alpha_1)\cdots(z-\alpha_n)}, \quad \beta_i \neq \alpha_j. \tag{5}$$

The formal expression for $r(z)$ is found by substituting successively $z = \beta_1, \beta_2, \cdots, \beta_n, \beta_{n+1}$, (where $\beta_{n+1} \neq \alpha_k$ is arbitrary) and solving for the a_k, which are then uniquely determined. This procedure is to be modified as usual for multiple points β_k. The difference between $r(z)$ and its formal expression (5) is a rational function whose numerator is a polynomial of degree n, whose denomi-

[5] Interpolation in this set of points has been studied in some detail. See for instance Runge, *Theorie und Praxis der Reihen*, Leipzig, 1904, pp. 126–142; Fejér, Göttingen Nachrichten, 1918, pp. 319–331; Walsh, Trans. Amer. Math. Soc., vol. 34 (1932), pp. 22–74.

nator is the denominator of $r(z)$, and which vanishes in all the points $\beta_1, \beta_2, \cdots, \beta_{n+1}$ and hence vanishes identically.

If $f(z)$ is analytic on and within the contour C, which contains the points $\beta_1, \beta_2, \cdots, \beta_{n+1}$ in its interior, then the sum $S_n(z)$ of the first $n+1$ terms of the formal expansion (3) of $f(z)$ is given by

$$(6)\quad f(z) - S_n(z) = \frac{1}{2\pi i}\int_C \frac{(z-\beta_1)\cdots(z-\beta_{n+1})(t-\alpha_1)\cdots(t-\alpha_n)f(t)dt}{(t-\beta_1)\cdots(t-\beta_{n+1})(z-\alpha_1)\cdots(z-\alpha_n)(t-z)},$$

z interior to C,

as we shall verify. Equation (6) is, by virtue of Cauchy's integral formula taken over C for the function $f(z)$, equivalent to

$$(7)\quad S_n(z) = \frac{1}{2\pi i}\int_C \left[1 - \frac{(z-\beta_1)\cdots(z-\beta_{n+1})(t-\alpha_1)\cdots(t-\alpha_n)}{(t-\beta_1)\cdots(t-\beta_{n+1})(z-\alpha_1)\cdots(z-\alpha_n)}\right]\frac{f(t)dt}{t-z},$$

z interior to C.

When the quantities in the square bracket are added by using the common denominator, the numerator vanishes identically in t and z for $t=z$, and hence that numerator is divisible by $t-z$. Thus, $S_n(z)$ as defined by (7) is a rational function of form (4) which by (6) coincides with $f(z)$ in the points (distinct or not) $\beta_1, \beta_2, \cdots, \beta_{n+1}$. These conditions determine $S_n(z)$ uniquely.

We note that (6) and (7) are valid at all points interior to C at which $S_n(z)$ is defined, even if points α_k lie interior to C.

4. *A convergence theorem*. By means of (6) we shall proceed to establish

THEOREM I.[6] *If $f(z)$ is analytic for $|z|<T$, if the points β_k are in modulus not greater than $B<T$, and if the points α_k are in modulus not less than A, then the formal expansion* (3) *of $f(z)$ converges to $f(z)$ for $|z|<(AT-BT-2AB)/(2T+A-B)$, uniformly for $|z|\leqq Z<(AT-BT-2AB)/(2T+A-B)$.*

If C is chosen as the circle $|t|=T'$, $B<T'<T$, and if $|z|=Z<T'$, $Z<A$, then we have

$$\left|\frac{z-\beta_k}{t-\beta_k}\right| \leqq \frac{Z+B}{T'-B}, \qquad \left|\frac{\alpha_k - t}{\alpha_k - z}\right| \leqq \frac{|\alpha_k|+T'}{|\alpha_k|-Z} \leqq \frac{A+T'}{A-Z}.$$

If M is suitably chosen we now have by (6)

$$|f(z)-S_n(z)| \leqq M\frac{(Z+B)^n(A+T')^n}{(T'-B)^n(A-Z)^n},$$

which approaches zero provided we have

$$\frac{(Z+B)(A+T')}{(T'-B)(A-Z)} < 1, \text{ or } Z < \frac{AT'-BT'-2AB}{2T'+A-B},$$

[6] Walsh, loc. cit.

when this last numerator is positive. This last inequality implies automatically $Z<T'$, $Z<A$, so Theorem I follows if we allow T' to approach T.

In particular we may set $A=\infty$; here the limit on Z reduces to $T-2B$, and the corresponding theorem is due to Méray (loc. cit.). We may also set $A=\infty$, $B=0$; the limit on Z is T, and the theorem is the usual one on the validity of the Maclaurin development of an analytic function.

The limit obtained for Z in Theorem I is natural and under the circumstances cannot be replaced by a larger limit. Indeed, let us choose $\beta_k=B$, $\alpha_k=-A$, $f(z)=1/(T-z)$. The reader may verify directly the formula

$$f(z) - S_n(z) = \frac{(z-B)^{n+1}(T+A)^n}{(T-B)^{n+1}(z+A)^n(T-z)}.$$

For the value $z=-Z=-(AT-TB-2AB)/(2T+A-B)$, this right hand member has the value $(-1)^{n+1}(Z+B)/[(T-B)\,(T-Z)]$ and hence fails to approach zero as n becomes infinite.

5. *Expansion of meromorphic functions.* Series of type (3) have another important property. Equation (3), where $f(z)$ is analytic in the points β_k, can also be written in the form

$$\begin{aligned} &\frac{(z-\alpha_1)(z-\alpha_2)\cdots(z-\alpha_n)}{(z-\beta_1)(z-\beta_2)\cdots(z-\beta_n)}\Bigg[f(z)-a_0-a_1\frac{z-\beta_1}{z-\alpha_1} \\ &\qquad -\cdots-a_{n-1}\frac{(z-\beta_1)\cdots(z-\beta_{n-1})}{(z-\alpha_1)\cdots(z-\alpha_{n-1})}\Bigg] \\ &\qquad = a_n+a_{n+1}\frac{z-\beta_{n+1}}{z-\alpha_{n+1}}+a_{n+2}\frac{(z-\beta_{n+1})(z-\beta_{n+2})}{(z-\alpha_{n+1})(z-\alpha_{n+2})}+\cdots \end{aligned} \tag{8}$$

for $n=1, 2, \cdots$. Each of these equations is again an equation of precisely type (3). Moreover, the method of computing the coefficients a_k as described in sections 1 and 3 yields the same results whether applied to (3) or (8), whether the β_k are distinct or not, as the reader can easily verify. The left-hand members of (8) are analytic in the points α_k and β_k (when suitably defined there) if the a_k are computed as described, and if $f(z)$ is analytic in the points α_k.

The advantage of equations (8) over equation (3) is twofold: (i) we can take out of the right-hand member of (3) any finite number of the points α_k and β_k, so it is clear that the validity of equation (8) or (3) depends on that validity when the points α_k and β_k are near their limit points. Under certain conditions the function $f(z)$ need not be analytic in *all* of the points β_k and may even be defined arbitrarily at certain of those points. (ii) If $f(z)$ has a pole of the first order in the point α_1, the left-hand member of equation (8) for $n=1$ is analytic at the point $z=\alpha_1$ (provided the function is suitably defined for $z=\alpha_1$); more generally, if $f(z)$ has a pole of any order for $z=\alpha$ and if α occurs among the α_k at least a number of times corresponding to its multiplicity, the function in the

left-hand member of (8) has at most a removable singularity at $z=\alpha$, provided n is sufficiently large. It is thus possible to study the expansion (3) of a *meromorphic* function $f(z)$ by studying the expansion of a suitably chosen *analytic* function, provided that all the poles of $f(z)$ (in a certain region) occur among the points α_k a number of times according to their multiplicities.

By applying these remarks to Theorem I, we shall prove

THEOREM II.[7] *If $f(z)$ is meromorphic for $|z|<T$, if the poles of $f(z)$ of modulus less than T belong to the sequence α_k which has no limit point of modulus less than A, and where each pole occurs in the sequence at least a number of times according to its multiplicity, and if finally the points β_k are distinct from the α_n and have no limit point of modulus greater than $B<T$, then the formal expansion* (3) *found by interpolation to $f(z)$ in the points β_k is valid for $|z|<(AT-BT-2AB)/(2T+A-B)$, $z\neq\alpha_k$. Convergence is uniform for z on any closed set interior to the circle $|z|=(AT-BT-2AB)/(2T+A-B)$ and containing no point α_k.*

Proof of Theorem II from Theorem I involves the choice of auxiliary numbers $A'<A$, $B'>B$, $T'<T$, $B'<T'$, choosing A', B', T' as the numbers which appear in the hypothesis of Theorem I, and applying Theorem I to (8) where n is so chosen that we have $|\beta_k|<B'$, $|\alpha_k|>A'$, $|\alpha_k|>T'$ for $k>n$. Theorem II is thus established with A, B, T replaced by A', B', T'; but A', B', T' can be allowed to approach A, B, T respectively and this yields Theorem II as stated.

The special case where A and T are infinite is of particular interest:

THEOREM III. *If $f(z)$ is a meromorphic function of z (that is, analytic except possibly for poles, at every finite point of the plane), if all the poles of $f(z)$ belong to the sequence $\alpha_1, \alpha_2, \alpha_3, \cdots \to\infty$, where every pole occurs at least a number of times corresponding to its multiplicity, and if the numbers β_k are distinct from the α_n and uniformly limited, then the formal expansion* (3) *found by interpolation in the points β_k is valid for all values of z other than the α_k. The series* (1) *converges uniformly in any closed limited region containing no point α_k.*

In particular if $f(z)$ is analytic at every finite point of the plane, and if we set $\alpha_k=\infty$, $\beta_k=\beta$, then (3) *is the Taylor expansion of $f(z)$ about the point $z=\beta$.*

The case of Theorem II where we set $\alpha_k=\infty$, and where β_k approaches zero yields a special case of (1) which is a generalization of Taylor's series, due to Bendixson:

If $f(z)$ is analytic for $|z|<T$, and if we have $\lim_{k\to\infty}\beta_k=0$, then the formal expansion (1) *of $f(z)$ converges to $f(z)$ for $|z|<T$, uniformly for $|z|\leqq Z<T$.*

6. *Invariant character of expansion.* Series (3) possesses a remarkable invariant property under linear transformation of the complex variable. Under such transformation, an analytic or meromorphic function remains analytic or meromorphic. Under such transformation, interpolation in n distinct or coincident points corresponds to interpolation in the n corresponding distinct or

[7] Walsh, Proc. Nat'l. Acad. Sc., vol. 18 (1932), pp. 165–171.

coincident points. The sum of the first $n+1$ terms of (3) (when the a_k are now considered arbitrary) is an arbitrary rational function of form (4), and such a function is transformed by a linear transformation into a function of the same type; the new rational function has poles (formally) in the n points which are the transforms of the points $\alpha_1, \alpha_2, \cdots, \alpha_n$.

The invariant property of the series (3) should correspond to certain convergence tests expressible in invariant form. This is indeed the case, as one can see by inspection of (6). A typical factor in the integrand can be written as

$$\frac{(z - \beta_{k+1})(t - \alpha_k)}{(t - \beta_{k+1})(z - \alpha_k)},$$

which is precisely the cross-ratio $(z, \beta_{k+1}, t, \alpha_k)$:

$$(z_1, z_2, z_3, z_4) = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_2 - z_3)(z_4 - z_1)}.$$

We shall now prove the following new theorem, a generalization of Theorem III expressed in invariant form:

THEOREM IV. *Let R_1, R_2, R_3 be arbitrary closed regions of the z-plane. Let R denote the closed point set which is the locus of the point t subject to the restriction*

$$|(z, \beta, t, \alpha)| \geqq 1 \tag{9}$$

when the independent variables z, α, β have the regions R_1, R_2, R_3 as their respective loci.

If the function $f(z)$ is analytic at every point of R, if the points α_k have no limit point exterior to R_2 and the points β_k have no limit point exterior to R_3, then the formal development (3) *of $f(z)$ converges to $f(z)$ uniformly for z in any closed subregion of R_1 containing no point α_k.*

If the function $f(z)$ is meromorphic at every point of R, if all the poles of $f(z)$ in R belong to the sequence α_k (with proper allowance for the multiplicities of those poles), and if the points α_k have no limit point exterior to R_2 and the points β_k no limit point exterior to R_3, then the formal development (3) *of $f(z)$ converges to $f(z)$ uniformly for z on any closed set in R_1 containing no point α_k.*

The case that R is the entire plane is trivial, for $f(z)$ must then be a constant, and the expansion (3) reduces to a single term. This case is henceforth excluded.

The locus R contains R_1 and R_3, for $t=z$ implies $(z, \beta, t, \alpha)=1$, and $t=\beta\neq\alpha$, $\beta\neq z$ implies $(z, \beta, t, \alpha)=\infty$.

The locus R (interpreted on the sphere instead of the complex plane) depends continuously on R_1, R_2, R_3, in the sense that a slight enlargement of R_1, R_2, R_3 causes only a slight enlargement of R, for (9) is a continuous relation, uniformly continuous in suitably chosen closed regions containing the regions R_1, R_2, R_3 respectively in their interiors. The locus R is a closed region (an open connected set plus its boundary points), for that is true of each of the loci

R_1, R_2, R_3. The function $f(z)$ is meromorphic in the closed region R, hence meromorphic in some closed region S containing R in its interior but containing no pole of $f(z)$ not a point α_k. Denote the boundary of S by C, which may be chosen as one or more rectifiable Jordan curves. There exist closed regions R_1', R_2', R_3' containing R_1, R_2, R_3 respectively in their interiors, such that the locus R' (defined by (9) when z, α, β have as loci R_1', R_2', R_3') lies interior to S. Then for z in R_1', α in R_2', β in R', t on C, we have uniformly for suitably chosen p,

$$|(z, \beta, t, \alpha)| \leqq p < 1;$$

the uniformity follows from the closure of the point sets involved.

In the proof of Theorem IV it is possible, by a linear transformation, to take S as a finite region. For suitably large index, the points α_k lie in R_2', the points β_k in R_3', and the left-hand member of (8) is analytic in the closed region S. Theorem IV now follows by the method of sections 4 and 5.

Theorem III is essentially the special case of Theorem IV corresponding to regions R_1, R_2, R_3 (hence also R) bounded by circles having as common center the origin. The effective determination of R when R_1, R_2, R_3 are given arbitrary regions is an interesting geometric problem which has been solved in certain cases.[8]

7. *Orthogonality property of series* (3). If C is an arbitrary rectifiable curve of the z-plane, the functions $\phi_j(z)$ and $\phi_k(z)$ are said to be *orthogonal on C* provided we have

$$\int_C \phi_j(z)\overline{\phi_k(z)}\, ds = 0, \qquad j \neq k.$$

As in the real domain, there is a close connection between orthogonality, and approximation in the sense of least squares. Let us prove

If the functions $f(z), \phi_1(z), \phi_2(z), \cdots, \phi_n(z)$ are continuous on C, if the functions $\phi_k(z)$ are orthogonal on C and if no function $\phi_k(z)$ is identically zero on C, then the linear combination

$$\lambda_1\phi_1(z) + \lambda_2\phi_2(z) + \cdots + \lambda_n\phi_n(z)$$

which minimizes the integral

$$\int_C \left| f(z) - [\lambda_1\phi_1(z) + \lambda_2\phi_2(z) + \cdots + \lambda_n\phi_n(z)]\right|^2 ds$$

is given by $\lambda_k = a_k$, where

$$a_k \int_C |\phi_k|^2 ds = \int_C \bar{f}\phi_k ds. \tag{10}$$

We have

$$\int_C [f - \sum \lambda_k\phi_k][\bar{f} - \sum \bar{\lambda}_k\bar{\phi}_k] ds = \int_C f\bar{f} ds - \sum \lambda_k \int_C \bar{f}\phi_k ds$$

[8] Cf. Walsh, loc. cit.

$$-\sum \bar{\lambda}_k \int_C f\bar{\phi}_k ds + \sum \lambda_k \bar{\lambda}_k \int_C \phi_k \bar{\phi}_k ds$$

$$= \int_C f\bar{f} ds - \sum a_k \bar{a}_k \int_C \phi_k \bar{\phi}_k ds + \sum (a_k - \lambda_k)(\bar{a}_k - \bar{\lambda}_k) \int_C \phi_k \bar{\phi}_k ds.$$

This last expression is clearly a minimum, considered as a function of the λ_k, when and only when we have $\lambda_k = a_k$.

If an infinite number of orthogonal functions $\phi_k(z)$ are given on C, the formal expansion of $f(z)$ on C in terms of the $\phi_k(z)$:

$$f(z) = a_1\phi_1(z) + a_2\phi_2(z) + \cdots, \tag{11}$$

where the a_k are given by (10), is then such that the sum of the first n terms of (11) is the linear combination of the functions $\phi_1, \phi_2, \cdots, \phi_n$ of best approximation to $f(z)$ on C in the sense of least squares.

The Maclaurin development of an arbitrary function analytic at the origin is precisely of form (11), in terms of the functions $1, z, z^2, \cdots$, where the a_k are given by (10) and where C is an arbitrary circle whose center is the origin and which has on or within it no singularity of $f(z)$. Indeed, on $C: |z| = R$, the orthogonal property is

$$\int_C z^j \bar{z}^k ds = \int_C z^j \frac{R^{2k}}{z^k} \frac{dz}{iz} = \frac{R^{2k}}{i} \int_C \frac{dz}{z^{k-j+1}} = 0, \quad j \neq k.$$

The formal development of $f(z)$ is

$$a_0 + a_1 z + a_2 z^2 + \cdots, a_k = \frac{1}{2\pi R^{2k}} \int_C f(z)\bar{z}^k ds = \frac{1}{2\pi i} \int_C \frac{f(z)}{z^{k+1}} dz. \tag{12}$$

If the points α_k lie exterior to $C: |z| = 1$, and if we choose the β_k as the sequence $0, 1/\bar{\alpha}_1, 1/\bar{\alpha}_2, 1/\bar{\alpha}_3, \cdots$, it is true that *the various terms of* (3) *are orthogonal on* C; the Maclaurin development is the special case $\alpha_k = \infty$. There are two formal expansions (3) of an arbitrary function $f(z)$ analytic on and within C, the one found by interpolation in these points β_k, the other found on C by (10) and (11). *These two formal expansions are identical, and are valid under suitable restrictions on* $f(z)$. For proofs of these statements and for further properties of series (3), the reader may refer to more detailed expositions to be found in the literature.[9]

[9] The references already given are useful in connection with sections 1–6, and the following references in connection with section 7: Malmquist, Comptes rendus du sixième congrès (1925) des mathématiciens scandinaves, Copenhagen 1926, pp. 253–259. Takenaka, Japanese Journal of Mathematics, vol. 2 (1926), pp. 129–145. (The formal work given by Takenaka is of much interest and in the main reliable, but the details of the reasoning are frequently inadequate; compare for instance, the application of "a theorem of Weierstrass," page 137.) Walsh, Trans. Amer. Math. Soc., vol. 34 (1932), pp. 523–556.

A PROOF OF WEIERSTRASS'S THEOREM[1]

By DUNHAM JACKSON, University of Minnesota

Weierstrass's theorem with regard to polynomial approximation can be stated as follows:

If $f(x)$ is a given continuous function for $a \leqq x \leqq b$, and if ϵ is an arbitrary positive quantity, it is possible to construct an approximating polynomial $P(x)$ such that

$$|f(x) - P(x)| < \epsilon$$

for $a \leqq x \leqq b$.

This theorem has been proved in a great variety of different ways. No particular proof can be designated once for all as the simplest, because simplicity depends in part on the preparation of the reader to whom the proof is addressed. A demonstration which follows directly from known facts about power series or Fourier series, for example, is not so immediate if a derivation of those facts has to be gone through first. A proof commonly regarded as among the simplest and neatest is the one due to Landau[2] in which an approximating polynomial is given explicitly by means of a certain type of "singular" integral. The purpose of this note is to present a modification or modified formulation of Landau's proof which is believed to possess further advantages of simplicity, at least from some points of view.

Let $f(x)$ be a given continuous function for $a \leqq x \leqq b$. Without essential loss of generality it can be supposed that $0 < a < b < 1$, since any finite interval whatever can be carried over into an interval contained in $(0, 1)$ by a linear transformation, under which any continuous function will go into a continuous function and any polynomial into a polynomial of the same degree. For convenience in the writing of the formulas which enter into the proof, let the function $f(x)$, supposed given in the first instance only for $a \leqq x \leqq b$, be defined outside the interval (a, b) as follows:

$$\begin{aligned} f(x) &= 0 && \text{for} \quad x \leqq 0, \\ f(x) &= \frac{x}{a} f(a) && \text{for } 0 < x < a, \\ f(x) &= \frac{1-x}{1-b} f(b) && \text{for } b < x < 1, \\ f(x) &= 0 && \text{for} \quad x \geqq 1. \end{aligned}$$

Then $f(x)$ is defined and continuous for all values of x. The question at issue is that of approximating it by means of a polynomial for values of x belonging to

[1] Presented to the Minnesota Section of the Association, May 13, 1933.

[2] E. Landau, *Über die Approximation einer stetigen Funktion durch eine ganze rationale Funktion*, Rendiconti del Circolo Matematico di Palermo, vol. 25 (1908), pp. 337–345; see also Courant-Hilbert, *Methoden der Mathematischen Physik*, vol. I, second edition, Berlin, 1931, pp 55–57.

the interval (a, b).

Let J_n denote the constant

$$J_n = \int_{-1}^{1} (1 - u^2)^n du,$$

and let

$$P_n(x) = \frac{1}{J_n} \int_0^1 f(t)[1 - (t - x)^2]^n dt.$$

The integrand in the latter integral is a polynomial of degree $2n$ in x with coefficients which are continuous functions of t, and the integral is for each value of n a polynomial in x of degree $2n$ (at most) with constant coefficients.

If $0 \leqq x \leqq 1$, the value of the integral is not changed if the limits are replaced by $-1+x$ and $1+x$, since $f(t)$ vanishes everywhere outside the interval $(0, 1)$, and vanishes in particular from $-1+x$ to 0 and from 1 to $1+x$:

$$P_n(x) = \frac{1}{J_n} \int_{-1+x}^{1+x} f(t)[1 - (t - x)^2]^n dt.$$

By the substitution $t - x = u$ this becomes

$$P_n(x) = \frac{1}{J_n} \int_{-1}^{1} f(x + u)(1 - u^2)^n du. \tag{1}$$

If the equation

$$1 = \frac{1}{J_n} \int_{-1}^{1} (1 - u^2)^n du$$

is multiplied by $f(x)$, this factor, being independent of u, may be written under the integral sign:

$$f(x) = \frac{1}{J_n} \int_{-1}^{1} f(x)(1 - u^2)^n du. \tag{2}$$

Hence, by subtraction of (2) from (1),

$$P_n(x) - f(x) = \frac{1}{J_n} \int_{-1}^{1} [f(x + u) - f(x)](1 - u^2)^n du. \tag{3}$$

The problem now is to show that the value of this expression approaches zero as n becomes infinite.

Let ϵ be any positive quantity. Since $f(x)$ is (uniformly) continuous there is a $\delta > 0$ (independent of x) such that $|f(x+u) - f(x)| \leqq \epsilon/2$ for $|u| \leqq \delta$. Let M be the maximum of $|f(x)|$. Then $|f(x+u) - f(x)| \leqq 2M$ for all values of u. For $|u| \geqq \delta$, $1 \leqq u^2/\delta^2$, and

$$|f(x + u) - f(x)| \leqq 2Mu^2/\delta^2.$$

For any value of u, one or the other of the quantities $\epsilon/2$, $2Mu^2/\delta^2$ is greater than or equal to $|f(x+u)-f(x)|$, and their sum therefore is certainly greater than or equal to $|f(x+u)-f(x)|$:

$$|f(x+u) - f(x)| \leqq \epsilon/2 + 2Mu^2/\delta^2$$

for all values of u. Consequently, for $0 \leqq x \leqq 1$,

$$\begin{aligned} |P_n(x) - f(x)| &\leqq \frac{1}{J_n}\int_{-1}^{1}(\epsilon/2)(1-u^2)^n du + \frac{1}{J_n}\int_{-1}^{1}\frac{2Mu^2}{\delta^2}(1-u^2)^n du \\ &= \epsilon/2 + \frac{2M}{\delta^2 J_n}\int_{-1}^{1} u^2(1-u^2)^n du. \end{aligned}$$

Let the last integral be denoted by J'_n. By integration by parts,

$$\begin{aligned} J_n' &= \int_{-1}^{1} u\cdot u(1-u^2)^n du = \left[-u\cdot\frac{(1-u^2)^{n+1}}{2(n+1)}\right]_{-1}^{1} \\ &\quad + \int_{-1}^{1}\frac{(1-u^2)^{n+1}}{2(n+1)}du = \frac{J_{n+1}}{2(n+1)}. \end{aligned}$$

But $J_{n+1} < J_n$, since $1-u^2<1$ throughout the interior of the interval of integration and hence $(1-u^2)^{n+1} = (1-u^2)(1-u^2)^n < (1-u^2)^n$. So

$$J_n' < \frac{J_n}{2(n+1)}, \qquad \frac{J_n'}{J_n} < \frac{1}{2(n+1)}.$$

It follows that as soon as n is sufficiently large

$$\frac{2MJ_n'}{\delta^2 J_n} < \epsilon/2$$

and consequently

$$|P_n(x) - f(x)| < \epsilon,$$

for $0 \leqq x \leqq 1$ and in particular for $a \leqq x \leqq b$. This is the substance of the conclusion to be proved.

The above proof has something in common with that of S. Bernstein,[3] the fundamental difference being that Landau's integral is used here instead of the algebraic formula for a binomial frequency distribution. It was in fact suggested to the writer, not by consideration of Bernstein's proof as such, but by a conversation with Professor W. L. Hart on the subject of Bernoulli's theorem. A noteworthy characteristic of Bernstein's proof is that it makes Weierstrass's theorem in effect a corollary of that of Bernoulli.

An alternative organization of the present method of proof is as follows. Let $f(x)$ at first be not merely continuous, but subject to the Lipschitz condition

[3] See e.g. Pólya and Szegö, *Aufgaben und Lehrsätze aus der Analysis*, vol. I, Berlin, 1925, pp. 66, 230.

$$(4) \qquad |f(x_2) - f(x_1)| \leqq \lambda |x_2 - x_1|.$$

If this condition is satisfied by $f(x)$ as originally defined for $a \leqq x \leqq b$, it will be satisfied, possibly with a different value of λ, when the definition is extended to cover all values of x. Then, in (3), $|f(x+u) - f(x)| \leqq \lambda |u|$, and

$$|P_n(x) - f(x)| \leqq \frac{\lambda}{J_n} \int_{-1}^{1} |u| (1 - u^2)^n du = \frac{2\lambda}{J_n} \int_0^1 u(1 - u^2)^n du.$$

Let $\delta = 1/n^{1/2}$, and let

$$I_1 = \int_0^\delta u(1 - u^2)^n du, \qquad I_2 = \int_\delta^1 u(1 - u^2)^n du,$$

so that $|P_n(x) - f(x)| \leqq 2\lambda(I_1 + I_2)/J_n$. In I_1, $u \leqq 1/n^{1/2}$, and

$$I_1 \leqq \int_0^\delta n^{-1/2}(1 - u^2)^n du \leqq \int_0^1 n^{-1/2}(1 - u^2)^n du = \tfrac{1}{2} n^{-1/2} J_n, \quad 2I_1/J_n \leqq 1/n^{1/2}.$$

In I_2, $1/u \leqq n^{1/2}$ and $u = u^2/u \leqq n^{1/2}u^2$, so that by application of the previous reckoning with J_n'

$$I_2 \leqq n^{1/2} \int_\delta^1 u^2(1 - u^2)^n du \leqq n^{1/2} \int_0^1 u^2(1 - u^2)^n du = \tfrac{1}{2} n^{1/2} J_n' \leqq \tfrac{1}{4} n^{1/2} J_n/(n + 1),$$

$$2I_2/J_n \leqq \tfrac{1}{2} n^{1/2}/(n + 1) < 1/n^{1/2}.$$

So it appears not merely that $|P_n(x) - f(x)|$ is less than a quantity independent of x which approaches zero as n becomes infinite, but also, more specifically,[4] that it is less than a quantity of the order of $1/n^{1/2}$. This is proved, to be sure, only for a function satisfying (4). But (4) is satisfied in (a, b) by any continuous function whose graph is a broken line made up of a finite number of straight line segments with finite slope, and as any function whatever that is continuous for $a \leqq x \leqq b$ can be approximated with any desired accuracy by a function of this special type, the general conclusion of Weierstrass's theorem follows immediately.

The method can be applied equally well to the proof of Weierstrass's theorem on the trigonometric approximation of a periodic continuous function, by the use of de la Vallée Poussin's integral[5]

$$\frac{1}{H_n} \int_{-\pi}^{\pi} f(t) \cos^{2n}\left(\frac{t - x}{2}\right) dt, \qquad H_n = \int_{-\pi}^{\pi} \cos^{2n} \frac{u}{2}\, du,$$

and to the proof of corresponding theorems on the approximate representation of continuous functions of more than one variable.

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[4] Cf. C. de la Vallée Poussin, *Sur l' approximation des fonctions d' une variable réelle et de leurs dérivées par des polynômes et des suites limitées de Fourier*, Bulletins de la Classe des Sciences, Académie Royale de Belgique, 1908, pp. 193–254; pp. 221–224.

[5] See de la Vallée Poussin, loc. cit., pp. 228–230.

QUESTIONS, DISCUSSIONS, AND NOTES

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The department of Questions and Discussions in the Monthly is open to all forms of activity in collegiate mathematics, including the teaching of mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.

ON THE VANISHING OF THE SUM OF THE NTH POWERS OF THE ROOTS OF A CUBIC EQUATION

By MORGAN WARD, California Institute of Technology

1. Suppose that

$$x^3 - Px^2 + Qx - R = 0, \tag{1.1}$$

P, Q, R rational integers, is a cubic equation with three distinct non-vanishing roots. Then it is a fundamental problem of considerable arithmetical interest to determine whether or not a given rational integer A may be represented as a sum of nth powers of the roots of (1.1), and—granted that such a representation is possible—to find out in how many ways it can occur.

More precisely, if α_1, α_2, α_3 are the roots of (1.1) and if $S_n = \alpha_1^n + \alpha_2^n + \alpha_3^n$ is the sum of their nth powers, we wish to solve the diophantine equation

$$S_x = A \tag{1.2}$$

in positive integers x, given P, Q, R and A.

The great difficulty of this general problem is at once apparent. For if we restrict the roots of (1.1) to be rational integers, then the special case $A = 0$ is the Fermat problem, and Fermat's conjecture is equivalent to asserting that the diophantine equation

$$S_x = 0 \tag{1.3}$$

can have only the solutions $x = 1$ and $x = 2$ if the roots of (1.1) are rational integers.

2. A simpler preliminary problem is to ascertain whether or not (1.2) and (1.3) can have an infinite number of solutions. In case all of the roots of (1.1) are real, the answer is apparent; *for any A, there are only a finite number of values of x satisfying* (1.2). For since the roots α are distinct, and $|\alpha_1\alpha_2\alpha_3| = |R| \geqq 1$, the magnitude of the arithmetically largest root of (1.1) is greater than one, and since S_n is clearly of the same order as the nth power of this root, $|S_n|$ tends to infinity with n.[1]

In case two of the roots of (1.1) are complex, nothing general seems to be known as to the number of solutions of (1.2). However, in the special case when $R = \pm 1$, Siegel[2] has shown by the use of Thue's theorem that (1.3) *can have*

[1] A trivial exception occurs if we have $\alpha_1 = -\alpha_2 > |\alpha_3|$, as then $S_{2n+1} = \alpha_3^{2n+1}$. Hence if $\alpha_3 = \pm 1$, $S_x = \pm 1$ will have an infinite number of solutions.

[2] Tohôku Journal, Vol. 20 (1921), p. 29.

only a finite number of solutions unless (1.1) *is of the form* $(x\pm 1)(x^2+1)$ *or* $(x\pm 1)(x^2\pm x+1)$.[1]

3. We can set ourselves the still more modest task of finding values of x for which (1.2) and (1.3) are insoluble—a common enough type of procedure in other problems in diophantine analysis.

For example, let p be a prime number, and r a positive integer. Then

$$S_{p^{r-1}} \equiv S_{p^{r-1}}^p \equiv (\alpha_1^{p^{r-1}} + \alpha_2^{p^{r-1}} + \alpha_3^{p^{r-1}})^p = \alpha_1^{p^r} + \alpha_2^{p^r} + \alpha_3^{p^r} \pmod{p}.$$

Since $S_1=P$, we see that[2]

$$S_{p^r} \equiv P \pmod{p}, \qquad (r = 0, 1, 2, \cdots).$$

Accordingly, if (1.2) has a solution for x a power of a prime p, we have the restriction $A\equiv P \pmod{p}$. In particular, (1.3) can have a solution in such a case only if $P\equiv 0 \pmod{p}$. Therefore if $P\not\equiv 0$, for a given cubic equation (1.1), S_x *can vanish for only a finite number of values of x which are primes or powers of primes.*

If $P=0$, we obtain a restriction on the value of A in (1.2), but no restriction upon x in (1.3). In this case, I shall conclude by proving the following theorem, the main point of novelty in the paper.

4. THEOREM. *Let* $\alpha_1, \alpha_2, \alpha_3$ *be the roots of an irreducible cubic equation*

$$x^3 + Qx - R = 0, \tag{4.1}$$

Q, R rational integers. Then if R is greater in absolute value than two, and prime to Q, the sum of the nth powers of the roots of (4.1), $S_n=\alpha_1^n+\alpha_2^n+\alpha_3^n$, *can never vanish if n is even, or if n is a prime.*

PROOF. By hypothesis, α_1, α_2 and α_3 are distinct and not zero, and

$$\alpha_1+\alpha_2+\alpha_3 = 0, \qquad \alpha_1\alpha_2+\alpha_2\alpha_3+\alpha_3\alpha_1 = Q, \qquad \alpha_1\alpha_2\alpha_3 = R, \tag{4.2}$$

$$|R| \geqq 3, \qquad (R, Q) = 1. \tag{4.3}$$

Consider first the case when (1) has only one real root. Denote it by α_1. Then we may write

$$\alpha_2 = re^{i\theta}, \qquad \alpha_3 = re^{-i\theta}, \qquad r > 0, \qquad 0 < \theta < 2\pi. \tag{4.4}$$

By (4.2), $\alpha_1 = -re^{i\theta}-re^{-i\theta} = -2r\cos\theta$. Therefore

$$\begin{aligned} S_n &= \alpha_1^n + \alpha_2^n + \alpha_3^n = (-2r\cos\theta)^n + (re^{i\theta})^n + (re^{-i\theta})^n \\ &= 2r^n\{\cos n\theta + (-1)^n 2^{n-1}(\cos\theta)^n\}. \end{aligned}$$

[1] We may if we please regard $S_0, S_1, S_2, \cdots, S_n, \cdots$ as a sequence (S) giving that particular solution of the difference equation $\Omega_{n+3}=P\Omega_{n+2}-Q\Omega_{n+1}+R\Omega_n$ with the initial values $S_0=3$, $S_1=P$, $S_2=P^2-2Q$, and ask more generally about the solutions of the diophantine equations $U_x=A$, $U_x=0$ for any particular rational integral solution $U_0, U_1, U_2, \cdots, U_n, \cdots$ of the difference equation. Siegel in the paper referred to studies the U_n as coefficients in the infinite series $\sum_{n=0}^{\infty} U_n t^n$. The two theorems just stated for carry over to the general sequence (U).

[2] Lucas, *Theorie des Nombres*, p. 422.

Hence $S_n=0$ when and only when

$$\cos n\theta + (-1)^n 2^{n-1}(\cos\theta)^n = 0.$$

Now by a familiar formula of elementary trigonometry,

$$\cos n\theta = (\cos\theta)^n\left\{1 - \binom{n}{2}\tan^2\theta + \binom{n}{4}\tan^4\theta - \binom{n}{6}\tan^6\theta + \cdots\right\},$$

the last term in the bracket being

$$(-1)^{(n/2)}\tan^n\theta \quad \text{or} \quad (-1)^{(n-1)/2}\binom{n}{n-1}\tan^{n-1}\theta$$

according as n is even or odd. Hence S_n vanishes when and only when $z=\tan^2\theta$ is a root of the algebraic equation

$$(4.5)\qquad F(z) = 1 + (-1)^n 2^{n-1} - \binom{n}{2}z + \binom{n}{4}z^2 - \binom{n}{6}z^3 + \cdots = 0.$$

From (4.4) we see that $i\tan\theta=(\alpha_2-\alpha_3)/(\alpha_2+\alpha_3)$, or

$$(4.6)\qquad \tan^2\theta = -\left(\frac{\alpha_2-\alpha_3}{\alpha_2+\alpha_3}\right)^2.$$

Next, assume that (1) has three real roots. Then two of them must be of the same sign. Denote the remaining root by α_1. If in the pair α_2, α_3, the root of greatest magnitude is α_2, we may write

$$(4.41)\qquad \alpha_2 = \pm re^{\theta}, \qquad \alpha_3 = \pm re^{-\theta}, \qquad r>0, \qquad \theta>0$$

where the upper sign is taken if α_2 and α_3 are both positive, and the lower sign if α_2 and α_3 are both negative. As in the first case, $\alpha_1=-2r\cosh\theta$ and proceeding exactly as before, we find that S_n vanishes when and only when $z=-\tanh^2\theta$ is a root of the algebraic equation (4.5), where

$$(4.61)\qquad \tanh^2\theta = \left(\frac{\alpha_2-\alpha_3}{\alpha_2+\alpha_3}\right)^2.$$

Now if n is even $=2k$, the leading coefficient of (4.5) becomes unity on multiplying through by $(-1)^k$, and the remaining coefficients are obviously rational integers. If n is an odd prime p, the leading coefficient of $F(z)$ is $(-1)^{(p-1)/2}p$ and the integers

$$\binom{p}{2}, \binom{p}{4}, \binom{p}{6}, \cdots$$

are all divisible by p. Furthermore the constant term $1-2^{p-1}$ of $F(z)$ is divisible by p by Fermat's theorem. Therefore on dividing $F(z)$ by $(-1)^{(p-1)/2}p$, we obtain again an equation with leading coefficient unity and rational integral coefficients. But any root of such an equation is an algebraic integer. Hence

referring to (4.6) and (4.61), we can state the result: *If S_n vanishes for n even or n an odd prime, it is necessary that* $\zeta = -(\alpha_2-\alpha_3)/(\alpha_2+\alpha_3)$ *be an algebraic integer.*

5. We finally show that the quantity ζ cannot be an algebraic integer by proving that the irreducible canonical equation with leading coefficient unity which it satisfies has non-integral coefficients.

First, since $(\alpha_2-\alpha_3)^2=(\alpha_2+\alpha_3)^2-4\alpha_2\alpha_3$ we obtain from (4.2) (iii) and (i) $\zeta=1-4R/\alpha_1^3$. Thus ζ is a root of the cubic equation

$$\Phi(z) = \left(z - 1 + \frac{4R}{\alpha_1^3}\right)\left(z - 1 + \frac{4R}{\alpha_2^3}\right)\left(z - 1 + \frac{4R}{\alpha_3^3}\right) = 0.$$

On multiplying out the right side of this expression and simplifying by the use of the relations (4.2) and (4.1), we obtain

$$\Phi(z) = z^3 + \left(9 + \frac{4Q^3}{R^2}\right)z^2 + \left(27 - \frac{8Q^3}{R^2}\right)z + 27 + \frac{4Q^3}{R^2} = 0.$$

This equation has rational integral coefficients when and only when $4Q^3\equiv 0$ (mod R^2) and hence never if $|R|>2$, and $(R, Q)=1$. It only remains to show that it is irreducible. If it were reducible, it would have at least one rational root, so that for $1\leqq i\leqq 3$, we would have $1-4R/\alpha_i^3$ rational, $\alpha_i^3=R-Q\alpha_i$ rational, or α_i rational, contradicting the assumed irreducibility of (4.1).

Note added in proof. Our analysis shows also that if $S_n=0$, n cannot be prime to R. For both $n\zeta^2$ and $R^4\zeta^2$ are algebraic integers.

ON THE GENERAL EQUATION OF THE PARABOLA

H. W. BAILEY, University of Illinois

Consider the general equation of the second degree

$$Ax^2 + 2Bxy + Cy^2 + 2Dx + 2Ey + F = 0.$$

While it is apparent that the various elements used in sketching this curve: directrices, foci, semi-axes, etc., are functions of the constants $A, \cdots, F$, yet the explicit expression in terms of these constants is so complicated algebraically as to be worthless practically. For example, the eccentricity appears as a root of the quartic equation

$$Je^4 - (I^2 + 4J)e^2 + (I^2 + 4J) = 0,$$

where $I=A+C$, $J=B^2-AC$. However, in the case of the parabola such expression is possible in very simple form. The purpose of this note is to give these explicit formulas when we take the equation of the parabola in the form

$$A^2x^2 + 2ACxy + C^2y^2 + 2Dx + 2Ey + F = 0.$$

Let the equation of the directrix and the coordinates of the focus be $\alpha x-\beta y-\gamma=0$ and (m, n) respectively. Using the general definition of a conic, the equation of this parabola may also be written

$$\beta^2x^2 + 2\alpha\beta xy + \alpha^2y^2 + 2[\alpha\gamma - m(\alpha^2 + \beta^2)]x$$
$$+ 2[-\beta\gamma - n(\alpha^2 + \beta^2)]y + [(m^2 + n^2)(\alpha^2 + \beta^2) - \gamma^2] = 0.$$

The invariants are then: $I = A^2 + C^2 = \alpha^2 + \beta^2$, $J = 0$, $\Delta = -(AE - CD)^2 = -I^2(\alpha m - \beta n - \gamma)^2$. We denote by δ and λ the quantities $AE - CD$ and $AD + CE$ respectively.

We then have to solve the following set of equations,

$$A^2 = \beta^2, \qquad C^2 = \alpha^2,$$
$$D = \alpha\gamma - mI,$$
$$E = -\beta\gamma - nI,$$
$$F = I(m^2 + n^2) - \gamma^2,$$

for α, β, γ, m, and n. The solutions are readily found to be

$$\alpha = C, \qquad \beta = A,$$
$$2\delta\gamma = FI - D^2 - E^2,$$
$$2\delta Im = CFI - D\delta - E\lambda,$$
$$2\delta In = -AFI - E\delta + D\lambda.$$

Consequently the equation of the directrix is

$$2\delta(Cx - Ay) = FI - D^2 - E^2.$$

Using the fact that the axis passes through the focus and is perpendicular to the directrix, the equation of the axis is found to be

$$I(Ax + Cy) + \lambda = 0.$$

Solving this equation with the equation of the parabola, the coordinates of the vertex, (h, k), are found to be

$$h = \{CFI^2 - \lambda(A\delta + EI)\}/2\delta I^2$$
$$k = \{-AFI^2 - \lambda(C\delta - DI)\}/2\delta I^2.$$

Forming the distance from the focus to the directrix and multiplying by 2 gives the length of the latus rectum, $2\delta/I^{3/2}$.

RECENT PUBLICATIONS

EDITED BY R. A. JOHNSON, Brooklyn College of the City of New York

All books for review should be sent directly to the editor of this department, at Brooklyn College, 66 Court Street, Brooklyn, N. Y., and not to any of the other editors or officers of the Association.

REVIEWS

Differential Equations. By Lester R. Ford. New York, McGraw-Hill Book Company, 1933. x+264 pages. \$2.50.

This text on differential equations definitely breaks away from the traditional form of introduction to the subject. When a new book on this branch of

mathematics comes off the press we usually look it over to see how much it resembles those already in print. This book is something entirely new and refreshing in its manner of presentation.

Many elementary texts on differential equations leave the student bewildered by the mass of special methods that are used for solving different types of equations. Also, accurate statements and rigorous proofs of existence theorems are usually avoided. As a result, the student acquires some skill and technique in handling the classical differential equations that occur in physics and elsewhere but he finds himself in a strange country when he takes a more advanced course that contains rigorous discussions.

The present text has overcome this difficulty by stressing in the first three chapters the geometrical and intuitive aspects, using lineal and circular elements, disks, and conical elements. Then the book in Chapters IV and V gives a rigorous treatment of existence theorems, using the method of successive approximations. No use is made of Cauchy's method of the calculus of limits and just a mention is included of the Cauchy-Lipschitz method. The first three chapters lead up to the later chapters by their discussions of direction fields, of solutions in series, of the Wronskian and linear dependence. The method of successive approximations leads naturally into the Chapters VI and VII on interpolation and numerical integration and solutions. These chapters, by the way, are unusual in an elementary text. They are extremely well done, thanks to the author's own ability in and contributions to these subjects.

Attention should be called to the treatments of finite differences and of the symbolic operators, also to the emphasis placed on whole families of solutions. Chapter VIII on linear equations contains the Gramian, the Wronskian and linear dependence, as well as symbolic methods. The next chapter on certain classical equations gives a good introduction to the hypergeometric, the Legendre, and the Bessel differential equations. In Chapter X on partial differential equations of the first order the distinction between complete and general solutions is well brought out, also the geometrical interpretations of solutions are emphasized. This will help the student to grasp the subject better.

In the preface, the author states: "The partial differential equation of the second order is a vexatious problem for the writer of an elementary text. What has been attempted here has been the presentation of a compact, connected, and (it is believed) teachable body of material which exhibits those elementary methods of solution which are of commonest use." Here again the author has succeeded.

The book concludes with a good index. It is well-printed, and its figures are interesting and very instructive. Many illustrative problems are worked out carefully in the text. Whereas many of the classical problems giving applications to geometry and physics are included, there are also to be found very modern and interesting problems such as the so-called parasite problem, the vibrating string and membrane, the conduction of heat, and the like. Some problems appear again and again to be solved by different methods or with different initial

conditions. The early part of the book contains plenty of problems, well scattered throughout the chapters. The latter part of the book might have more problems than it has. Also a few answers now and then would help to give the student confidence.

There are only a very few misprints, such as c' in Figure 10, a in Fig. 17, a comma at the middle of page 102, the lower limit a at the bottom of page 127, $f'(a)$ in (23) on page 134, $e^{\mu m x}$ in (12) on page 169, etc. There seem to be remarkably few places where the author has omitted a step that the student would need. One such place is on page 120, where the discussion of Newton's interpolation formula is given for n an integer and then the illustrative example uses $n=1.4$. Also on page 111 the student would have hard work seeing that (20) is "precisely the condition that this be an equation in f and z alone."

The style of the book is forceful, almost conversational, and decidedly refreshing. Many a teacher feels that if only an author could put into print his explanations to his classes, the result would be a good, teachable text. Unfortunately the language of writing is much more stilted than the language of speech. The author has come pretty close to writing as he would talk. Notice the use of the word except instead of unless at the bottom of page 115; also the sentence on page 173 that reads "A symbolic method in which the integrations are hitched abreast instead of tandem avoids this difficulty"; also such section headings as "Aids to Good Guessing" and "On the Making of Rules." Moreover it is well to note how the special cases are grouped under general methods with the comment that an energetic and intelligent student could keep on forever discovering new rules for integrating particular types of equations.

Parts of the text have been taught by the author to different classes of students. The whole text has been used by him in a year's course. A good semester's course could be offered using selected portions of the book and putting thereby the emphasis wherever the teacher should desire. Also the book could be studied without the help of a teacher. In every way this is a very good text on differential equations.

A. D. Campbell

Theory of Functions as Applied to Engineering Problems. By R. Rothe, F. Ollendorff, and K. Pohlhausen. (Authorized Translation by Alfred Herzenberg.) Cambridge, Mass., Technology Press, Massachusetts Institute of Technology, 1933. x+190 pages. $3.50.

This book is a translation of a series of lectures delivered at the Berlin Institute of Technology in its Winter Session of 1929–1930 and published in Germany by Julius Springer in 1931. The lectures were designed originally to introduce engineers to the beautiful and powerful methods of the theory of functions of a complex variable. In such an undertaking, the lecturers succeeded admirably. The English translation will serve the same purpose for American engineers, especially for those who wish to delve into the theoretical side of such subjects as electrical engineering, hydraulics, and aeronautics.

A perusal of this book will convince the reader that a thorough grounding in advanced as well as elementary mathematics and physics is essential for the solution of problems in modern engineering. No longer is a first course in physics and a first course in the calculus, plus the stamp of a first degree in engineering, sufficient training for a first-class engineer (even though some engineering schools in the United States still feel that such training is all that is needed for a teacher of mathematics to engineering students).

If this book is read and understood by the general run of German engineers, the reviewer doffs his hat to them and to the engineering schools from which they have graduated. The better-trained engineers in the United States will find this a most valuable book to have in their libraries, as well as a most stimulating and interesting book.

The book should appeal also to mathematicians. More and more is it becoming necessary for a mathematician to acquire at least a familiarity with and a sympathy for the view-point of physics, engineering, chemistry and the other subjects where mathematics is being applied. The mathematician and the mathematical physicist are becoming one and the same person. Mathematics as the language of science should be as well-known in all these fields of knowledge as English in our every-day life. Therefore, the mathematician should be able to speak to engineers and others in mathematical language they will understand, and then he can help them acquire more of this language for further use in their professions. (On the other hand, our engineers should be better mathematicians and physicists and not just users of hand-books whose misprints may prove their ruin.) Hence, this book should be in the mathematician's library alongside such books as Mellor's *Higher Mathematics for Students of Physics and Chemistry*, Karapetoff's little volumes on mathematics for engineering students, Root's *Engineering Mathematics*, Perry's books, and other such texts.

The first eighty-one pages of this book are given up to an exposition by R. Rothe of the elementary theory of the complex variable. In the preface we find this statement, "While the physicist—by virtue of his education—often comes in contact with the theory of functions, the engineer usually lacks a sufficient knowledge of these mathematical methods. There exist, it is true, a few procedures of applied mathematics especially adapted to technical problems, and these procedures are often based on the theorems of function theory. The development of theoretical engineering, however, makes it advantageous to free these special procedures from their symbols and to consider them—from a higher point of view—as a branch of general mathematics." On page 61 in the text, Rothe mentions the so-called Heaviside Operational Calculus as an example of the above statement, calling attention to the fact that this symbolic notation was originated by Leibniz, has been known since the time of Lagrange, and is discussed in detail in such books as Boole's *Treatise on the Calculus of Finite Differences*.

Rothe's treatment of the complex variable is necessarily brief, but quite thorough, and consists of an original sort of mixture of the viewpoints and

methods of such books as Burkhardt-Rasor's and Goursat-Hedrick's. It is interesting to see how he economizes on space and yet manages to cover so much essential material. Just a list of the main topics treated will surprise the reader. For example, he courses over some parts of analytic functions, mapping, line-integrals, divergence, circulation and potential, residues, Taylor's and Laurent's series, the so-called hook-integrals, a bit of linear differential equations, Riemann surfaces, as well as other subjects. His treatment of these topics is vigorous and refreshing and should prove interesting to all who read the book, from beginners clear through to experts in the subject. His methods of proof are quite entertaining; e.g., his proof of the fundamental theorem about the integrals of analytic functions of a complex variable.

All through the book, the translator has kept the spirit of the original, even to using the German notation and writing very often in English that reads like German. The drawings are numerous and striking in their excellence, the print is clear and legible. In every way the book is a fine piece of printing, and a credit to the publishers. There are some few misprints and omissions of parts of symbols and formulas, such as in (25) on page 15, (95) on 56, just above (144) on 67, at the bottom of page 72, Fig. 32 on 77, just above and below (183) on 80, bottom of 82, (76) on 124, (77) on 125, (22) on 150, (5) on 171.

Even though the second half of the book consists of lectures on applications delivered by other men than Rothe, nevertheless the book hangs together as well as if one man had written it all. Thus, Rothe stresses such topics as impulse functions and Schwarz' mapping of polygons and these topics are used to great advantage later on in the applications. The applications given are to electric and magnetic fields, two dimensional fields of flow, the field distribution in the neighborhood of edges, the complex treatment of electric and thermal transient phenomena, and the spreading of electric waves along the earth.

The lectures on these subjects are so well done that even a mathematician who knows little or nothing of the physics and engineering involved can follow the discussions with great interest and profit. A good bibliography is given at the end of each lecture. Of particular interest is the clever use of mapping (as well as of velocity potential and stream functions) in solving problems of flow (including those of airplane wing design) and the use of impulse functions for problems involving transient phenomena. Here again the drawings help considerably toward an understanding of the subjects treated.

The reader will surely agree with the translator when he says in his preface that "American engineering owes a debt of gratitude to President Karl T. Compton of the Massachusetts Institute of Technology and to the late Mr. John R. Freeman for their support of this project to make available in English a valuable engineering treatise." We should add that we all owe a debt of gratitude also to the original lecturers and to the translator, Alfred Herzenberg. Once again, that splendid institution, Massachusetts Institute of Technology, has sponsored a very worthwhile publication.

A. D. CAMPBELL

Conjugate Functions for Engineers. By Miles Walker. Oxford University Press, 1933. 116 pages. $4.75.

The author presents "a simple exposition of the Schwarz-Christoffel transformation applied to the solution of problems involving two-dimensional fields of force and flux." He seeks to make clear to engineers the mathematics involved in the use of conjugate functions to engineers' problems. For this reason he has developed the subjects from the beginning and illustrates very clearly the concepts of streamlines and equipotential lines, flow of current about corners, sudden reductions in width of channel, the magnetic field in an air-gap, the flux in the case of a pole-piece with a rounded tip, the flow of electricity in a large sheet where the boundary is partly circular, etc.

The interest to mathematicians lies in the presentation of the Schwarzian transformations which map these various graphs on simpler figures.

W. D. CAIRNS

MATHEMATICS CLUBS

EDITED BY F. M. WEIDA, The George Washington University, Washington, D.C.

All reports of club activities, suggestions and topics for club programs, and material of interest to clubs should be sent to F. M. Weida, The George Washington University, Washington, D.C. All manuscripts should be typewritten, with double spacing, and with margins at least one inch wide. All club activity manuscripts for the academic year 1933–1934 should be submitted for publication not later than June 1, 1934.

CLUB ACTIVITIES

1932–1933

THE PI MU EPSILON MATHEMATICAL FRATERNITY

Pi Mu Epsilon is an academic fraternity in institutions of university grade. Its primary aim is the advancement of mathematics and scholarship. It is a living, active, working fraternity of scholars in which the members are engaged in study and research and the exchange of ideas in the field of mathematical science.

Pi Mu Epsilon of the University of Oklahoma

The officers for the year 1932–1933 were: Professor Dora McFarland, Director; Esther Gassett, Vice Director; Mildred Dolezal, Secretary; Balfour S. Whitney, Treasurer; Professor J. O. Hassler, Librarian. These officers were elected in May 1932 by ballot.

There were forty-eight members on the roll of this chapter, fourteen of whom are faculty members, thirteen graduate students in mathematics or natural science, and twenty-one undergraduate students. One election was held in March at which time seventeen new members were admitted. The initiation dinner which took the place of one regular meeting was held at the Faculty Club in April.

Regular meetings were held on the second and fourth Thursdays of each month at 7:30 P.M. After the business meeting and the formal program, refreshments were served and informal discussions were encouraged.

The meetings and programs were as follows:

September 29, 1932: "Complex numbers" by Miss Mildred Dolezal and Miss Mildred Bleckley.

October 13, 1932: "Some practical applications of complex numbers" by Mr. Ernest Handley; "An introduction to quaternions" by Miss Mary Louise Kropp.

October 27, 1932: "Quaternions" by Miss Esther Gassett and Mrs. Edith Townes.

November 10, 1932: "Quaternions and their applications" by Mr. Julian Evans and Mr. Earl LaFon.

December 8, 1932: "Social meeting—A Christmas party was held at the Women's Building."

January 12, 1933: "The algebra of quaternions" by Professor Dora McFarland.

February 9, 1933: "The history of quaternions" by Mr. B. S. Whitney and Mr. Dewey McKnelly.

February 23, 1933: "Oblique coordinates" by Mr. Jack Laudermilk; "Cylindrical coordinates" by Mr. Henry Harms; "Spherical coordinates" by Mr. S. B. Ingerson.

March 9, 1933: Election of members.

March 23, 1933: "Homogeneous coordinates" by Miss Betty LeComte; "Line coordinates" by Miss Rosella Dorsett; "Trihedral coordinates" by Mr. Ralph Dorsett.

April 7, 1933: Initiation banquet.

April 29, 1933: "Plücker coordinates" by Miss Esther Gassett; "Tetrahedral coordinates" by Mr. Stephen Brixey.

May 11, 1933: "Trisection of an angle" by Miss Dorothy Huff; "A machine for trisecting an angle" by Mr. Harold Feldstein; Election of officers for 1933–1934.

The aim of the club is scholarship for individual members in all subjects and especially in mathematics; the advancement of the science of mathematics and the mutual and personal advancement of its members. The requirements for eligibility to membership are a general average of "B" and an average between "A" and "B" in mathematics.

MILDRED DOLEZAL, *Secretary*

Pi Mu Epsilon of the University of Oregon

The officers for 1932–1933 were: Kenneth I. Kienzle, Director; Holly Fryer, Vice Director; Eileen Hickson, Secretary; John Millican, Assistant Secretary; Harriet Holbrook, Treasurer; Professor Edgar E. DeCou, Permanent Secretary.

The meetings and programs were as follows:

October 28, 1932: "The history of mathematical journals and societies in America" by Professor W. E. Milne.

November 10, 1932: Regular business meeting.

December 2, 1932: "A special differential equation of the second order" by Holly Fryer; "The development of mathematics in America" by Professor E. E. DeCou.

February 1, 1933: "Mathematics and the physical sciences at Oxford" by Mr. Robert Jackson, Rhodes Scholar from the University of Oregon.

March 9, 1933: A joint chapter was formed with Oregon State College due to a reorganization of the University.

April 14, 1933: "The ancient and modern Abacus" by Harriet Holbrook; "Disintegration of the lithium atom" by Louis Fendrich.

May 16, 1933: Business meeting.

May 27, 1933: Annual initiation and installation of officers. This event was followed by a banquet.

EDGAR E. DECOU, *Permanent Secretary*

LOCAL MATHEMATICS CLUBS

The Mathematics Club of Rutgers University

The officers for 1932–1933 were: John H. MacDonough, President; Leonard S. Stout, Secretary-Treasurer; Professor Emory P. Starke, Faculty Sponsor.

The meetings and programs were as follows:

January 12, 1933: "A discussion of the physical meaning of the theory of least squares as applied to curve tracing through observation of a group of points obtained from an experiment" by Mr. Leonard Rusby; "Pot-Pourri" by Professor Emory P. Starke.

February 8, 1933: "Complex number" by Professor W. B. Fite of Columbia University. We had as our guests the Mathematics Club of the New Jersey College for Women.

February 23, 1933: At this meeting, Mr. Joseph P. Catlin, Jr., representing the student paper, spoke on the topics: "Newton's method of determining the roots of a polynomial expression in one variable by successive approximations" and "To construct a line equal,very approximately, to the circumference of a given circle." Professor Richard Morris, representing the faculty paper, spoke on the topic: "The tetrahedron and its circumscribed parallelopiped."

March 16, 1933: "To find the locus of the point of intersection of the two perpendicular tangents drawn first to a parabola and secondly to an ellipse" by Mr. John H. MacDonough; "Jacobi's development of a certain definite integral" by Mr. C. R. Wilson.

April 7, 1933: We were the guests of the Mathematics Club of the New Jersey College for Women. Professor Anna Pell Wheeler spoke on the topic: "The theory of numbers."

May 18, 1933: "The volume of a tetrahedron as a determinant" by Mr. Leonard S. Stout; "A problem taken from the works of Archimedes" by Professor Emory P. Starke. The officers for 1933–1934 were elected.

LEONARD S. STOUT, *Secretary-Treasurer*

The Mathematics Club of Eastern Illinois State Teachers College

The club meets bi-weekly and is open to all students of the college interested in mathematics. The purpose of the club is to further interest in mathematics, bring students interested in mathematics together and to supplement the college courses in mathematics. Three editions of an eight paged mimeographed paper called "The Discriminant" were published during the year. The paper contained articles of general mathematical interest as well as articles of personal interest to the members of the club.

The officers for the year 1932–1933 were: Mr. John Black, President; Mr. Richard Provines Vice President; Miss Marie Schnepper, Secretary-Treasurer.

The meetings and programs were as follows:

September 28, 1932: "Queerest mathematics before the Greeks" by Mr. Golden Flake; "The use of mathematics" by Professor E. H. Taylor.

October 12, 1932: Annual Fall Term wiener roast.

October 26, 1932: "Map projections" by Mr. John J. Black; "Conic sections" by Professor H. H. Heller.

November 9, 1932: "The use of mathematics in Chemistry;" by Mr. Robert Wiseman; "Functional relationships in mathematical training" by Mr. Richard Provines.

December 7, 1932: "Jazzing up our mathematics" by Miss Marie Schnepper; "History and use of the slide rule" by Mr. Clifford Cole.

December 13, 1932: Special meeting called to elect a staff for the paper called "The Discriminant."

December 21, 1932: "Mathematical fallacies" by Miss Clara Balmer; "Properties of squares and cubes in arithmetical numbers" by Mr. Walter Treece.

January 11, 1933: "Romantic aspect of numbers" by Mr. Clarence Taylor; "Warped surfaces" by Mr. Victor Patrick.

January 25, 1933: "Development of the number system" by Professor W. C. Cook.

February 8, 1933: "Poetry and mathematics" by Miss Francis McCormick; "Origin of geometry" by Miss Esther McCandlish.

March 8, 1933: "History of mathematical notation" by Mr. Milton Baker; "Modern calculating machines" by Professor H. H. Heller.

March 22, 1933: Annual ciphering contest for College and High School.

April 5, 1933: "Mathematics in China" by Miss Esther Schubert; "The notion of infinity" by Professor E. H. Taylor.

April 26, 1933: "History of *Pi*" by Miss Wilma Nuttall; "One unknown and two unknowns" by Mr. Walter Treece; "Mathematics in the future" by Miss Thelma Quicksall.

May 24, 1933: "Geometry humanized." This topic was presented in the form of a play by the tenth grade under the direction of Miss Gertrude Hendrix. The officers for 1933–1934 were also elected.

MARIE SCHNEPPER, *Secretary-Treasurer*

The Mathematics Club of the New Jersey College for Women

The officers for 1932–1933 were: Edith Rapp, President; Armie Apamian, Vice President; Dorothy Meyer, Secretary; Molly Bruce, Treasurer. One change was made when, upon the resignation of the Secretary, Elizabeth Potts was elected to this position in February.

Our meetings were held bi-monthly and were planned by a committee consisting of Professor Richard Morris, Faculty Adviser; Miss Edith Rapp, President; Miss Armie Apamian, Vice President. The total enrollment for the year was fifty-nine.

As stated in the constitution, the purpose of the club is "to cultivate a deeper interest in mathematics and to promote a wider knowledge of the field." At the meetings of the club, papers and reviews were presented aimed at the realization of this end.

The topics discussed during the year were as follows: "Miscellaneous problems" by the Misses Apamian, Petrie, Bourath, Berkow, Smith, and Schliemann; "Some historic curves" by the Misses Apamian and Bosshardt, and Mr. Robert Walter; "The application of mathematics in Physics" by Dr. Wilfred Jackson; "Properties of the tetrahedron inscribed in a parallelopiped" by the Misses Venook and Rovner; "Algebraic problems" by the Misses Groff, Rosin and Nolf; "Derivation of the formula for the differentiation of a determinant" by Miss Rosenberg; "Stewart's theorem and pertinent problems" by Miss Phillips; "A problem related to orthologic triangles" by Dr. Richard Morris; "The hypergeometric series of Gauss" by Miss Edna Smith" "Time savers in mathematical calculations" by Miss Nolf.

In addition to the above, two outside mathematicians spoke to the club: Dr. W. Benjamin Fite of Columbia University spoke on "The complex number"; and Dr. Anna Pell Wheeler of Bryn Mawr College spoke on "Numbers."

Two customary social functions, a tea and a Christmas party were held. In the Spring, the annual dance was given by the club.

DOROTHY MEYER, *Secretary*

The Mathematics Club of the Woman's College of the University of North Carolina

The officers for 1932–1933 were: Julia McLendon, President; Eleanor Shelton, Vice President; Fay Dellinger, Secretary-Treasurer; Miss Cornelia Strong, Faculty Adviser.

The meetings and programs were as follows:

September 1932: Business meeting.

October 1932: "Life of Lewis Carroll" by Beatrice Roberts and Virginia Allen; "The Alice books" by Julia McLendon. The program was followed by a social hour.

November 1932: "Construction of the nine-point circle" by Mildred Boatman; "History of the magic square" by Lottie Lee Kennedy.

December 1932: "Theories concerning the Bethlehem Star" by Katherine Nowell: "The eclipse of 1932" by Miss Cornelia Strong. The program was followed by a social hour.

February 1933: Initiation of new members. Tableaux from the history of mathematics were presented.

March 1933: Reports of experiences in the training school by senior members engaged in practice teaching.

April 1933: "Mathematics in the Greensboro High School" by Miss Ione Grogan, Head of the mathematics department of the Greensboro High School.

May 1933: Picnic and election of officers for the academic year 1933–1934.

FAY DELLINGER, *Secretary*

The Mathematics Club of Northwestern University

The purpose of the club is to promote an active interest in the development of mathematics and to foster social intercourse between the members of the faculty and advanced mathematics students. Faculty members, graduate students, and seniors majoring in mathematics are eligible for membership. Undergraduates who have completed a minor are eligible for associate membership.

The officers are elected each semester. Those for the first semester were as follows: Dr. Norman Rutt, Faculty Adviser; Willard Adcock, President; Willard Grant, Vice President; Winifred Berglund, Secretary; Jeannette Smith, Treasurer. The officers for the second semester were as follows: Dr. Norman Rutt, Faculty Adviser; Irwin Perlin, President; Willard Grant, Vice President; Jeannette Smith, Secretary; Helen Carter, Treasurer.

The meetings and programs were as follows:

October 6, 1932: Election of officers. "Some problems that illustrate the calculus of variations" by Professor H. A. Simmons.

October 20, 1932: "Vector analysis" by Dean E. J. Moulton.

November 15, 1932: "Regular tile patterns" by Professor D. R. Curtiss.

November 29, 1932: "The theory concerning the origin of the real number system" by Mr. Irwin Perlin.

December 15, 1932: "Summation of series" by Dr. H. L. Garabedian.

February 21, 1933: Election of officers. "Minimum numbers" by Miss Jeannette Smith.

March 21, 1933: "Transcendental numbers" by Professor H. S. Wall.

April 11, 1933: "Ideal numbers" by Professor Lois Griffiths.

May 2, 1933: "Quasi-analytic functions" by Mr. Irwin Perlin.

May 16, 1933: "Representations of continuous functions by means of polynomials" by Mr. Willard Adcock.

JEANNETTE SMITH, *Secretary*

The Mathematics Club of the University of Virginia

The Echols Mathematics Club of the University of Virginia was organized October 13, 1931, to "promote better fellowship among its members and to foster a wider interest in the subject of mathematics at the University of Virginia." The meetings for the academic year 1932–1933 were scheduled for each second and fourth Thursday.

The officers for 1932–1933 were: W. T. Puckett, Jr., President; T. L. Wade, Jr., Vice President; M. W. Aylor, Secretary-Treasurer.

The meetings and programs were as follows:

October 26, 1932: "Linear vector equations" by Professor W. H. Echols.

November 10, 1932: "The Poncelet-Steiner and Mascheroni construction theorems" by H. W. Eves.

December 1, 1932: "Desargues configuration" by Professor C. M. Sparrow.

January 12, 1933: "The triangle and its related circles" by Professor B. Z. Linfield.

January 26, 1933: "Some invariant properties of conics" by J. W. Blincoe.

February 10, 1933: "The introduction of the calculus to England" by Professor Scott Buchanan.

February 23, 1933: "Elementary theorems and linear equations in finite differences" by I. T. Shapiro.

April 10, 1933: "The construction of a function through differential processes" by Professor W. H. Echols.

April 27, 1933: "Some notes on tensors" by Ed. Callis. Election of officers for 1933–1934.

May 18, 1933: "A process of mapping" by Professor J. J. Luck.

We served refreshments at each meeting.

M. W. AYLOR, *Secretary*

PROBLEMS AND SOLUTIONS

Elementary Problems

Edited by Otto Dunkel, H. L. Olson, and W. F. Cheney, Jr.

Send all communications about Elementary Problems and Solutions to W. F. Cheney, Jr., Dept. Box 35, Storrs, Connecticut.

The Department of Elementary Problems welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. Problems may be submitted unaccompanied by their solutions.

PROBLEMS FOR SOLUTION

E 93. *Proposed by H. T. R. Aude, Colgate University.*

Find the locus of the centers of the circles in a plane which pass through a given point and are orthogonal to a given circle.

E 94. *Proposed by E. P. Starke, Rutgers University.*

If three integers, a, b and c, satisfying $a+b=c$, together contain each of the nine digits from one to nine just once, show that c is a multiple of nine such that $450<c<1000$. Also show that, with three exceptions, if c satisfies this necessary condition (and is made up of three distinct, non-zero digits) there exist values of a and b satisfying the original hypothesis.

E 95. *Proposed by Harry Langman, Cooper Union, New York, N. Y.*

Show that $(2n)!/(2^n n!)$ is a positive integer if n is.

E 96. *Proposed by R. A. Johnson, Brooklyn College, Brooklyn, N. Y.*

The captain of a man-of-war saw, one dark night, a privateersman crossing his path at right angles, and at a distance ahead of c miles. The privateersman was making a miles an hour, while the man-of-war could only make b miles an hour. The captain's only hope was to come as close to the privateersman as possible, and to disable him by one or two well-directed shots; so the ship's lights were put out and her course altered in accordance with this plan. Through what angle must the ship's course have been changed in order to secure the nearest approach, and when and where did it occur? (This is a modification of a problem appearing on page 126 of Osgood's *Introduction to the Calculus.*)

E 97. *Proposed by Churchill Eisenhart and H. N. Russell, Jr., Princeton University.*

In a certain hypothetical state, no one-digit automobile license numbers are issued, and no license number begins with a zero. Assuming that the numbers are issued in numerical order, and that the number 29,000 has just been given out, show that if a palindromic number is seen, it is more likely to be a five-digit number than not.

E 98. *Proposed by Maud Willey, Long Beach, Mississippi.*

$B_0, A_1, A_2, A_3, \cdots$ are equally spaced points on a line, q. A circle is drawn

with radius $B_{i-1}A_i$ and center A_i, and the ends of its diameter perpendicular to q are labeled B_i and C_i ($i=1, 2, 3, \cdots$). What is the curve of lowest degree through the points B_i and C_i?

E 99. *Proposed by Arnold Dresden, Swarthmore College.*

In the following problem, each letter represents one of the digits from zero through nine, and it is required to so identify them that they will constitute a correctly worked out problem in long division.

$$
\begin{array}{r}
a\ b)c\ d\ d\ e\ f\ g(h\ i\ f\ j \\
\underline{c\ c\ h} \\
e\ e \\
\underline{a\ b} \\
h\ d\ f \\
\underline{d\ e\ c} \\
h\ a\ g \\
\underline{h\ h\ j} \\
c
\end{array}
$$

This problem is from the Haagsche Post, with appreciative acknowledgment.

SOLUTIONS

E 65 [1933, 606]. *Proposed by J. M. West, Pennsylvania State College.*

If the vertices of a triangle taken counterclockwise have the abscissas x_1, x_2 and x_3, and if the slopes of the opposite sides are m_1, m_2 and m_3, then prove that the area equals $\frac{1}{2}(x_1-x_2)(x_1-x_3)(m_2-m_3)$, as well as either of the two similar expressions obtainable from this by cyclic permutation of the subscripts.

Solution by Simon Vatriquant, Athénée Royale d'Ixelles, Brussels.

The area of the triangle is given by the classic formula

$$A = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

or, subtracting the first row from each of the others and changing the signs of these rows,

$$A = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_1 - x_2 & y_1 - y_2 & 0 \\ x_1 - x_3 & y_1 - y_3 & 0 \end{vmatrix}.$$

Expanding this determinant, we get

$$A = \tfrac{1}{2}[(x_1 - x_2)(y_1 - y_3) - (x_1 - x_3)(y_1 - y_2)]$$

or

$$\begin{aligned} A &= \tfrac{1}{2}(x_1 - x_2)(x_1 - x_3)[(y_1 - y_3)/(x_1 - x_3) - (y_1 - y_2)/(x_1 - x_2)] \\ &= \tfrac{1}{2}(x_1 - x_2)(x_1 - x_3)(m_2 - m_3). \end{aligned}$$

The similar expressions obtainable by cyclic permutation of subscripts must also represent the area, since the original determinant is not altered in value by such permutation of subscripts.

Also solved by L. M. Bauer, T. C. Esty, S. E. Field, J. B. Hall, A. D. Hestenes, E. W. Holt, I. L. Miller, E. P. Starke, C. W. Trigg and the proposer.

E 66. [1933, 606]. *Proposed by J. Rosenbaum, The Milford School, Milford, Connecticut.*

Prove that the sum of the squares of the medians of a tetrahedron equals four-ninths of the sum of the squares of the edges. (A median of a tetrahedron is a line joining a vertex with the centroid of the opposite face.)

Solution by T. C. Esty, Amherst College.

Let the vertices of the tetrahedron be A, B, C and D. With D as origin, let $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ be the vectors to the points A, B, and C.

If we denote by $\boldsymbol{a}'$, $\boldsymbol{b}'$, $\boldsymbol{c}'$ and $\boldsymbol{d}'$ the vectors of the centroids of the faces opposite A, B, C and D respectively, we may write

(I) $\boldsymbol{a}' = (\boldsymbol{b} + \boldsymbol{c})/3$, $\boldsymbol{b}' = (\boldsymbol{c} + \boldsymbol{a})/3$, $\boldsymbol{c}' = (\boldsymbol{a} + \boldsymbol{b})/3$, and $\boldsymbol{d}' = (\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c})/3$.

The vector medians from the tetrahedron vertices to their opposite face centroids are then

$$\text{(II)} \qquad \boldsymbol{a}' - \boldsymbol{a},\ \boldsymbol{b}' - \boldsymbol{b},\ \boldsymbol{c}' - \boldsymbol{c}, \text{ and } \boldsymbol{d}'.$$

If the primed vectors in (II) are replaced by their values from (I) and the results in (II) squared and added, the result will be

$$\text{(III)} \qquad \frac{4}{9}[3(\boldsymbol{a}^2 + \boldsymbol{b}^2 + \boldsymbol{c}^2) - 2(\boldsymbol{a}\cdot\boldsymbol{b} + \boldsymbol{b}\cdot\boldsymbol{c} + \boldsymbol{c}\cdot\boldsymbol{a})].$$

Now the vector edges are $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$, $\boldsymbol{a}-\boldsymbol{b}$, $\boldsymbol{b}-\boldsymbol{c}$, and $\boldsymbol{c}-\boldsymbol{a}$. If these are squared and added, the result will be

$$\text{(IV)} \qquad 3(\boldsymbol{a}^2 + \boldsymbol{b}^2 + \boldsymbol{c}^2) - 2(\boldsymbol{a}\cdot\boldsymbol{b} + \boldsymbol{b}\cdot\boldsymbol{c} + \boldsymbol{c}\cdot\boldsymbol{a}).$$

Comparison of (III) with (IV) shows that the sum of the squares of the medians of a tetrahedron equals four-ninths of the sum of the squares of the edges.

Also solved by Ruth S. Berkow, A. D. Hestenes, Theodore Lindquist, Roy MacKay, E. P. Starke, C. W. Trigg, Simon Vatriquant and the proposer.

E 67. [1933, 606]. *Proposed by E. C. Kennedy, University of Texas.*

Give a scheme for writing down mechanically the sides of an unlimited number of dissimilar right triangles whose sides are integers. After the first set, the values are to be written down, not merely indicated, without any calculations whatever. No addition, subtraction, multiplication, division, involution or evolution, mental or otherwise, is allowed.

One way is to start with the triangle whose sides are 21, 220 and 221.

Solution by Maud Willey, Long Beach, Mississippi.

Since the triangle contains a right angle, $a^2+b^2=c^2$, where c is the hypotenuse and a and b the other two sides.

Assume that $c=b+1$. Then $a^2=(b+1)^2-b^2=2b+1$, must be a perfect square if the three sides of the triangle are integers. If $2b+1=4n^2+4n+1$, and if n is an integer, then $b=2n^2+2n$ is also an integer, and so is $a=2n+1$, $c=2n^2+2n+1$. Any positive, integral value of n will now give integers for a, b and c which satisfy $a^2+b^2=c^2$. If n is a positive integral power of ten, the numbers a, b and c can be written down without any sort of arithmetic, by merely inserting zeros. For the first six powers of ten used for n, we have

a	b	c
21	220	221
201	20200	20201
2001	2002000	2002001
20001	200020000	200020001
200001	20000200000	20000200001
2000001	2000002000000	2000002000001.

Note by the proposer. Similar sets of irreducible right triangles may be written by the mere insertion of zeros, starting with such triangles as $69^2+260^2=269^2$ or $41^2+840^2=841^2$; and there are also several sets of reducible right triangles such as $44^2+240^2=244^2$, $84^2+880^2=884^2$.

Also solved by F. L. Manning, Simon Vatriquant and the proposer.

E 68. [1933, 606]. *Proposed by W. F. Cheney, Jr., Connecticut State College.*

In the triangle ABC, D is the midpoint of BC. The equilateral triangles, ABP, ACQ and ADR are drawn in the plane of triangle ABC, the vertices of each being listed counterclockwise. Prove that R is the midpoint of PQ.

I. *Solution by Roy MacKay, Albuquerque, New Mexico.*

The proposed theorem may be generalized as follows: If D is any point on the side BC of the triangle ABC and if similar polygons, $ABP_1P_2\cdots$, $ACQ_1Q_2\cdots$, and $ADR_1R_2\cdots$ are constructed in the plane of triangle ABC on AB, AC and AD as corresponding sides (with vertices listed counterclockwise

in order of subscripts), then R_i divides P_iQ_i in the same ratio that D divides BC.

Proof. Let P, Q and R be any three corresponding vertices. Then triangles ABP, ACQ and ADR are similar. Hence $AB/AP=AD/AR$. Also $\angle BAD = \angle PAR$, since $\angle BAP \pm \angle DAP = \angle DAR \pm \angle DAP$. Consequently triangles BAD and PAR are similar, so $\angle BDA = \angle PRA$, and $BD/PR=AD/AR$. Likewise triangles ADC and ARQ are similar, so $\angle ADC = \angle ARQ$, and $AD/AR=DC/RQ$. Hence PRQ is a straight line and $BD/DC=PR/RQ$.

II. *Solution by J. R. Musselman, Western Reserve University.*

If the notation of the problem as stated is changed by replacing A, B, C, P and Q by A_1, A_2, A_3, A_{12} and A_{13} respectively, and small a's with corresponding subscripts are used to represent the complex numbers which plot at the points designated by the capitals, then the following solution by complex numbers offers interesting sidelights on this problem.

Let D be the point on A_2A_3 dividing it in the ratio $m:n$. Then the coordinates of A_{12}, A_{13} and R are

$$-wa_1 - w^2a_2, \qquad -wa_1 - w^2a_3, \qquad \text{and} \qquad -wa_1 - \frac{w^2(na_2+ma_3)}{m+n},$$

where w is an imaginary cube root of unity. From this it is apparent that R is on the line $A_{12}A_{13}$ and divides it in the ratio of $m:n$. If $m=n$, R is the midpoint and the problem is solved, for its coordinate is then

$$-wa_1 - \frac{w^2(a_2+a_3)}{2}.$$

Suppose points R_2 and R_3 located by constructions similar to the one above by interchanging the subscripts 1 and 2 first, and then 2 and 3, so that a fairly symmetric construction appears about the three sides of the original triangle $A_1A_2A_3$. (Let us call the original R, R_1.) The coordinates of R_2 and R_3 will be $-wa_2-w^2(a_3+a_1)/2$ and $-wa_3-w^2(a_1+a_2)/2$ respectively. Then we can prove that the triangle $R_1R_2R_3$ is similar to $A_1A_2A_3$, has the same centroid, and has seven-fourths of its area.

Also if equilateral triangles $A_1A_2A_{21}$, $A_1A_3A_{31}$ and A_1DR_1' are drawn in the plane of triangle $A_1A_2A_3$, the vertices of each being listed clockwise (instead of counterclockwise), and if D is the midpoint of A_2A_3, then R_1' is the midpoint of $A_{21}A_{31}$. By similar constructions we obtain the points R_2' and R_3'. Now the triangle $R_1'R_2'R_3'$ is similar to $A_1A_2A_3$, has the same centroid, and seven-fourths of its area.

The six points, A_{13}, A_{21}, A_{32}, A_{31}, A_{12}, and A_{23} are of interest. The area of the triangle $A_{13}A_{21}A_{32}$ is five-halves that of $A_1A_2A_3$ plus $s^2\sqrt{3}/8$, where $s^2=\overline{A_1A_2}^2+\overline{A_2A_3}^2+\overline{A_3A_1}^2$. Similarly, the area of the triangle $A_{31}A_{12}A_{23}$ is five-halves that of $A_1A_2A_3$ minus $s^2\sqrt{3}/8$. Hence the sum of the areas of the triangles $A_{13}A_{21}A_{32}$ and $A_{31}A_{12}A_{23}$ is five times the area of the triangle $A_1A_2A_3$.

The hexagon $A_{12}A_{32}A_{31}A_{21}A_{23}A_{13}$ has twice the area of the triangle $A_1A_2A_3$. The hexagons $A_{12}A_{21}A_{32}A_{13}A_{31}A_{23}$, $A_{12}A_{31}A_{13}A_{21}A_{32}A_{23}$ and $A_{12}A_{31}A_{23}A_{32}A_{13}A_{21}$ are of the M type described in volume forty of this MONTHLY, 1933, p. 157. Their areas are $(\overline{A_1A_2}^2+\overline{A_3A_1}^2)\sqrt{3}/4$, $(\overline{A_2A_3}^2+\overline{A_3A_1}^2)\sqrt{3}/4$, and $(\overline{A_1A_2}^2+\overline{A_2A_3}^2)\sqrt{3}/4$ respectively, and the sum of these three areas is $s^2\sqrt{3}/2$.

Also solved by L. M. Bauer, T. C. Esty, S. E. Field, E. L. Harp, E. W. Holt, Olga Larson, G. R. Livingston, B. D. Roberts, J. Rosenbaum, E. P. Starke, C. W. Trigg, Simon Vatriquant, Maud Willey, E. N. Yeager and the proposer.

E 69. [1933, 607]. *Proposed by Raphael Robinson, University of California at Berkeley.*

Instead of a product of powers, a^bc^d, a printer accidentally prints the four-digit number, $abcd$. The value is however the same. Find the number and show that it is unique.

Editor's Note. This problem may be found in Dudeney's *Amusements in Mathematics* (Thomas Nelson and Sons), page 20.

Solution by C. W. Trigg, Cumnock College, Los Angeles.

1. Since $a^bc^d=abcd$, neither a^b nor c^d can have more than four digits. Hence the highest possible powers for any single digits in the problem are 9^4, 8^4, 7^4, 6^5, 5^5, 4^6, 3^8, 2^9 and 1^9.

2. Neither a nor c is zero, as that would make $abcd$ zero.

3. If $a=1$, or if $b=0$, then $abcd=c^d$. Similarly, if $c=1$ or $d=0$, then $abcd=a^b$. Examination of the expanded powers which have four digits shows that none meets these conditions. Therefore $a\neq 1$, $b\neq 0$, $c\neq 1$, and $d\neq 0$.

4. If one of a^b and c^d has four digits, the other is less than ten and so must be n^1, 2^2, 2^3 or 3^2. But no product of an eligible four-digit expanded power by any one of these numbers meets condition (1), so these possibilities are accordingly ruled out.

5. Since $abcd$ cannot end in zero, and since neither a nor c is one, $2000<abcd<10{,}000$. Since $abcd$ has four digits, the two factors cannot each be less than 31, for $31^2<1000$.

6. A test of the possible products formable from the remaining eligible expanded factors by multiplying the units digits and noting that the resultant units digit is not the same as either exponent, eliminates the majority of possibilities. Complete multiplication disposes of all others except $2^5\cdot 9^2=32\cdot 81=2592$, which is therefore the unique solution.

Also solved by Florence E. Allen, W. E. Buker, E. P. Starke, Simon Vatriquant and the proposer.

Advanced Problems

Send all communications about Advanced Problems and Solutions to Otto Dunkel, Washington University, St. Louis, Mo. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide.

Problems containing results believed to be new or extensions of old results are especially sought. The editorial work would be greatly facilitated, if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known textbooks or results found in readily accessible sources, will not be proposed as problems for solution in this department. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

PROBLEMS FOR SOLUTION

3679. *Proposed by J. M. Feld, Brooklyn College.*

Prove that for any positive integer p

$$p!\,h^p \sum_{k=0}^{n-1} (a + kh)^{p-1}$$

$$= \begin{vmatrix} (a+nh)^p - a^p & \binom{p}{2}h^2 & \binom{p}{3}h^3 & \cdots\binom{p}{p-1}h^{p-1} & h^p \\ (a+nh)^{p-1} - a^{p-1} & \binom{p-1}{1}h & \binom{p-1}{2}h^2 & \cdots\binom{p-1}{p-2}h^{p-2} & h^{p-1} \\ (a+nh)^{p-2} - a^{p-2} & 0 & \binom{p-2}{1}h & \cdots\binom{p-2}{p-3}h^{p-3} & h^{p-2} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ (a+nh)^2 - a^2 & 0 & 0 & \cdots\ \binom{2}{1}h & h^2 \\ (a+nh) - a & 0 & 0 & \cdots\ 0 & h \end{vmatrix}.$$

3680. *Proposed by William Hoover, Columbus, Ohio.*

A given ellipse moves in a plane so that it is always tangent to a fixed straight line at a given point. Derive the equation of the locus of its center.

3681. *Proposed by Elmer Schuyler, Bay Ridge High School, Brooklyn, N. Y.*

Construct the triangle ABC, given its orthocenter, and the mid-points of two of its sides.

3682. *Proposed by Mannis Charosh, New Utrecht High School, Brooklyn, N. Y.*

If a prime p has the form $p=4k+3$, and m is the number of quadratic non-residues less than $p/2$, prove that

(a) $$1\cdot 3\cdot 5 \cdots (p-2) \equiv (-1)^{m+k} \text{ mod. } p,$$

(b) $$2\cdot 4\cdot 6 \cdots (p-1) \equiv (-1)^{m+k+1} \text{ mod. } p.$$

3683. *Proposed by Raphael Robinson, University of California at Berkeley.*

Show that the sum of the medians of a simplex in n dimensions is smaller than $2/n$ and greater than $(n+1)/n^2$ times the sum of the edges of the simplex, and that these are the *best limits* that can be given.

This is a generalization of problem 3618, solution in this issue.

3684. *Proposed by W. H. Rasche, Virginia Polytechnic Institute.*

On the surface of a sphere given three points A, B, C; find the position of a fourth point P on the spherical surface such that $AP+BP+CP$ is a minimum, AP, BP, CP being arcs of great circles.

3685. *Proposed by Martin Rosenman, Brooklyn, New York.*

If A and B are two points in 3-space, we may write $A>B$, $A=B$ or $A<B$ according as the number of coordinates of A which exceed the corresponding coordinates of B is greater than, equal to, or less than the number of coordinates of B which exceed the corresponding coordinates of A. Thus $(5, 3, 1)>(4, 2, 6)=(1, 2, 7)$. Prove or disprove that there exist n points $A_1 \cdots A_n$ with any prescribed relations between A_i and A_j, $i\neq j$. (Note that we may have $A>B>C>A$.)

SOLUTIONS

3612 [1933, 298]. *Proposed by H. Grossman, New York.*

Prove that the ratio of the sum of the hth, $(h+k)$th, $(h+2k)$th, etc. coefficients to the sum of all the coefficients in the expansion of $(a+b)^n$ converges to the limit $1/k$, as n approaches ∞, where $h=1, 2, 3, \cdots, k$; and k is a positive integer.

Solution by Raphael Robinson, University of California at Berkeley.

Let ϵ be a primitive kth root of unity. Then if h is an integer, $0\leqq h<k$, we have

$$\sum_{\kappa=0}^{k-1}\epsilon^{-\kappa h}(1+\epsilon^\kappa)^n=\sum_{\kappa=0}^{k-1}\epsilon^{-\kappa h}\sum_{\nu=0}^{n}\binom{n}{\nu}\epsilon^{\kappa\nu}=\sum_{\nu=0}^{n}\binom{n}{\nu}\sum_{\kappa=0}^{k-1}(\epsilon^{\nu-h})^\kappa=k\sum_{\lambda=0}^{\infty}\binom{n}{h+\lambda k}.$$

To justify the last step, let δ be the greatest common divisor of $\nu-h$ and k. Then $\epsilon^{\nu-h}$ is a primitive (k/δ)th root of unity. Hence the inner sum in the next to the last step is the sum of all the (k/δ)th roots of unity, each counted δ times. But the sum of the (k/δ)th roots of unity is 0 unless $k/\delta=1$, in which case it is 1. But in this case $\delta=k$, i.e., k divides $\nu-h$, so that $\nu=h+\lambda k$. The upper limit has been written ∞ but the terms vanish of themselves as soon as $h+\lambda k>n$.

Hence

$$\lim_{n\to\infty}\frac{\sum_\lambda\binom{n}{h+\lambda k}}{\sum_\nu\binom{n}{\nu}}=\lim_{n\to\infty}\frac{\frac{1}{k}\sum_{\kappa=0}^{k-1}\epsilon^{-\kappa h}(1+\epsilon^\kappa)^n}{2^n}=\frac{1}{k}\sum_{\kappa=0}^{k-1}\epsilon^{-\kappa h}\lim_{n\to\infty}\left(\frac{1+\epsilon^\kappa}{2}\right)^n=\frac{1}{k},$$

since $|1+\epsilon^\kappa|/2<1$ unless $\kappa=0$. This agrees with the statement of the problem except that we have considered the $(h+1)$th, $(h+k+1)$th, $(h+2k+1)$th, etc., coefficients for $h=0, 1, 2, \cdots, k-1$.

Note by the Editors. The last step in the first display may also be verified in another manner. If $\nu-h$ is a multiple of k, the inner sum in the previous step is obviously k. If it is not a multiple of k, then the sum is

$$\frac{\epsilon^{(\nu-h)k}-1}{\epsilon^{\nu-h}-1}=0,$$

since the numerator is zero, and, ϵ being a primitive kth root, the denominator is not zero.

Solved also by R. MacKay.

3613 [1933, 298]. *Proposed by V. F. Ivanoff, San Francisco, Calif.*

Prove that

$$(R_x\cos\theta)^{2/3}+(R_y\cos\phi)^{2/3}+(R_z\cos\psi)^{2/3}=2R^{2/3},$$

where R is the radius of curvature of a given curve at the given point (x_1, y_1, z_1); R_x, R_y, R_z are the radii of curvature of the projections of this curve on the coordinate planes YOZ, XOZ and XOY at the points (y_1, z_1), (x_1, z_1) and (x_1, y_1), respectively; and θ, ϕ and ψ are the angles between the binormal to the curve and the coordinate axes.

Solution by M. L. Vest, Morgantown, W. Va.

Let the given curve be

$$x=f_1(t),\quad y=f_2(t),\quad z=f_3(t).$$

Then

$$\cos\theta=R\frac{dyd^2z-dzd^2y}{ds^3},\quad \cos\phi=R\frac{dzd^2x-dxd^2z}{ds^3},\quad \cos\psi=R\frac{dxd^2y-dyd^2x}{ds^3},$$

or, using the notation: $A=dyd^2z-dzd^2y$; $B=dzd^2x-dxd^2z$; $C=dxd^2y-dyd^2x$; we have

$$\cos\theta=\frac{RA}{ds^3},\qquad \cos\phi=\frac{RB}{ds^3},\qquad \cos\psi=\frac{RC}{ds^3}. \tag{1}$$

Now, the projection of the curve on the coordinate plane YOZ is the plane curve $y=f_2(t)$, $z=f_3(t)$. The radius of curvature of this plane curve at the point (y_1, z_1) can be written as

$$R_x=\frac{(dy^2+dz^2)^{3/2}}{dyd^2z-dzd^2y}=\frac{(dy^2+dz^2)^{3/2}}{A}.$$

Similarly,

$$R_y = \frac{(dz^2 + dx^2)^{3/2}}{B}, \qquad R_z = \frac{(dx^2 + dy^2)^{3/2}}{C}.$$

Then, using these values and those in (1),

$$\begin{aligned}(R_x \cos\theta)^{2/3} &+ (R_y \cos\phi)^{2/3} + (R_z \cos\psi)^{2/3} \\ &= \frac{R^{2/3}(dy^2 + dz^2)}{ds^2} + \frac{R^{2/3}(dz^2 + dx^2)}{ds^2} + \frac{R^{2/3}(dx^2 + dy^2)}{ds^2} \\ &= \frac{2R^{2/3}(dx^2 + dy^2 + dz^2)}{ds^2} = 2R^{2/3}.\end{aligned}$$

Solved also by E. F. Allen and S. Vatriquant.

Note by the Editors. For problems of this nature there is considerable advantage in the use of vectors, since with a slight knowledge of the simple rules for vector combinations, there is no necessity for recalling the complicated formulae of differential geometry expressed in rectangular coordinates. Let $\boldsymbol{t}$ be the unit vector tangent to the curve at a point taken in the direction of increasing arc length s; $\boldsymbol{n}$, the unit vector principal normal directed toward the center of plane curvature; $\boldsymbol{b}$, the unit vector binormal taken with the direction so that $\boldsymbol{t}, \boldsymbol{n}, \boldsymbol{b}$ form a right-hand system, i.e., so that $\boldsymbol{t}=\boldsymbol{n}\times\boldsymbol{b}$, $\boldsymbol{n}=\boldsymbol{b}\times\boldsymbol{t}$, $\boldsymbol{b}=\boldsymbol{t}\times\boldsymbol{n}$. The only formula that we now need is one which is almost obvious. It is $K\boldsymbol{n}=\boldsymbol{t}'$, where $K=R^{-1}$ is the curvature, and the accent means differentiation with respect to s. If x, y, z for the point of the curve are expressed in terms of s, and $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ are unit vectors forming a right-hand system parallel to the x, y, z axes, respectively, then the vector $OP=\boldsymbol{r}=x\boldsymbol{i}+y\boldsymbol{j}+z\boldsymbol{k}$. We have at once

$$\boldsymbol{t} = x'\boldsymbol{i} + y'\boldsymbol{j} + z'\boldsymbol{k}, \tag{1}$$

$$K\boldsymbol{n} = x''\boldsymbol{i} + y''\boldsymbol{j} + z''\boldsymbol{k}, \tag{2}$$

$$K\boldsymbol{b} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ x' & y' & z' \\ x'' & y'' & z'' \end{vmatrix} = A\boldsymbol{i} + B\boldsymbol{j} + C\boldsymbol{k}. \tag{3}$$

If the parameter is $t\neq s$, then there are slight changes in the left sides: they are in this case (1)′ $s'\boldsymbol{t}$; (2)′ $K(s')^2\boldsymbol{n}+s''\boldsymbol{t}$; (3)′ $K(s')^3\boldsymbol{b}$; where $s'=ds/dt$.

For the projection on the xy-plane we may choose the direction of the arc length s_z so that $\boldsymbol{t}_z, \boldsymbol{n}_z, \boldsymbol{k}$ form a right-hand system of unit vectors. Then from (1)′ result three equations such as $(s_z')^2=(x')^2+(y')^2$, and from (1) we then have

$$2 = (s_x')^2 + (s_y')^2 + (s_z')^2. \tag{4}$$

From (3) results $K\boldsymbol{b}\cdot\boldsymbol{k}=C=K\cos\psi$; also from (3)′ we have $K_z(s_z')^3=C$. Hence $R(s_z')^3=R_z\cos\psi$, and there are two more similar equations. These with (4) give the desired result.

We may also obtain a result for 3447 [1930, 381], a solution of which is printed [1931, 343]. From the vector product $\boldsymbol{n}_z=\boldsymbol{k}\times\boldsymbol{t}_z$ and (1)′, we derive

$$s_z' t_z = x'i + y'j,$$
$$s_z' n_z = x'j - y'i.$$

Then from (2), $Ks_z' n \cdot n_z = x'y'' - y'x'' = C$, and this with a result above gives $K_z(s_z')^3 = Ks_z' \cos\gamma$, where $\cos\gamma = n \cdot n_z$. Thus, if $s_z' \neq 0$, we have $R(s_z')^2 = R_z \cos\gamma$. This with two other similar equations and (4) give

$$2R = R_x \cos\alpha + R_y \cos\beta + R_z \cos\gamma.$$

If $s_z' \neq 0$, we have also $s_z' = t \cdot t_z$; and if s_z' approaches zero at the given point, the curve must at the given point be perpendicular to the xy-plane.

3615 [1933, 299]. *Proposed by B. D. Roberts, New Mexico Normal University.*

A dust storm contains particles of two kinds identical except as to color, brown and yellow particles existing in the ratio of 3:2. If five particles of this dust enter my eye at random, determine the probability that two of them are brown and the other three are yellow.

Solution by F. L. Manning, Ursinus College, Collegeville, Pa.

The probability that any one particle be brown is .6, and that it be yellow is .4.

The probability that the five particles arrive at the eye in the order (*bbyyy*) is the product of the two probabilities given above or $(.6)^2(.4)^3$. These particles may arrive in other orders to make a total of two brown and three yellow. The number of such ways is the number of arrangements of five things of which two are alike and three are alike. Therefore, the required probability is

$$\frac{5!}{2!\,3!}(.6)^2(.4)^3 = \frac{144}{625} = .2304.$$

This result can be arrived at immediately by taking the fourth term of the binomial expansion $(.6+.4)^5$.

All this is on the assumption that there is an infinite number of dust particles in the air. Then the problem becomes like one in which marbles are drawn from an urn and replaced each time so that the probabilities on successive drawings remain constant.

Solved also by S. Vatriquant, Maud Willey, and the proposer.

A Note by the Editors. We may regard the five particles as entering the eye one after the other, and in this case (*bbyyy*) is the order in time of entry. Or we may regard the five particles as entering simultaneously, and in this case the order is that of position.

If we assume that $5n$ is the total number of particles, the probability is

$$\binom{3n}{2}\binom{2n}{3}\bigg/\binom{5n}{5},$$

and for $n \to \infty$, we obtain the above result.

3618 [1933, 363]. *Proposed by N. A. Court, University of Oklahoma.*

The sum of the medians of a tetrahedron (i.e. the lines joining the vertices to the centroids of the opposite faces) is smaller than two thirds and greater than four ninths of the sum of the edges of the tetrahedron.

Solution by Frank Ayres, Jr., Dickinson College.

Consider the tetrahedron $A_1A_2A_3A_4$ for which the sum of the edges is P; and let the centroid of the face F_i, opposite A_i, having the perimeter P_i, be denoted by B_i. Let the required sum be represented by S. Since

$$A_iB_i < B_iB_j + A_iB_j, \quad i \neq j,\ i, j = 1, 2, 3, 4, \tag{1}$$

we obtain by summing both sides for all values of i and j admissible

$$3S < 2\sum_{i<j} B_iB_j + \sum_j\left(\sum_i A_iB_j\right). \tag{2}$$

Now

$$\sum_{i<j} B_iB_j = \frac{1}{3}\sum_{i<j} A_iA_j = \frac{1}{3}P, \qquad \sum_i A_iB_j < \frac{2}{3}P_j, \tag{3}$$

and hence

$$3S < \frac{2}{3}P + \frac{4}{3}P, \quad \text{or} \quad S < \frac{2}{3}P. \tag{4}$$

Again

$$A_iB_i + A_jB_j > A_iA_j + B_iB_j = \frac{4}{3}A_iA_j. \tag{5}$$

Hence

$$3S = \sum_{i<j}(A_iB_i + A_jB_j) > \frac{4}{3}\sum_{i<j} A_iA_j = \frac{4}{3}P,$$

$$S > \frac{4}{9}P.$$

Solved also by J. W. Clawson, G. S. Jones, R. MacKay, A. Pelletier, J. Rosenbaum, H. D. Ruderman, W. P. Udinski, and Maud Willey.

Note by the Editors. Some of the less obvious steps of the above solution may be verified as follows: In the face $A_kA_hA_i$ containing B_j, let A_iB_j cut A_hA_k in M_{ij}. Then

$$A_iB_j = \frac{2}{3}A_iM_{ij} < \frac{2}{3}\,\frac{A_iA_h + A_iA_k}{2},$$

$$\sum_{i<j} A_iB_j < \frac{2}{3} P_j.$$

Also A_jB_i passes through M_{ij}, and B_iB_j is parallel to and one third the length of A_iA_j. If A_iB_i and A_jB_j meet in N_{ij}, then two inequalities from the triangles $A_iA_jN_{ij}$ and $B_iB_jN_{ij}$ give a single inequality

$$A_iB_i + A_jB_j > A_iA_j + B_iB_j.$$

NEWS AND NOTICES

Readers are invited to contribute to the general interest of this department by sending items to Professor J. H. Weaver, Ohio State University, Columbus, Ohio.

A banquet was recently given in honor of three members of the staff of the department of mathematics of the University of Wisconsin. The men so honored were Professor Charles Slichter, Dean of the Graduate School, Professor E. B. Skinner, and Emeritus Professor E. B. Van Vleck. A number of representatives from other universities were present. The visitors included Professor G. D. Birkhoff of Harvard, Professors G. A. Bliss and E. P. Lane, of Chicago, and Professor T. F. Holgate, of Northwestern University. Dean Slichter has served the University of Wisconsin for forty-seven years, Professor Skinner for forty-one years, and Professor Van Vleck for twenty-eight years. The total record of service is one hundred sixteen years.

Professor Gilbert Ames Bliss has been awarded the Martin A. Ryerson Distinguished Service Professorship for his brilliant and fruitful service to the University of Chicago. Of the nine Distinguished Service Professorships at the University of Chicago, two are held by mathematicians, the Eliakim Hastings Moore Distinguished Service Professorship having been awarded to Leonard Eugene Dickson several years ago.

E. W. Brown, Emeritus Professor of Mathematics at Yale University, read a paper, "Time and Determination," before the American Philosophical Society on February 2nd.

The Edison Medal of the American Institute of Electrical Engineers has been awarded to Professor A. E. Kennelly of Harvard University for his meritorious achievements in electrical science, electrical engineering, and electrical arts, as exemplified by his contributions to the theory of electrical transmissions and the development of international electrical standards.

The Gold Medal of Villanova College has been awarded to the Abbé Georges Le Maître of the University of Louvain, visiting professor of astronomy at the Catholic University, Washington.

Sir Arthur Stanley Eddington has been appointed Messenger lecturer at Cornell University for 1934. He will deliver a series of twelve lectures there on

"New Pathways of Science," in April and May.

Doctor Joseph Eugene Rowe, who resigned the presidency of Clarkson College in June 1932, has since that time been engaged in research in the social sciences at Johns Hopkins University and in Baltimore. He has been appointed a member of the Board of Veterans Appeals for Maryland.

Professor C. D. Killebrew, of the department of mathematics at the University of Alabama, died March 9, 1934. He was a member of the Mathematical Association.

E. R. Mathews, University Fellow in mathematics at the University of California, Berkeley, died recently.

J. M. Poor, professor of astronomy at Dartmouth College and head of the Shattuck Observatory, died December 11, 1933, at the age of sixty-two.

Professor Arthur Ranum, of the department of mathematics at Cornell University, died on February 28, 1934, in his sixty-fourth year. He was a charter member of the Mathematical Association.

Professor Paul L. Saurel, for many years head of the department of mathematics at the College of the City of New York, died in Paris, January 21, 1934, at the age of sixty-two. He was a charter member of the Mathematical Association.

The following courses in mathematics are announced for the summer, 1934.

Columbia University, July 9 to August 17. In addition to the courses listed in the April number of this MONTHLY, Dr. Vera Sanford will give a course, History of mathematics.

The University of North Carolina, first term, June 13 to July 24; second term July 25 to August 31. In addition to the usual courses through the calculus, the following courses will be offered: By Professor Winsor: Introduction to modern geometry; History of mathematics. By Professors Linker and Hill: Differential equations. By Professors Browne and Lasley: Analytic projective geometry. By Professors Hoyle and Mackie: Theory of functions of a complex variable. By Professor Browne: Modern higher algebra. By Professors Hoyle and Lasley: Differential geometry. By Professor Munch: Teaching of mathematics.

University of Southern California, first term, June 18 to July 27. The following advanced courses are offered: By Professor L. D. Ames: History of mathematics; Introduction to modern geometry; Theory of probability and statistics. By Professor L. E. Gurney: Modern higher algebra; Mathematical astronomy. Second term, July 28 to August 31. By Associate Professor D. V. Steed: Vector analysis; Riemannian geometry; Seminar (subject to be announced during first term).

The Carus Mathematical Monographs

The CARUS MONOGRAPHS are already fulfilling their mission as intended by the generous donor, MRS. MARY HEGELER CARUS, and her son, DR. EDWARD H. CARUS.

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It is believed that the ASSOCIATION is rendering a great service to mathematics by this enterprise, and a liberal support from the membership constitutes an appropriate vote of confidence in the undertaking.

MONOGRAPHS THUS FAR PUBLISHED

No. 1. *Calculus of Variations*, by PROFESSOR G. A. BLISS. (First Impression, 1925; Second Impression, 1927.)

No. 2. *Analytic Functions of a Complex Variable*, by PROFESSOR D. R. CURTISS. (First Impression, 1926; Second Impression, 1930.)

No. 3. *Mathematical Statistics*, by PROFESSOR H. L. RIETZ. (First Impression, 1927; Second Impression, 1929.)

No. 4. *Projective Geometry*, by PROFESSOR J. W. YOUNG. (First Impression, 1930.)

No. 5. *History of Mathematics in America before 1900*, by PROFESSOR DAVID EUGENE SMITH and DOCTOR JEKUTHIEL GINSBURG. (Ready for distribution, Mar. 1.) Special price to members, $1.25 each.

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BOOKS FOR REVIEW should be addressed to R. A. JOHNSON, Brooklyn College, 66 Court Street, Brooklyn, N.Y.

BUSINESS CORRESPONDENCE should be addressed to the SECRETARY-TREASURER of the Association, W. D. CAIRNS, 33 Peters Hall, Oberlin, Ohio.

CHANGE OF ADDRESS: Members should send notice of any change of address to the SECRETARY-TREASURER, W. D. CAIRNS, Oberlin, Ohio, before the 10th of each month.

MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Eighteenth Summer Meeting of the Association, Williamstown, Mass., Sept. 3-4, 1934.

The following is a list of the sections of the Association, with dates of those section meetings which have been scheduled for 1934 and reported to the Secretary.

ALLEGHENY MOUNTAIN, Pittsburgh, Pa., Feb. 10; Washington, Pa., May 5.
ILLINOIS, Jacksonville, May 4-5.
INDIANA, La Fayette, May 11-12.
IOWA, Des Moines, April 20-21.
KANSAS, Topeka, Mar. 17.
KENTUCKY, May.
LOUISIANA-MISSISSIPPI, Jackson, Miss., Mar. 23-24.
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Williamsburg, Va., May.
MICHIGAN, Ann Arbor, Mar. 17.
MINNESOTA, Northfield, May 12.
MISSOURI.
NEBRASKA, Crete, Apr. 27.
OHIO, Columbus, Apr. 5.
OKLAHOMA, Oklahoma City, Feb. 9.
PHILADELPHIA, Philadelphia, Dec. 1.
ROCKY MOUNTAIN, Colorado Springs, Apr. 20-21.
SOUTHEASTERN, University, Ala., Mar. 30-31.
SOUTHERN CALIFORNIA, Riverside, Mar. 3.
TEXAS, Apr. 17.
WISCONSIN, Oshkosh, May 5.

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Ernest Williams, B.S. (Birmingham- Southern) Instr., Alabama Poly. Inst., Auburn, Ala.

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W. D. Cairns, *Secretary*

THE MARCH MEETING OF THE SOUTHERN CALIFORNIA SECTION

The fourteenth regular meeting of the Southern California Section was held at Riverside Junior College, Riverside, California, Saturday, March 3, 1934. Professor O. W. Albert presided.

The attendance was forty-two, including the following twenty-two members of the Association: O. W. Albert, L. D. Ames, Harry Bateman, Clifford Bell, E. T. Bell, P. H. Daus, Iva B. Ernsberger, Raymond Garver, Harriet E. Glazier, G. H. Hunt, C. G. Jaeger, Glenn James, Mary N. Keith, G. A. Linhart, G. F. McEwen, Lena E. Reynolds, G. E. F. Sherwood, Morgan Ward, L. E. Wear, W. M. Whyburn, Clyde Wolfe, Euphemia R. Worthington.

The following officers were elected for the next year: Chairman, Prof. L. E. Wear, California Institute of Technology; Program Committee, Prof. D. V. Steed, University of Southern California, and Prof. C. G. Jaeger, Pomona College. The next meeting was tentatively scheduled for March 2, 1935, at the University of Southern California.

The following five papers were read:

1. "Tests of reality of regularities indicated in sequences of observations" by Professor G. F. McEwen, professor of physical and dynamic oceanography, The Scripps Institution of Oceanography of the University of California.

2. "Reducibility criteria for polynomials" by Professor Morgan Ward, California Institute of Technology.

3. "On sums involving combination numbers" by Professor P. H. Daus, University of California at Los Angeles.

4. "Selective functions and operations" by Professor Harry Bateman, California Institute of Technology.

5. "The mathematical interpretation of the classification of sand particles by means of differential screens" by Doctor G. A. Linhart, Riverside Junior College.

Abstracts of the papers follow, the numbers corresponding to the numbers in the list of titles.

1. Suppose the magnitudes of a series of observations exhibit a regularity, that is, a continuous increase or decrease, a maximum, or a minimum. Is this indicated regularity real, that is, is it true of the universe sampled or is it the result of sampling from a random sequence?

Two cases are considered. First, only the sequence of values is available, without any information as to how it was obtained. Second, a series is formed from averages of groups selected in a known systematic way from a series of individual observations.

Methods appropriate to each case are derived. The first involves the ratio of the number of permutations having the indicated regularity, to the total number and is expressed in terms of binomial coefficients. The second depends upon determining the theoretical frequency distribution of the means of samples from that of the individual observations. The primary object is to provide additional methods of testing the reality of cycles.

2. Tschirnhausen transformations are employed to show that the presence of an irreducible factor of degree four or less for any polynomial is equivalent to the presence of a linear factor in a certain associated polynomial.

3. If in the binomial expansion of $(1+x)^n$ we set x equal to various complex roots of unity, and then combine these relations, we obtain sums involving the binomial coefficients. The use of such sums was illustrated by finding the power series expansion formed from the multiplication of two known series.

4. A function $f(m, n)$ of two integers m, n is said to be selective when its value is zero when $m \neq n$, and is different from zero when $m = n$. Many examples of selective functions are given. An operation f_n, which operates on a function $g(x)$, is said to be selective, when $f_m f_n g(x)$ is zero for a special value of x when $m \neq n$ and is different from zero when $m = n$. Examples of selective operations are given and uses are found for them.

5. In a recent article in *Industrial and Engineering Chemistry* experimental results are published dealing with "the variation in screen analysis with position studied for sand suspended in water. The coarse material is found to concentrate beneath the paddle, while the fine material is distributed irregularly through the mass of liquid. The use of the simple agitator as a classifier is suggested."

The results thus obtained lend themselves readily to mathematical interpretation which leads to some very interesting results.

P. H. DAUS, *Secretary*

RECOMMENDATIONS CONCERNING DEMONSTRATIVE GEOMETRY AND ADVANCED MATHEMATICS FROM THE ASSOCIATION OF TEACHERS OF MATHEMATICS IN NEW ENGLAND

At a meeting held in Cambridge, January 20, 1934, at which 80 members were present, the Association passed the following votes concerning geometry, for communication to the Committee on Geometry of the National Council of Teachers of Mathematics and to the Commission on Mathematics of the College Entrance Examination Board. These votes grew out of a series of resolutions which were framed by two committees of this Association, one committee representing the eastern part of New England, the other the Connecticut Valley Branch of this Association.

1. The requirement in geometry, as officially defined, should neither explicitly nor by implication suggest any teaching order, nor should full credit on examination be denied on the ground that any one theorem must necessarily precede some other theorem. (Yea 31, Nay 5)

2. Trivial demonstrations can be excluded by some such regulation as the following: "When a theorem is proved merely by showing it to be an obvious corollary of another theorem, the proof of that other theorem is required." (Yea 40, Nay 2)

3. The requirement should be stated in the form of a list of theorems arranged by topics but without implying any definite teaching order. (Yea 40, Nay 1) There should be starred propositions if the examination is to cover book theorems. (Yea 15, Nay 10)

4. The use of algebra and of trigonometric ratios should be encouraged for demonstrations as well as for problems of computation. For example, the statement and the proof of the theorems referring to the third side of a triangle when the other two sides and their included angle are given become in algebraic form not only simpler but more comprehensive. (Algebra: Yea 45, Nay 0. Trigonometry: (Yea 30, Nay 2)

5a. It should no longer be permissible to ignore incommensurable magnitudes; some form of the theory of limits, or some fairly adequate treatment of irrational numbers should be required in geometry. (Yea 35, Nay 4)

5b. This requirement should be enforced by *questions set on examination* from time to time. (Yea 10, Nay 30)

6. Some effort should be made to bring to the attention of pupils the invaluable logical content of geometry, and thus make that content more readily available in matters not connected with geometry. (Yea 35, Nay 0)

7. Pupils should be encouraged to quote a theorem in a general and comprehensive form, rather than in a special case: for example C47* instead of C48 or C49 and C30* instead of C42. The notation C47*, etc., refers to Document 108 of the College Entrance Examination Board. (Yea 20, Nay 8)

8. Teachers should be advised not to require extreme formality in demon-

stration, or to insist on such rigmaroles as "$AB=AB$ by identity," or on reference to axioms of algebra which are never referred to in their regular algebraic work. (Yea 40, Nay 2)

9. No formal distinction should be made for examination purposes between "book theorems" and "originals." The purpose of this change is to give greater freedom in teaching. (Yea 40, Nay 3)

10. Theorems which require superposition for their proof may be postulated. (Yea 35, Nay 4)

11. The usual list of postulates may be extended. (Yea 25, Nay 2)

12. Some terms ostensibly defined ought to be taken as undefined. (Yea 15, Nay 5)

13. It is recommended that much of the factual content of plane and solid geometry be taught as informal geometry in grades 7, 8, 9. (Yea 25, Nay 1)

14. Some pertinent ideas from solid geometry ought to be exhibited concurrently with a first course in demonstrative geometry. (Yea 30, Nay 0)

15. Geometry should be taught so as to obtain transfer of logical training to non-geometric situations. (Yea 15, Nay 0)

16. The logical structure of geometry should be emphasized. (Yea 20, Nay 1)

At a meeting in Cambridge on March 10, 1934, at which 48 members were present, the Association passed the following votes concerning advanced mathematics, for communication to the Commission on Mathematics of the College Entrance Examination Board.

1. A full year course consisting of trigonometry, solid geometry, and advanced algebra is to be preferred to the present practice of offering half-year courses in only two of these three subjects. (Only one dissenting vote)

2. Several votes were taken to determine the sense of the meeting concerning the amount of time to be devoted to each subject in such a course.

(a) With respect to trigonometry, only 3 were opposed to giving less than a half-year course in addition to the instruction in trigonometry now commonly included in elementary algebra; 25 would accept a reduction to 40 percent of the year; and 16 of these latter felt that 33 percent of the year was sufficient for the trigonometry.

(b) With respect to solid geometry, a few desired at least 40 percent of the year; 5 would not oppose a reduction to 25 percent of the year; but the majority wanted at least 33 percent of the year for this subject.

(c) With respect to advanced algebra, the theory of equations was unanimously regarded as the most important topic; next in importance came permutations, combinations, and probability, with pertinent reference to the binomial theorem and compound interest. With 40 percent of the year alloted to trigonometry, and 33 percent to solid geometry, there remains 27 percent of the year for these topics in algebra.

3. (a) Only 6, of whom 5 were secondary school teachers, favored mention of the derivative in connection with the theory of equations.

(b) A committee of the Connecticut Valley Branch of this Association

favored a year course in advanced mathematics 40 percent of which should be devoted to trigonometry, and about 20 percent to solid geometry, making possible an allotment of 30 percent of the year to algebra, including certain topics from analytic geometry, and an allotment of 10 percent to the study of slopes and areas by means of differentiation and integration.

4. The College Entrance Examination Board is requested to consider the desirability of replacing the present examinations in advanced mathematics by a single examination on an undivided year course in advanced mathematics, the paper to be so constructed that questions must be answered in each subject while permitting sufficient option to allow teachers some latitude in selecting material for instruction.

A THEOREM ON VOLTERRA INTEGRAL EQUATIONS OF THE SECOND KIND WITH DISCONTINUOUS KERNELS

By W. M. RUST, Jr., Rice Institute

An integral equation is one in which the unknown function occurs under one or more signs of integration just as a differential equation involves one or more derivatives of the unknown function. And whereas a differential equation has in general infinitely many continuous solutions involving arbitrary constants that must be determined by the boundary conditions, the more familiar integral equations have, with proper restrictions, one and only one continuous solution. A typical integral equation is

$$(A) \qquad u(x) = f(x) + \int_0^x K(x, y)u(y)dy$$

where $u(x)$ is the unknown function and $f(x)$ and $K(x, y)$ are known functions.

The type of equation represented by (A) is called a Volterra integral equation of the second kind, because the first systematic treatment of it was given by Volterra. A Volterra integral equation of the first kind differs from (A) in that the left hand member is zero rather than $u(x)$, thus the unknown function occurs *only* under the sign of integration. A different type of integral equation has a fixed upper limit of integration instead of a variable one, this is called Fredholm's integral equation.[1]

The classical result for equation (A) is stated by the following theorem.

THEOREM I. *If $f(x)$ is bounded and continuous for $a \leqq x \leqq b$ and if $K(x, y)$ is bounded and continuous for $a \leqq y \leqq x \leqq b$, equation (A) has one and only one solution $u(x)$ bounded and continuous for $a \leqq x \leqq b$.*

[1] A brief and readable account of the more elementary and useful portions of the theory of integral equations is given in Bôcher's *Introduction to the Study of Integral Equations.* A more extensive treatment is found in Volterra's *Leçons sur les Équations Intégrales.* A fairly complete bibliography extending to 1927 is contained in the German edition of Vivanti's *Integralgleichungen.*

The proof of this theorem is to be found in any of the references given above; moreover it can be shown that the solution is

$$u(x) = f(x) + \int_0^x k(x, y)f(y)dy, \tag{B}$$

where

$$k(x, y) = \sum_{i=1}^{\infty} K_i(x, y),$$

the functions $K_i(x, y)$ being known as the *iterated functions* and defined by the equations

$$K_1(x, y) = K(x, y) \tag{1}$$

$$K_i(x, y) = \int_x^y K_1(x, z)K_{i-1}(z, y)dz. \tag{2}$$

From this definition of the iterated functions it can easily be shown by mathematical induction that

$$K_{m+i}(x, y) = \int_x^y K_m(x, z)K_i(z, y)dz. \tag{3}$$

Theorem I admits certain generalizations. From the form of the solution given by (B) we see that if the solution $u(x)$ is to be continuous, the function $f(x)$ must be continuous, but that it might be possible to have $K(x, y)$ discontinuous and still have a continuous solution. This is true; one such generalization is to have $K(x, y)$ bounded and with regularly distributed discontinuities. Another generalization is to have $K(x, y)$ of the form $G(x, y)/(x-y)^\alpha$, where $G(x, y)$ is bounded and continuous in the region considered and $0<\alpha<1$. A generalization in a direction not considered here is to admit solutions with certain sorts of discontinuities; such a generalization was discussed by G. C. Evans.[1]

The generalization which forms the main subject of this paper is proven by a method of some intrinsic interest. It is stated by the following theorem.

THEOREM II. *The Volterra integral equation*

$$u(x) = f(x) + \int_0^x K(x, y)u(y)dy \tag{A}$$

has one and only one solution $u(x)$ that is bounded and continuous for $a \leqq x \leqq b$ if $f(x)$ is bounded and continuous for $a \leqq x \leqq b$ and if $K(x, y)$ satisfies the following two conditions for $a \leqq y \leqq x \leqq b$:

(i) *If $r(x)$ is bounded and continuous for $a \leqq x \leqq b$ so is $\int K(x, y)r(y)dy$.*

[1] Transactions of the American Mathematical Society, vol. 11 (1910), p. 429.

(ii) *For n great enough the iterated function* $K_{2n}(x, y)$ *is bounded and continuous*[1] *for* $a \leqq y \leqq b$.

For the proof of this theorem we need four Lemmas.

LEMMA I. *If* $r(x)$ *is bounded and continuous so is* $\int_0^x K_m(x, y)r(y)dy$.

The proof is by mathematical induction. Condition (i) is the Lemma stated for $m=1$. Suppose it holds for $m-1$. We see that

$$\int_0^x K(x, z)dz \int_0^z K_{m-1}(z, y)r(y)dy = \int_0^x K_m(x, y)r(y)dy$$

by a change of order of integration in the lefthand member and an application of the definition (2). But condition (i) tells us that the left hand member is bounded and continuous and hence the right hand member is, and the Lemma is established.

By the use of the iterated functions we define the following set of $n+1$ equations.

$$(0) \qquad u_0(x) = f(x) + \int_0^x K_1(x, y)u_0(y)dy$$

$$(1) \qquad u_1(x) = f(x) + \int_0^x K_1(x, y)f(y)dy + \int_0^x K_2(x, y)u_1(y)dy$$

. . . .

$$(k) \qquad u_k(x) = f(x) + \sum_{m=1}^{2k-1} \int_0^x K_m(x, y)f(y)dy + \int_0^x K_{2^k}(x, y)u_k(y)dy$$

. . . .

$$(n) \qquad u_n(x) = f(x) + \sum_{m=1}^{2n-1} \int_0^x K_m(x, y)f(y)dy + \int_0^x K_{2^n}(x, y)u_n(y)dy.$$

LEMMA II. *Equation* (n) *has one and only one bounded, continuous solution.*

This follows from Theorem I since by the condition on $f(x)$ and condition (ii) and Lemma I the known functions are bounded and continuous.

LEMMA III. *A bounded, continuous solution of equation* (k) *is a solution of equation* $(k-1)$.

Proof. Let $u_k(x)$ be a bounded, continuous solution of equation (k). Then the function $e(x)$ defined by

$$(a) \quad \epsilon(x) + u_k(x) = f(x) + \sum_{m=1}^{2k-1-1} \int_0^x K_m(x, y)f(y)dy + \int_0^x K_{2^{k-1}}(x, y)u_k(y)dy$$

[1] From condition (i) it follows that if any iterated function is bounded and continuous so are all subsequent ones.

is bounded and continuous.

Subtracting (k) from (a) gives

$$
(b)\qquad
\begin{aligned}
e(x) &= -\sum_{m=2^{k-1}}^{2^k-1}\int_0^x K_m(x, y)f(y)dy + \int_0^x K_{2^k-1}(x, y)u_k(y)dy \\
&\qquad - \int_0^x K_{2^k}(x, y)u_k(y)dy \\
&= -\sum_{m=2^{k-1}+1}^{2^k-1}\int_0^x K_m(x, y)f(y)dy - \int_0^x K_{2^k}(x, y)u_k(y)dy \\
&\qquad + \int_0^x K_{2^k-1}(x, y)[u_k(y) - f(y)]dy.
\end{aligned}
$$

In (b) replace $[u_k(y)-f(y)]$ by its value from equation (a), this gives

$$
(c)\qquad
\begin{aligned}
e(x) = &-\sum_{m=2^{k-1}+1}^{2^k-1}\int_0^x K_m(x, y)f(y)dy - \int_0^x K_{2^k}(x, y)u_k(y)dy \\
&+ \int_0^x K_{2^k-1}(x, y)dy\Bigg[- e(y) + \sum_{m=1}^{2^{k-1}-1}\int_0^y K_m(y, z)f(z)dx \\
&+ \int_0^y K_{2^k-1}(y, z)u_k(z)dz\Bigg].
\end{aligned}
$$

When we use the property (3) of the iterated functions all the terms in the right hand member cancel except the one involving $e(x)$ giving

$$
(d)\qquad e(x) = -\int_0^x K_{2^k-1}(x, y)e(y)dy,
$$

and so

$$
\begin{aligned}
\int_0^x K_{2^k-1}(x, y)e(y)dy &= -\int_0^x K_{2^k-1}(x, y)dy\int_0^y K_{2^k-1}(y, z)e(z)dz \\
&= -\int_0^x K_{2^k}(x, y)e(y)dy,
\end{aligned}
$$

so that

$$
e(x) = \int_0^x K_{2^k}(x, y)e(y)dy.
$$

By repeating this process we finally have

$$
(e)\qquad e(x) = \int_0^x K_{2^n}(x, y)e(y)dy.
$$

But since $K_{2^n}(x, y)$ is bounded, by condition (ii), say less than M, we have

$$(f) \qquad | e(x) | \leqq M \int_0^x | e(y) | dy,$$

which can only be true, for a bounded, continuous $e(x)$, if $e(x) \equiv 0$.

But if $e(x) \equiv 0$, the equation (a) shows that $u_k(x)$ is a solution of equation $(k-1)$ and the Lemma is established. We now prove the converse of this Lemma.

LEMMA IV. *A solution of equation* $(k-1)$ *is a solution of equation* (k).

This follows at once if we write (k) in the form

$$\begin{aligned} u_k(x) = f(x) &+ \sum_{m=1}^{2^{k-1}-1} \int_0^x K_m(x, y) f(y) dy \\ &+ \int_0^x K_{2^{k-1}}(x, y) dy \left[f(y) + \sum_{m=1}^{2^{k-1}-1} \int_0^y K_m(y, z) f(z) dz \right. \\ &+ \left. \int_0^y K_{2^{k-1}}(y, z) u_k(z) dz \right]. \end{aligned}$$

If $u_k(x)$ is a solution of equation $(k-1)$ the expression in brackets in the last term reduces to $u_k(x)$ by equation $(k-1)$ and the equation becomes equation $(k-1)$ and so is satisfied.

The proof of our theorem now follows by mathematical induction. Lemmas II and III show that equation (A), which is the same as equation (0), has a bounded, continuous solution, namely the solution of equation (n). Lemma IV shows that if equation (A) has more than one bounded, continuous solution, equation (n) would have more than one bounded, continuous solution in contradiction to Lemma II. The theorem is thus established.

TRANSPOSITION OF INDICES IN MULTIPLE-LABELED DETERMINANTS

By RUFUS OLDENBURGER, Case School of Applied Science

Consider a quatrilinear form

$$F = \sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{k=1}^{p} \sum_{l=1}^{q} a_{ijkl} x_i y_j z_k w_l.$$

Making the non-singular transformations

$$B: \ z_k = \sum_{s=1}^{p} b_{ks} u_s$$

$$C: w_l = \sum_{t=1}^{q} c_{lt} v_t$$

on F we obtain the form

$$G = \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{p}\sum_{l=1}^{q}\sum_{s=1}^{p}\sum_{t=1}^{q} a_{ijkl} b_{ks} c_{lt} x_i y_j u_s v_t.$$

Let a matrix D be defined by

$$D = (b_{ks}c_{lt}) = \begin{pmatrix} b_{11}c_{11} & b_{11}c_{12} \cdots b_{11}c_{1q} & b_{12}c_{11} \cdots b_{1p}c_{1q} \\ \vdots & \vdots & \vdots \\ b_{11}c_{q1} & b_{11}c_{q2} \cdots b_{11}c_{qq} & b_{12}c_{q1} \cdots b_{1p}c_{qq} \\ b_{21}c_{11} & b_{21}c_{12} \cdots b_{21}c_{1q} & b_{22}c_{11} \cdots b_{2p}c_{1q} \\ \vdots & \vdots & \vdots \\ b_{21}c_{q1} & b_{21}c_{q2} \cdots b_{21}c_{qq} & b_{22}c_{q1} \cdots b_{2p}c_{qq} \\ \vdots & \vdots & \vdots \\ b_{p1}c_{q1} & b_{p1}c_{q2} \cdots b_{p1}c_{qq} & b_{p2}c_{q1} \cdots b_{pp}c_{qq} \end{pmatrix}.$$

If T_1 denotes the *partition kl* of the indices of D, T_2 the partition st, then D is a matrix of the form $(d_{T_1T_2})$ where T_1 and T_2 are respectively row and column indices ranging over the number pairs 11, 12, $\cdots$, $1q$, 21, 22, $\cdots$, $2q$, $\cdots$, $p1$, $\cdots$, pq. Matrices of the type D are used by Weyl[1] in his treatment of product spaces, and properties of inverses of matrices of type D, and M given below, have been studied by Rice.[2]

Let $|D|$ represent the determinant of D. The author has shown elsewhere[3] that $|D| = |b_{ks}|^q |c_{lt}|^p$. Evidently $|D| \neq 0$. Array the coefficients of F in a matrix[4] M given by

$$M = (a_{ij,kl}) = \begin{pmatrix} a_{1111} & a_{1112} \cdots a_{111q} & a_{1121} \cdots a_{11pq} \\ a_{1211} & a_{1212} \cdots a_{121q} & a_{1221} \cdots a_{12pq} \\ \vdots & \vdots & \vdots \\ a_{1n11} & a_{1n12} \cdots a_{1n1q} & a_{1n21} \cdots a_{1npq} \\ a_{2111} & a_{2112} \cdots a_{211q} & a_{2121} \cdots a_{21pq} \\ \vdots & \vdots & \vdots \\ a_{mn11} & a_{mn12} \cdots a_{mn1q} & a_{mn21} \cdots a_{mnpq} \end{pmatrix}.$$

---

[1] H. Weyl, *Gruppentheorie und Quantenmechanik*, Leipzig, 1931, p. 80ff.

[2] L. H. Rice, *Adjoint and inverse determinants and matrices*, Journal of Mathematics and Physics, vol. 5 (1925), pp. 55–64.

[3] In the author's paper entitled "*Composition and rank of n-way matrices and multilinear forms*" to appear in an early issue of the Annals of Mathematics.

[4] Matrices of type M are used by P. A. M. Dirac, *Principles of Quantum Mechanics*, p. 70, to represent physical states.

Let the coefficients of G be arrayed in a matrix E given by

$$E = (e_{ij,st}) = \begin{pmatrix} \sum_{k,l} a_{11kl}b_{k1}c_{l1} & \sum_{k,l} a_{11kl}b_{k1}c_{l2} & \cdots & \sum_{k,l} a_{11kl}b_{kp}c_{lq} \\ \vdots & \vdots & & \vdots \\ \sum_{k,l} a_{1nkl}b_{k1}c_{l1} & \sum_{k,l} a_{1nkl}b_{k1}c_{l2} & \cdots & \sum_{k,l} a_{1nkl}b_{kp}c_{lq} \\ \sum_{k,l} a_{21kl}b_{k1}c_{l1} & \sum_{k,l} a_{21kl}b_{k1}c_{l2} & \cdots & \sum_{k,l} a_{21kl}b_{kp}c_{lq} \\ \vdots & \vdots & & \vdots \\ \sum_{k,l} a_{mnkl}b_{k1}c_{l1} & \sum_{k,l} a_{mnkl}b_{k1}c_{l2} & \cdots & \sum_{k,l} a_{mnkl}b_{kp}c_{lq} \end{pmatrix}.$$

Evidently $E=MD$. Since D is non-singular M and E are of the same rank, whence this rank is an invariant of M under the non-singular linear transformations B and C. Further, if the product of the orders of the ranges of i and j is equal to the product of the orders of the ranges of k and l, then M is square. Hence if $mn=pq$,

$$|E| = |M||D|.$$

But $|D| = |B|^q|C|^p$, where $B=(b_{ks})$, $C=(c_{lt})$. Hence $|E| = |M|\,|B|^q|C|^p$. Therefore M is a relative invariant under the transformations B and C.

Determinants of the type $|M|$ and $|D|$ are called *multiple-labeled* determinants. We have shown above how such determinants arise in transformations on multilinear forms where they possess invariant properties, and factor into determinants of lower orders.

Let partitions of indices T_1 and T_2 be given by $T_1=i_1i_2\cdots i_s$ and $T_2=i_{s+1}\cdots i_p$ respectively. Let the indices $i_1, i_2, \cdots, i_p$ range over the values $(1, 2, \cdots, n_1)$, $(1, 2, \cdots, n_2)$, $\cdots$, $(1, 2, \cdots, n_p)$ respectively, where $n_1n_2\cdots n_s=n_{s+1}n_{s+2}\cdots n_p$. We define a multiple-labeled determinant $|A|=|a_{T_1T_2}|$ to be the determinant obtained by arraying numbers $a_{T_1T_2}$ in a square array, T_1 being the index of the rows and T_2 being the index of the columns of $|A|$. Evidently the determinants $|M|$, $|E|$, and $|D|$ mentioned in the above paragraphs are special cases of $|A|$. It is to be noted that any determinant may be multiple-labeled.

It may be observed that, as one passes down any column of M, the first subscript is 1 while the second ranges from 1 to n, then the first subscript is 2 while the second ranges from 1 to n and so forth. We may say that the rows are *ordered* first according to the second subscript and second according to the first subscript. One may thus, in general, consider a matrix A whose rows are ordered first according to the sth subscript, second according to the $(s-1)$th subscript, $\cdots$, jth according to the $(s-j+1)$th subscript, and so on. $T_1=i_1i_2\cdots i_j\cdots i_k\cdots i_s$ thus also serves to indicate the way in which the rows of A are ordered. If now a matrix A' is formed by ordering the rows of A jth ac-

cording to the $(s-k+1)$th subscript, kth according to the $(s-j+1)$th subscript and otherwise just as in A, we call A' the *transpose* of A (with respect to i_j and i_k). The way in which the rows are ordered we indicate now by $T_1' = i_1 i_2 \cdots i_k \cdots i_j \cdots i_s$ and say that T_1' is obtained from T_1 by the *transposition* of i_j and i_k in T_1. Note that T_1' is merely a means of designating the way in which the rows are ordered. Even though we write symbolically $|A'| = |a_{T_1'T_2}|$, T' is not necessarily,[1] for every set of values of the i's the subscript of some element of A or A'. *Throughout this paper the term "transpose" or "transposition" will be used only in the above sense.* A transposition of indices in our sense is an interchange in the ordinary sense only when the ranges of the two indices is the same. The following example will emphasize the distinction:

$$|N| = |a_{i_1 i_2 i_3}| = \begin{vmatrix} a_{111} & a_{112} & a_{113} & a_{114} & a_{115} & a_{116} \\ a_{121} & a_{122} & a_{123} & a_{124} & a_{125} & a_{126} \\ a_{131} & a_{132} & a_{133} & a_{134} & a_{135} & a_{136} \\ a_{211} & a_{212} & a_{213} & a_{214} & a_{215} & a_{216} \\ a_{221} & a_{222} & a_{223} & a_{224} & a_{225} & a_{226} \\ a_{231} & a_{232} & a_{233} & a_{234} & a_{235} & a_{236} \end{vmatrix}$$

where $T_1 = i_1 i_2$, $T_2 = i_3$. The transpose of $|N|$ obtained by letting $T_1' = i_2 i_1$ is given by

$$|N'| = \begin{vmatrix} a_{111} & a_{112} & a_{113} & a_{114} & a_{115} & a_{116} \\ a_{211} & a_{212} & a_{213} & a_{214} & a_{215} & a_{216} \\ a_{121} & a_{122} & a_{123} & a_{124} & a_{125} & a_{126} \\ a_{221} & a_{222} & a_{223} & a_{224} & a_{225} & a_{226} \\ a_{131} & a_{132} & a_{133} & a_{134} & a_{135} & a_{136} \\ a_{231} & a_{232} & a_{233} & a_{234} & a_{235} & a_{236} \end{vmatrix}.$$

It is the purpose of this paper to determine the effect of such transpositions of indices in the row or column partitions of a multiple-labeled determinant on the value of this determinant, its minors and cofactors.

THEOREM I: *If a single transposition of indices is made in the partition associated with the rows or columns of a multiple-labeled determinant $|A|$, then the determinant $|A|$ remains unchanged in value or changes sign according as D is even or odd, where D is $\frac{1}{4}$ of the product of the order of the determinant $|A|$ by the orders diminished by 1 of the ranges of the transposed indices.*

Let $T_1 = \alpha i_j \beta i_k \rho$ where α, β, ρ are partitions of indices (which may be vacuous). Let the product of the orders of all of the ranges of the indices in α, β, ρ be denoted by m. If one of these partitions is vacuous (that is contains no in-

[1] If, for example, $n_k > n_j$, there is no element whose jth subscript is n_k.

dices) let its order be 1. If $T=i_ji_k$ then $m=1$. Further let $P_u^v=n_un_{u+1}\cdots n_{v-1}n_v$. We define $P_u^v=1$ if $u=v+1$. By assigning to each index in α, β, ρ fixed values in all possible ways we can divide the rows in $|a_{T_1T_2}|$ and $|a_{T_1'T_2}|$ into m sets with i_j, i_k varying in each set over their respective ranges. Let $\alpha=\alpha_1$, $\beta=\beta_1$, $\rho=\rho_1$ determine one such set, α_1, β_1, ρ_1 denoting the partitions α, β, ρ where fixed values have been assigned to all of the indices in these partitions. The set is composed of the following n_j blocks where a single element, e.g. $a_{\alpha_1 1\beta_1 1\rho_1 T_2}$, is used to denote a row of $|A|$:

$$\begin{bmatrix} a_{\alpha_1 1\beta_1 1\rho_1 T_2} \\ a_{\alpha_1 1\beta_1 2\rho_1 T_2} \\ \vdots \\ a_{\alpha_1 1\beta_1 n_k\rho_1 T_2} \end{bmatrix}$$
$$\begin{bmatrix} a_{\alpha_1 2\beta_1 1\rho_1 T_2} \\ \vdots \\ a_{\alpha_1 2\beta_1 n_k\rho_1 T_2} \end{bmatrix}$$
$$\vdots$$
$$\begin{bmatrix} a_{\alpha_1 n_j\beta_1 1\rho_1 T_2} \\ \vdots \\ a_{\alpha_1 n_j\beta_1 n_k\rho_1 T_2} \end{bmatrix}$$

The transpose of the above array is given by

$$\begin{bmatrix} a_{\alpha_1 1\beta_1 1\rho_1 T_2} \\ a_{\alpha_1 2\beta_1 1\rho_1 T_2} \\ \vdots \\ a_{\alpha_1 n_j\beta_1 1\rho_1 T_2} \end{bmatrix}$$
$$\begin{bmatrix} a_{\alpha_1 1\beta_1 2\rho_1 T_2} \\ \vdots \\ a_{\alpha_1 n_j\beta_1 2\rho_1 T_2} \end{bmatrix}$$
$$\vdots$$
$$\begin{bmatrix} a_{\alpha_1 1\beta_1 n_k\rho_1 T_2} \\ \vdots \\ a_{\alpha_1 n_j\beta_1 n_k\rho_1 T_2} \end{bmatrix}$$

Let us denote these arrays by R_1 and R_2 respectively.[1] If interchanges of

[1] Note that the blocks in R_1 do not necessarily correspond to those in R_2. If for example $2n_j>n_k>n_j$, the n_kth element of R_1 is $a_{\alpha_1 1\beta_1 n_k\rho_1 T_2}$ and n_kth element of R_2 is $a_{\alpha_1\mu\beta_1 1\rho_1 T_2}$ where $\mu=n_k-n_j$. Also there are n_k blocks in R_2 and only n_j in R_1.

$a_{\alpha_1 2\beta_1 1\rho_1 T_2}$ are made in such a way that the relative order of the remaining rows is not affected, the number of interchanges of rows necessary to bring $a_{\alpha_1 2\beta_1 1\rho_1 T_2}$ of R_1 to its position in R_2 is (n_k-1). Similarly it requires $(\sigma-1)$ (n_k-1) interchanges to bring $a_{\alpha_1\sigma\beta_1 1\rho_1 T_2}$ of R_1 into its position in R_2 for $\sigma=3, \cdots, n_j$. Let $S_1(n_k)$ be the total number of interchanges necessary to bring all elements of the first block of R_2 from R_1 into their positions in R_2. Then

$$S_1(n_k) = (n_k-1) + 2(n_k-1) + \cdots + (n_j-1)(n_k-1) = \tfrac{1}{2}n_j(n_j-1)(n_k-1).$$

The remaining elements now form an array similar to R_1 except that the first element in each block is omitted. Hence the same process can be applied to $a_{\alpha_1 1\beta_1 2\rho_1 T_2}, \cdots, a_{\alpha_1 n_j\beta_1 2\rho_1 T_2}$. Let S_2 be the number of interchanges necessary to bring these elements into position. It is seen that

$$S_2 = S_1(n_k-1) = \tfrac{1}{2}n_j(n_j-1)(n_k-2).$$

Similarly $S_{n_k-1}=\frac{1}{2}n_j(n_j-1)$ and $S_{n_k}=0$. The total number of interchanges necessary to obtain one set of $|a_{T_1'T_2}|$ from one set of $|a_{T_1T_2}|$ by this method is

$$S^* = \sum\nolimits_{\nu=1}^{n_k-1} S_\nu.$$

Now $S^*=\frac{1}{4}n_k(n_k-1)n_j(n_j-1)$, whence

$$|a_{T_1T_2}| = (-1)^{mS^*}|a_{T_1'T_2}|.$$

Hence $|A|=|A'|$ if

$$D = \tfrac{1}{4}P_1^s(n_j-1)(n_k-1) \equiv 0 \pmod 2;$$

otherwise $|A|=-|A'|$. The same argument holds for columns.

If $n_j=n_k=n$, $D=\frac{1}{4}mn^2(n-1)^2$. Now $D\equiv\frac{1}{2}mn(n-1) \pmod 2$, whence $D_1=\frac{1}{2}mn(n-1)$ may be substituted for D in the above theorem. In the example given above $D=3$, whence $D\equiv 1 \pmod 2$, and $|N|=-|N'|$.

Let A_1 and A_1' denote the cofactors of the element $a_{T_1T_2}$ in the displays $|a_{T_1T_2}|$ and $|a_{T_1'T_2}|$ respectively. The theory of inverse matrices gives $A_1=(-1)^k A_1'$ if $|a_{T_1T_2}|=(-1)^k|a_{T_1'T_2}|$. Hence

THEOREM 2: *The cofactors A_1 and A_1' of the same element $a_{T_1T_2}$ in the determinants $|A|$ and $|A'|$ are equal or equal to the negatives of each other according as D of Theorem 1 is even or odd.*

THEOREM 3: *The unsigned minors of an element in the multiple-labeled determinant $|A|$ and the transpose $|A'|$ of $|A|$, where $|A'|$ is obtained from $|A|$ by a single transposition of indices as in Theorem 1, are equal or the negatives of each other according as $D\equiv d \pmod 2$ or $D\equiv d+1 \pmod 2$ where D is defined as in Theorem 1 and d is given below.*

PROOF. We shall assume that $k>j$. The element $a_{i_1\cdots i_s i_{s+1}\cdots i_p}$ is in the rth row of $|A|=|a_{T_1T_2}|=|a_{i_1\cdots i_s, i_{s+1}\cdots i_p}|$ where $r=\sum_{t=1}^{s-1}(i_t-1)P_{t+1}^s+i_s$. When the i_j and i_k indices are transposed the above element moves to the pth row of

$|A'|$, where

$$p = \sum_{t=1}^{j-1} (i_t - 1)P_{t+1}^{s} + (i_k - 1)\frac{n_j}{n_k} P_{j+1}^{s} + \frac{n_j}{n_k} \sum_{t=j+1}^{k-1} (i_t - 1)P_{t+1}^{s}$$
$$+ (i_j - 1)P_{k+1}^{s} + \sum_{t=k+1}^{s-1} (i_t - 1)P_{t+1}^{s} + i_s.$$

The difference between r and p is given by

$$d = P_{k+1}^{s}\Bigg[(i_j - i_k)(P_{j+1}^{k} - 1) \tag{1}$$
$$+ (n_k - n_j)\left\{(i_k - 1)P_{j+1}^{k-1} + \sum_{t=j+1}^{k-1} (i_t - 1)P_{t+1}^{k-1}\right\}\Bigg].$$

The theorem follows from equation (1) and Theorem 2.

In the previous illustration the formulas for $|N|$ and $|N'|$ give $d=(2i_1-i_2-1)$. Hence the congruences of Theorem 3 imply the equality of the minor of $a_{i_1i_2i_3}$ in $|N|$ with the minor of $a_{i_1i_2i_3}$ in $|N'|$ if i_2 is even; if i_2 is odd the minor of $a_{i_1i_2i_3}$ in $|N|$ is equal to the negative of the minor of $a_{i_1i_2i_3}$ in $|N'|$.

THE RICHARD PARADOX[1]

By ALONZO CHURCH, Princeton University

A system of symbolic logic must begin with a list of undefined symbols, a list of formal axioms, and a list of rules of inference.

Let us call any finite sequence of the undefined symbols of the system a *formula*. Then each of the formal axioms is a formula. And each of the rules of inference states an operation which enables us, out of given formulas, to obtain new ones. The theorems of the system are the formulas which can be obtained from the formal axioms by a finite number of applications of the rules of inference.

It would seem natural to require that the list of undefined symbols and the list of formal axioms should each be finite, but, as a matter of fact, in most of the systems of symbolic logic which have actually been proposed, one or both of these lists is enumerably infinite.

Now it is well known that in any particular system the set of all formulas is enumerable. For we may arrange the list of undefined symbols in a fixed order, and then define the *grade* of a formula to be the least positive integer n, such that the formula does not contain more than n symbols and does not contain any symbol beyond the nth in the list of undefined symbols. The set of all formulas of a particular grade is then always finite. And therefore, after fixing

[1] An address delivered at the meeting of the Mathematical Association in Cambridge, Mass., Dec. 30, 1933.

upon a rule for ordering formulas of the same grade (as is easily done), we can enumerate the set of all formulas by arranging them in the order of their grades.

For convenience, let us use the phrase *function of positive integers* to mean a single-valued function of one variable, $f(x)$, which takes on a value which is a positive integer, whenever x takes on a value which is a positive integer. We can prove, by a familiar argument, that the set of all functions of positive integers is not enumerable. For, if $f_1(x)$, $f_2(x)$, $f_3(x)$, $\cdots$ is any enumeration of functions of positive integers, then $1+f_x(x)$ is a function of positive integers not included in the enumeration.

Since, in any system of symbolic logic, the set of all formulas is enumerable, whereas the set of all functions of positive integers is not enumerable, it seems to follow that, in the case of any system of symbolic logic, there exists a function of positive integers such that there is no formula which stands for it. And surely the existence of a function of positive integers which has no representation as a formula in the system means that the system is inadequate even for elementary number theory.

The Richard paradox can be said to consist in the following problem. How is it possible that a system of symbolic logic, in which the set of all formulas is enumerable, should be adequate for any branch of mathematics which deals with the members of a non-enumerable set (in particular for elementary number theory)?

Given a system of symbolic logic, let us try to construct the function of positive integers such that there is no formula in the system that stands for it. What we must do is first to enumerate all formulas, by the method which we have described, and then, going through this enumeration, to pick out in order those formulas which stand for functions of positive integers. The result is an enumeration of all formulas which stand for functions of positive integers. And if we let $f_n(x)$ be the function of positive integers represented by the nth formula in this enumeration, then $1+f_x(x)$ is the function of positive integers such that there is no formula in the system that stands for it.

But this function $1+f_x(x)$ is not, in general, defined in such a way that it is always possible to calculate its value for a given positive integer x. For, in the process of going through the list of all formulas and picking out those which stand for functions of positive integers, we may at some stage find a formula about which we do not know whether or not it stands for a function of positive integers. For example, we may find a formula whose intuitive meaning is, "The least positive integer n, greater than x, such that the equation $u^n+v^n=w^n$ has a solution in positive integral values of u, v, w." And we could not determine whether this formula stood for a function of positive integers without first proving, or disproving, Fermat's last theorem. Indeed, to be sure of always being able to determine whether a given formula stands for a function of positive integers, we must have discovered a method of procedure which would enable us to solve any problem of number theory whatever. Therefore the infinite sequence (about which we have been talking) of all formulas which stand for func-

tions of positive integers almost certainly is not such an infinite sequence that it is possible to calculate as many terms of it as we please. And therefore the function $1+f_x(x)$ has not been defined in a way which could be called constructive, but has merely been proved by an indirect argument to exist.

Now a particular system of symbolic logic could be adjudged inadequate only in the presence of a particular function of positive integers, which could be defined intuitively, but had no formula in the system to stand for it. We require of our formal system merely that it shall be adequate to define any function of positive integers which can be defined intuitively. And hence an existence proof which cannot be supported with an effective construction has no significance for our present problem.

Hence it appears to be possible that there should be a system of symbolic logic containing a formula to stand for every definable function of positive integers, and I fully believe that such systems exist.

But surely a function of positive integers which cannot be defined by any means whatever is no function of positive integers at all. If you agree, then it seems to follow that the non-enumerable set of all functions of positive integers can be put into one-to-one correspondence with a subset of the enumerable set of all formulas of an adequate symbolic logic. In fact, we are presented with the alternative of supposing that there is no adequate system of symbolic logic or of supposing that an enumerable set can contain a non-enumerable subset. Of the two alternative suppositions the latter is clearly preferable and (as we have seen) by no means untenable.

Let us turn, however, to another aspect of the problem of the possible adequacy of a system of symbolic logic. In order that a system be adequate it is necessary not only that it contain a formula to stand for every function of positive integers, but also that, in the case of every formula f which stands for a function of positive integers, the formal theorem $N(x) \supset_x N(f(x))$ shall be provable; where $N(x)$ is the formula whose intuitive meaning is "x is a positive integer," and $\supset_x$ is the formula which stands for the relation of implication between propositional functions, so that $N(x) \supset_x N(f(x))$ is to be read, "x is a positive integer implies that $f(x)$ is a positive integer."

But in the case of any system of symbolic logic the set of all provable theorems is enumerable.

This is most easily seen in the case of a system of the simplest sort, for which the number of formal axioms and the number of rules of inference are alike finite.[1] For in such a case the number of theorems provable in n steps is finite, for any fixed value of n. By inspection of the rules of inference and the formal axioms we can obtain a complete list of the theorems provable in one step. And, in general, by inspection of the rules of inference, the formal axioms, and the finite

[1] We assume that with a fixed premise or premises and a fixed rule of inference the formula obtained as conclusion is unique (a suggestion due to J. B. Rosser). This property by no means holds of all systems which have actually been proposed. But at the possible cost of increasing the number of rules of inference, it can always be made to hold, without essentially altering the system.

list of theorems provable in not more than $n-1$ steps, we can obtain a complete list of the theorems provable in n steps. Hence we may enumerate all the theorems of the system by enumerating first the formal axioms, then the theorems provable in one step, then the theorems provable in two steps, and so on.

In the case of a system of a more complicated sort, for which the number of rules of inference, or the number of formal axioms, or both, are infinite, an evident modification of the foregoing method will still enable us to obtain an enumeration of all the formal theorems.

Out of this enumeration of all theorems select those which have the form $N(x) \supset_x N(f(x))$. This gives an enumeration $N(x) \supset_x N(f_1(x))$, $N(x) \supset_x N(f_2(x))$, $N(x) \supset_x N(f_3(x))$, $\cdots$. And hence we obtain an enumeration $f_1(x)$, $f_2(x)$, $f_3(x)$, $\cdots$, of all formulas about which we can prove the formal theorem that they are functions of positive integers; whereas the set of all (intuitively definable) functions of positive integers is not enumerable.

Since the enumeration of all formal theorems is effective, and since there is a uniform procedure by which we can recognize whether any given formula has the form $N(x) \supset_x N(f(x))$, it follows that in the case of any particular system of symbolic logic, we can actually carry out the enumeration $f_1(x)$, $f_2(x)$, $f_3(x)$, $\cdots$ to as many terms as we care to. Hence we cannot escape from the present dilemma in the same way that we did before.

Since the set of all formulas about which we can prove the formal theorem that they are functions of positive integers is (effectively) enumerable, while the set of all functions of positive integers is not enumerable, we conclude, either that there is some function of positive integers which is definable intuitively but about which we cannot prove the formal theorem that it is a function of positive integers, or else that there is some formula about which we can prove the formal theorem that it is a function of positive integers but which on the basis of the intuitive meanings given to our undefined terms does not stand for a function of positive integers. That is, briefly, every system of symbolic logic either is inadequate to prove all theorems which are intuitively true or else suffices to prove theorems which are intuitively false.

Or, if we prefer not to assume the existence of an intuitive logic which is right in an absolute sense, then it is sufficient to observe that in any system of symbolic logic not hopelessly inadequate there would be a formal equivalent of the intuitive argument which we have just set forth, and hence that this system of symbolic logic would contain the formal theorem that this same system, regarded objectively, was either insufficient or over-sufficient. This means that the assumption that this system of symbolic logic was a true and complete representation of what is logically correct would defeat itself.

This, of course, is a deplorable state of affairs. It plainly implies that the whole program of the mathematical logician is futile.

For in the presence of such a situation, not only is it impossible to obtain a single set of postulates which would lead to all mathematics (as, for example, the authors of *Principia Mathematica* would do), but it is even impossible to

obtain a set of postulates adequate to a particular branch of mathematics, such as number theory, analysis, or Euclidean plane geometry. That is, provided we assume, as I think we must, that a satisfactory formalization of, say, Euclidean plane geometry, means formalizing, not only the geometric terms, such as "point" and "line," which appear in its propositions, but also the logical terms, such as "if," "and," "is."

Indeed, if there is no formalization of logic as a whole, then there is no exact description of what logic is, for it is in the very nature of an exact description that it implies a formalization. And if there is no exact description of logic, then there is no sound basis for supposing that there is such a thing as logic.

Under these circumstances, a definition, for instance of Euclidean plane geometry, as that body of propositions which follows logically from a certain set of axioms, is altogether vague and unsatisfactory, because any attempted definition of the adverb "logically" is necessarily incomplete. We are led to despair of the currently accepted search for mathematical rigor, which amounts essentially to an appeal from the realm of spatial and other intuitions to the realm of logic.

Fortunately, however, there is a way out of this condition of nihilism. The theorem which led us to such pessimistic conclusions does not really apply to all systems of symbolic logic but only to systems which satisfy certain conditions. And one of these conditions is, either that there shall be a unique symbol for implication between propositional functions, or that there shall be a set of symbols for implication and an effective way by which we can always determine whether a given formula is one of the symbols for implication. For, in the contrary case, there would be no effective way of picking out from a list of theorems those which had the form $N(x) \supset_x N(f(x))$, and hence we could escape from our second dilemma in the same way that we did from our first one.

Therefore we seek a system of symbolic logic in which the notion of implication between propositional functions is obtained by definition, and in which there are a variety of notions of implication, obtainable by different definitions. In the case of each definition we desire that it shall be possible by an intuitive argument to prove the character of the defined symbol as an implication symbol. But there shall be no uniform means of determining whether a given formula is an implication symbol.

A system of this sort not only escapes our unpleasant theorem that it must be either insufficient or oversufficient, but I believe that it escapes the equally unpleasant theorem of Kurt Gödel to the effect that, in the case of any system of symbolic logic which has a claim to adequacy, it is impossible to prove its freedom from contradiction in the way projected in the Hilbert program. This theorem of Gödel is, in fact, more closely related to the foregoing considerations than appears from what has been said.

As I speak, I have in mind a particular set of postulates for symbolic logic, whose freedom from contradiction can be proved, and which lead to a non-enumerable multiplicity of definitions of implication, in the manner we desire.

It seems probable that the system of logic which results from these postulates is adequate at least for elementary number theory, but how far it is adequate for analysis there is at present no safe basis for conjecture.

Apparently, however, in view of the theorem of Gödel, and of the difficulties arising in connection with the Richard paradox, a system of symbolic logic of this kind is the most general which can be regarded as satisfactory from our present point of view. If it be true that no system of this kind can lead to analysis, then it seems to follow that the indictment against the soundness of analysis which is contained in the Richard paradox must be allowed to stand.

THE THEORY OF THE CHESHIRE CAT[1]

By D. E. RICHMOND, Williams College

"This time the Cheshire Cat vanished quite slowly, beginning with the end of the tail, and ending with the grin which remained some time after the rest of it had gone. 'Well! I've often seen a cat without a grin,' thought Alice, 'but a grin without a cat! It's the most curious thing I ever saw in all my life!' "

I must now confess that I am going to talk about mathematics, which I have chosen to characterize as the theory of the Cheshire Cat, or better, the theory of the Cheshire Cat's Grin. The appropriateness of this characterization will, I trust, soon become apparent. It is no accident that I should find a text for my remarks in Lewis Carroll. It is well known that in real life Carroll, or rather Dodgson, was a mathematician and a logician. In his nonsense books he has contrived to say a great deal of mathematical and logical importance. I have here a little book[2] on the philosophy of mathematics, in which practically all of the references are to the works of Lewis Carroll.

This delightful little book begins as follows: "The view that the fundamental principles of logic consist solely of the law of identity was held by Leibniz, Drobisch, Uberweg and Tweedledee. Tweedledee, it may be remembered, remarked that certain identities 'are' logic." The reference given is to *Through the Looking Glass.* "'Contrariwise,' continued Tweedledee, 'if it was so, it might be; if it were so, it would be: but as it isn't it ain't. That's logic.' " There are two other references on the law of identity both to *Sylvia and Bruno.* "The professor said: 'The day is the same length as anything that is the same length as *it*' " and "Bruno observed that when the Other Professor lost himself, he should shout. He'd be sure to hear hisself, 'cause he couldn't be far off. " I think that this combination of high seriousness and nonsense is almost unique.

But to return to the Cheshire Cat, I should like to suggest that the progress of mathematics has left us with something almost as unsubstantial as the grin without the cat. The content of mathematics like the body of the cat fades

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[1] Delivered as a public lecture at Williams College, February, 1932.

[2] Jourdain, *The Philosophy of Mr. Bertrand Russell.*

farther and farther into the background so that at last we have a form almost without content, a pattern or structure without appreciable body. In any other subject, except perhaps theoretical physics where the fading is taking place under our eyes, this would be called destructive criticism. But paradoxically enough, the less substantial mathematics becomes, the more powerful it is. The less restricted its content is, the more universally applicable are its results. I hope to succeed in making this clear.

A mathematician is peculiarly handicapped. In other sciences one can point to something concrete and say, "That is the sort of thing with which I am concerned." The astronomer has his stars, the geologist his rocks, the biologist his plants and animals. The chemist and physicist have somewhat greater difficulties but they too can manage to be quite concrete. Of course in all these sciences there is much more than meets the eye. Nevertheless, there are obvious places to get started. But what has the mathematician to show? Only a page of symbols or perhaps some diagrams. What *is* the subject matter of mathematics? What is the mathematician trying to do with his x's and y's, his $+$'s and $-$'s?

To this question there is an ancient and respectable answer which has been embalmed in all the dictionaries. Mathematics is the science of space and number. On the one hand it is concerned with points, lines, circles, in a word with geometrical figures; on the other, with numbers and their properties. Now this answer has a certain inherent plausibility and it seemed satisfactory until quite recently. If you catch a mathematician at work, the chances are that you will find him dealing with just such things. But we have come to see that these things are only the materials with which the mathematician works. To suppose that they tell the whole story would be to miss the point of the last century of mathematical progress. It would be to fail to appreciate one of the major discoveries of modern times.

These things are no more mathematics than the artist's paint is art. The artist is attempting to do something with his materials; so too the mathematician. Both are creating forms: the artist visual forms; the mathematician, forms of thought, logical structures. To sum it up in a phrase, mathematics is the *art of building logical structures*. Each branch of mathematics is a great piece of logical architecture.

But so far we have been too vague. We must give more precision to our statement of the case. Let me ask you for a moment to think back to the days when you studied plane geometry. There you started with certain preliminary axioms or postulates. (I shall use these terms indiscriminately.) These postulates were said to represent self-evident truths. You also met what purported to be definitions of points, lines and planes. These were the sorts of things with which you were to work. On this foundation of axioms and definitions there was erected an impressive structure of theorems or propositions which were supposed to follow logically from the starting point. Here in Euclid's system we have the first great example of mathematical architecture. So impressive was this geometry, that 2000 years passed before anyone imagined that there could be any other.

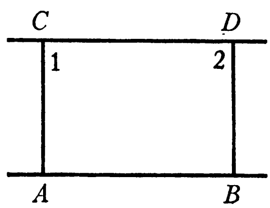
What held up progress more than anything else was the idea that the axioms represented self-evident truths. It may surprise some of us to learn that it is possible to doubt the axioms. Isn't it obvious that two points determine a straight line, or that two lines have at most one point in common? Surely it would seem so. But "obvious" is the most dangerous word in mathematics. It is fortunate that Euclid's successors did find one of his axioms to be far from self-evident. I refer to the famous parallel postulate.

We have quoted the postulate that two lines have at most one point in common. This leaves an alternative, a loophole. When do they have one point in common and when none? If two lines have no point in common they are said to be parallel. Now the parallel postulate asserts that through a given point there exists one and only one line parallel to a given line.

There is internal evidence to show that Euclid himself felt that this postulate was not particularly obvious for he did not put it at the beginning with the others but introduced it only when he could no longer dispense with it. And many a high school boy and girl must have felt, and quite rightly, that it was not nearly as self-evident as some of the first theorems in the book which seemed to demand proof.

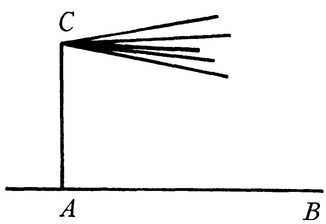
The attempts to prove Euclid's parallel postulate from the others began in Euclid's own time and continued down to the beginning of the last century without success. It was shown that this postulate could be replaced by a number of others which were equivalent to it, but these equivalent postulates were no more self-evident than Euclid's. Thus if one is willing to take it as self-evident that the sum of the angles of a triangle is equal to two right angles one can prove the parallel postulate. But I suppose that no one would regard this statement as obvious.

Among the equivalent postulates was one which was destined to play an important part in the development of mathematics. Suppose that we erect two perpendiculars to the same line, AB, mark off equal distances and join their ends. We naturally expect that in the resulting figure all four angles are right angles, but we cannot prove that $\angle 1$ and $\angle 2$ are right angles without the use



of the parallel postulate. The best that we can do is to show that $\angle 1$ equals $\angle 2$. Now if it is assumed that one of them, say $\angle 1$, is a right angle, it is easy to prove that CD is parallel to AB and that it is the only parallel to AB through C. That is to say, the assumption that $\angle 1$ is a right angle is equivalent to Euclid's parallel postulate.

Now the Russian mathematician Lobachevsky tried an intellectual experiment. He assumed for the sake of argument that $\angle 1$ was less than a right angle, and proceeded to develop the consequences of this assumption and the other Euclidean postulates. Surely sooner or later one should get a contradiction. At least so it would seem. But strangely enough this didn't happen at all. Lobachevsky built up a geometry on this modified basis which seemed just as good logically as Euclid's. In this new geometry, the sum of the angles of a triangle turns out to be less than two right angles, and many other surprising theorems result. Through C there are an infinite number of parallels to AB instead of only one. But no contradiction develops.



Then Riemann took the remaining alternative. Suppose that $\angle 1$ is more than a right angle. What then? Here too, Riemann found that he could go on and on. He got many theorems which differ from those of our school geometries, but his results never contradicted each other. In Riemann's case, the sum of the angles of a triangle turns out to be greater than two right angles and there are no parallel lines. Any two straight lines must intersect each other.

Here then we have three geometries, all equally good logically, one the Euclidean and two new ones called the non-Euclidean geometries. One's first reaction to this situation is to say that the Euclidean one is the right one. It is indeed obvious that these three geometries cannot apply to the same things. A Euclidean straight line cannot be the same as a non-Euclidean one. But perhaps some would go further and question the right to apply the term straight line to the objects which obey the assumptions of Lobachevsky and Riemann.

Of course we have the classic answer of Humpty Dumpty: "When *I* use a word, it means just what I choose it to mean—neither more nor less." Choice of words is a question of taste or convenience, not logical necessity. But all the same we feel that there *is* a reason for preferring Euclid's system. The reason is that in order to make the alternatives at all plausible, we have to draw a figure in which the so-called straight lines do not look right. That is to say we have in our heads a picture of a straight line to which the new geometries do not seem to do justice.

I suppose that it has always been realized that the geometers' points and lines are ideal objects which cannot be exhibited in what we choose to call real life. Thus according to Euclid a line has neither breadth nor thickness. No one has even seen such a thing except in the mind's eye. A line which can be drawn

on a piece of paper or the blackboard suggests the geometers' line but it certainly *is not* one. His points and lines are ghostly sorts of things but they do represent an abstract from something in experience.

To return to our new geometries, we feel that they do not fit into experience, or the abstract from experience which we make so naturally. At this point we realize that Euclid was pretty vague in his definition of a straight line. "A *line* which lies evenly between the points in itself is a *straight* line." Evidently the reader was supposed to know what was meant. In fact, these statements of Euclid's are not definitions at all. They are merely rough indications. One cannot define everything. But now we come to a turning point in the history of thought.

This consists in the realization that geometry as it has come down to us is a hybrid, whose father was logic and whose mother usually goes under the name of geometric intuition. Euclid used logic but in conjunction with diagrams or figures. And if Euclid's geometry is to be preferred to the others, it is only on the basis of our geometric intuition, *not* on the basis of any explicitly stated properties of straight lines. Insofar as diagrams are really necessary for an argument, to that extent must we admit that this argument does not follow strictly from what was stated in the postulates.

But as soon as this hybrid character was realized, it was decided that the logical part was the characteristically mathematical part. The attempt was therefore made to separate the *looks* from the *logic*. The latter is pure mathematics. The hybrid is applied mathematics, or if you prefer, the simplest branch of physics. The objects or ideal objects to which the straight lines refer (like straight edges and light rays) are physical concepts. Right at the beginning then we find the close linkage which has characterized these two subjects.

Of course as soon as the attempt was made to discard the appeal to intuition, to abstract the logical meat, it developed that Euclid had failed to state all of his assumptions. However, these defects have since been remedied, among others by Hilbert, usually considered to be the greatest living mathematician. It has now been possible to exhibit the logical structure so abstractly that no one would dream what it was all about. In this formulation a blind man would be at no disadvantage. Indeed, it becomes clear that if you use no other facts about points, lines, and planes than those explicitly stated in the postulates it cannot possibly make any difference whether they look like the things which we are accustomed to call by these names or not. We could call them thingamabobs, what you may callems and whats its, or more briefly x's, y's and z's. If the propositions follow from the postulates on purely logical grounds they will continue to do so if we change the names. Of course no one would ever have thought of the logical structure if there had not been an interpretation first. But that is another matter. Once it is built up we can dispense with the intuitive scaffolding and leave only a body of propositions[1] which follow on purely deductive grounds from an initial set of propositions. From this point of view, the initial proposi-

[1] Strictly, propositional forms or "propositional functions."

tions cannot be self-evident truths because we have completely disregarded the entities to which they might apply. They are merely what we agree to start with, our logical foundation.

Now this seems like a singularly futile viewpoint, I have no doubt. The points and lines were shadowy enough to begin with! Why make it worse? The Cheshire Cat is disappearing and leaving the grin, but why not keep the cat? The answer can be given in a few words. Because one can then proceed to try the grin onto other things. The analogy is a bit forced because after all there is no occasion to try a grin on anything else. But in the case of logic, there is an occasion. We can say and it is useful to say that any sorts of things for which the initial propositions are true will be such that all the rest of them are true.

We now come within hailing distance of the famous *bon mot* of Bertrand Russell. "Mathematics is that science in which we do not know what we are talking about, nor whether what we are saying is true." I have noted that books which contain this statement are always marked with expressions of approval. This statement is however not to be interpreted as a confession of the incompetence of the mathematician or of the uselessness of his subject.

Russell has restated the idea in this somewhat more enlightening form. "Pure mathematics consists entirely of such assertions as that, if such a proposition is true of *anything*, then such and such another proposition is true. It is essential not to discuss whether the first proposition is really true, and not to mention what the anything is of which it is supposed to be true."

It is worth pointing out that the viewpoint expressed by Russell's dictum is just what science needs. The scientist does not know whether his hypotheses are true or not. But mathematics puts in his hands a host of consequences which would follow if they were true. Some of these consequences can be tested. If the test is unfavorable the original hypothesis was false.

In our terms, the theoretical physicist is attempting to make out the cosmic facial expression. He tries on the different expressions which mathematics furnishes to see which one fits. At present he is trying on a form of mathematics developed by Hilbert and his school, called "The Mathematics of Function Space." It is interesting to note in passing that the beings of which the physicists speak (electrons, atoms, electromagnetic waves) are sharing in the abstract tendency which we are discussing. They have ceased to be the easily visualized things that we thought they were. They possess properties embodied in mathematical assumptions but they are increasingly inconvenient to imagine.

Russell's formulation brings out a characteristic of mathematics which may be plainly seen even in its most embryonic form, by which I mean arithmetic. Thus $2+2=4$ whether we are referring to apples, mountains or ideas. Arithmetic, like logic, is completely indifferent to the sort of thing to which one may apply it. In algebra we see further illustrations. Thus $x+y=y+x$, which says that for any two numbers whatever, you get the same result if you add in one order as you do in the other. We note again the sublime indifference expressed by *any*. This indifference is the secret of the power of mathematics.

It is a consequence of Russell's definition, that most of what goes by the name of mathematics is really applied mathematics, or if you wish, the most fundamental branch of physics. Thus, very little pure mathematics is taught in any undergraduate college. We do draw diagrams on the board. There is some appeal to intuition. Nevertheless Russell's definition places the emphasis in the right place, I think. It is fairer to characterize a subject by its adult condition than by its childhood or adolescent characteristics. In any case, it is necessary to make some sort of distinction between the study of logical structure and the study of objects of space intuition.

At all events the abstract viewpoint furnishes the key to what has happened recently. Once set free from the idea of axioms as self-evident truths, mathematicians were free to try hosts of new sets of postulates bearing little or no relation to those of Euclid. We now have hundreds of geometries and algebras which scarcely would have been thought of before the adoption of the abstract viewpoint. Thus we have four-dimensional geometries, which seem very mystifying until one realizes that no one intends that they be grasped intuitively. The point is that these higher dimensional geometries bear certain formal analogies to two and three dimensional ones. The mathematician's symbolism fortunately enables him to dispense with space intuition in cases of this sort. The non-mathematician does not have these logical tools and he is likely to think that he is asked to do tricks with his imagination, which no mortal can do.

Aside from the tremendous stimulus to mathematics which came from the removal of the obstacle of the obvious or self-evident, or rather as a result of it, we must count it as one of the great virtues of the new view that it has led to the creation of impressive logical systems which have stood ready for the use of the physicist. Relativity would have been simply unthinkable except in terms of the four-dimensional geometries of Riemann, Ricci, and Levi-Civita, geometries worked out of course with no thought of application. The new quantum mechanics rests at least as far as one of its formulations goes, on a geometry in a space of infinitely many dimensions. In another aspect it uses a pathological kind of algebra in which it makes a difference which way you multiply. (ab is not the same as ba.) As long as one thought of algebra as applying exclusively to numbers, such an algebra would never have been imagined. And if it hadn't been thought of, we should know less about atoms than we do.

In this connection, it is pertinent to remark that the formal point of view has done excellent service in modern quantum theory. The quantum mechanics of Heisenberg and the wave mechanics of Schrödinger have proved to be two cats with the same grin. They represent different exemplifications of the same structural scheme. Indeed, as the work of Dirac, Weyl and von Neumann shows, for many purposes there are distinct advantages in dealing with the abstract relationships quite apart from either of the more concrete interpretations.

To summarize, we have seen that essentially the mathematician is a builder of logical air-castles which unlike most air-castles turn out to be useful. The materials he uses are in the first instance the objects of his geometrical intuition

and of his number sense, but in the second instance symbols which have no definite meaning but stand for anything which may satisfy his assumptions.

Somewhat paradoxically, the deliberate effort to strip away the intuitive content has contributed in no small measure to the development of mathematics and to its fruitfulness. The role of the mathematician now stands out in sharp relief. It is his function to explore the vast field of logical possibility. It is the task of the scientist to find which of these logical possibilities is suited to those aspects of the actual world of which he aims to give an account.

NOTE ON RATIONAL CURVES WITH TRIGONOMETRIC PARAMETER

By R. M. WINGER, University of Washington

In deriving the equation of a curve it is frequently convenient to express the coordinates of the tracing point in terms of trigonometric functions of a variable angle. The equation before us, the next question concerns the nature of the curve. In this note we shall prove two theorems of which the following are instances:

An epi- or hypo-cycloid is a rational curve if the radii of the two circles are commensurable.

The cyclic-harmonic curves

$$\rho = a \cos n\theta + k$$

are rational if n is a rational number.[1]

THEOREM 1. *If the non-homogeneous coordinates of all points on a curve in space of n dimensions are rational functions of trigonometric functions of rational multiples of an angle θ, the curve is rational.*[2]

We define a rational curve as one, the non-homogeneous coordinates of whose points are expressible as rational functions of a single parameter. There is no loss in generality if we suppose the coordinates to be rectangular. Further it will suffice to deal with the sine and cosine since the other trigonometric functions are expressible rationally in terms of them. Suppose now that

$$(1)\quad x_i = R_i(\sin a_1^{(i)}\theta, \sin a_2^{(i)}\theta, \cdots, \sin a_j^{(i)}\theta, \cos b_1^{(i)}\theta, \cdots, \cos b_k^{(i)}\theta), \qquad i = 1, \cdots, n,$$

where R_i denote rational functions and each a and b is a rational number in the form p/q, p and q integers. We are to show that x_i can be expressed as rational functions of a single parameter. If $t = \tan(\phi/2)$ and n is an integer, we have from

[1] R. E. Moritz, *The general theory of cyclic-harmonic curves*, Annals of Mathematics, vol. 23 (1921), p. 39.

[2] For the converse of a special case, see Hilton, *Plane Algebraic Curves*, p. 153.

trigonometry.

$$\sin \phi = 2t/(1 + t^2), \qquad \cos \phi = (1 - t^2)/(1 + t^2) \tag{2}$$

and

$$\begin{aligned} \sin n\phi &= n \cos^{n-1} \phi \sin \phi - \binom{n}{3} \cos^{n-3} \phi \sin^3 \phi + \binom{n}{5} \cos^{n-5} \phi \sin^5 \phi - \cdots \\ \cos n\phi &= \cos^n \phi - \binom{n}{2} \cos^{n-2} \phi \sin^2 \phi + \binom{n}{4} \cos^{n-4} \phi \sin^4 \phi - \cdots. \end{aligned} \tag{3}$$

Thus $\sin n\phi$ and $\cos n\phi$ are rational functions of t for every integer n. Now if m is the least common denominator of all the fractions a and b in (1), by writing $\theta = m\phi$, each rational multiple of θ becomes an integral multiple of ϕ. Hence by (2) and (3), each coordinate x_i is expressible as a rational function of t, which proves the theorem. If all of the trigonometric functions were present in (1) we could make x_i rational in t by taking m as the least common denominator of all the rational multiples of θ and putting $\theta = m\phi$ as before.[1]

THEOREM 2. *If in the polar equation of a curve ρ is a rational function of the trigonometric functions of rational multiples of θ, then the curve is rational.*

Suppose the polar equation of a curve is

$$\rho = R(\sin a_1\theta, \sin a_2\theta \cdots . \cos b_1\theta \ \cos b_2\theta \cdots . \tan c_1\theta \cdots) \tag{4}$$

where R is a rational function of all six trigonometric functions and $a_i, b_i, c_i, \cdots$ are rational numbers in the form p/q, p and q integers. Then the parametric equations of the curve are

$$x = \rho \cos \theta, \qquad y = \rho \sin \theta. \tag{5}$$

But by Theorem 1, ρ is a rational function of t and by (3) and (2) so also are $\sin \theta$ and $\cos \theta$ if $t = \tan (\phi/2)$ and $\theta = m\phi$, where m is the least common denominator of the fractions $a_i, b_i, c_i, \cdots$. Hence x and y are rational functions of a single parameter and the curve is rational.

Corollary 1. *The inverse of* (4) *with respect to a circle with center at the pole is a rational curve.*

For if r is the radius of the circle of inversion and (ρ', θ) is a point inverse to (ρ, θ), the equation of the inverse curve is

$$\rho' = r^2/\rho. \tag{6}$$

Thus ρ' obviously satisfies the conditions of the theorem.[2]

[1] Theorem 1 is also valid if trigonometric functions are replaced by hyperbolic functions. The proof is virtually the same, utilizing the formulas analogous to (2) and (3) for hyperbolic functions.

[2] That the inverse of any rational curve is rational also follows from the fact that the genus of a curve is invariant under a quadratic transformation.

Corollary 2. *The conchoid, link k, of a rational curve in form* (4) *is a rational curve.*

For the equation of such a conchoid is

$$\rho' = \rho + k. \tag{7}$$

Indeed we may say more generally that from any rational curve in form (4) may be derived a second rational curve

$$\rho' = R(\rho) \tag{8}$$

by taking for the right side of (8) any rational function of the ρ in (4).

Our theorems furnish a direct proof that a large proportion of the curves treated in the elementary books are rational, although many of them are known to be rational on other grounds. As very special cases may be mentioned the conics, numerous cubics (including the cubical and semicubical parabolas, the cissoid, the strophoid, the witch and the folium of Descartes) the conchoid of Nicomedes and the algebraic trochoids, which include epi- and hypo-cycloids, rose curves and limaçons—together with the inverse and the conchoids of each. Moreover the form of proof indicates the method of actually writing down the parametric equations of the curves in rational form, given the usual polar or parametric equations.

QUESTIONS, DISCUSSIONS AND NOTES

EDITED BY R. E. GILMAN, Brown University, Providence, Rhode Island

The department of Questions and Discussions in the Monthly is open to all forms of activity in collegiate mathematics, including the teaching of mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.

ON SIMILAR TRIANGLE TRANSFORMATIONS

By W. H. ECHOLS, University, Va.

1. It is a pleasure to see that Professor Musselman and others are interested in the similar triangle transformations as shown by their recent notes in the Monthly.

These constructions were introduced in this MONTHLY[1] by W. H. Echols, and somewhat elaborated in two dissertations[2] by R. L. Echols in 1928–29.

The transformation of a triangle, or of any figure, by a linear transformation into another similar figure is old and familiar. But it takes on a new significance

[1] This MONTHLY, vol. 30 (1923), pp. 120, 301.

[2] *Roots of Circulants and Application to the Roots of Polynomials*, and *A Correspondence in Geometry and Some Properties of Pseudo-homochiral and Pseudo-heterochiral Configurations;* dissertations by R. L. Echols. Printed by Williams and Wilkins Co., Baltimore, Md., and distributed by the University of Virginia library.

if a figure I, a polygon, be divided into component triangles so that adjacent triangles have a side in common, and a different linear transformation be applied to each component triangle.

The writer has been accustomed in lectures on function theory for a number of years to approach the algebra of this subject somewhat as follows.

2. In a given figure I two points z_1, z_2 transform into one point W_{12} in a figure II. Or, what is the same thing, a directed vector segment z_2-z_1 goes over into the vector point w_{12} independent of the direction of the vector z_2-z_1 so that $w_{12}\equiv w_{21}$. Thus if a, b, c are the vector sides of a given triangle abc taken in positive order (interior on the left), then representing corresponding vector points by capital letters A, B, C,

$$aA + bB + cC = 0, \qquad (a + b + c = 0) \tag{1}$$

is the equation of *all* triangles ABC homochiral (directly similar) to the given triangle abc. The three points A, B, C representing a solution of (1) can be chosen arbitrarily as to size and position of the triangle ABC but its shape must be invariant. If one point A be assigned fixed then B and C are free to move, but if one describes any path (arc) the other must describe a similar arc of a similar curve.

We refer to a given figure I as a vector or directed segment figure and the corresponding figure II as a point figure. If I consists of two triangles having a side in common, then II can always be so chosen that its corresponding two triangles have their vertices corresponding to the common side in I coincident. This is the transformation to be employed in more complicated rectilinear figures.

In illustration consider the simple case of a directed vector quadrilateral with its diagonals given as in Fig. I.

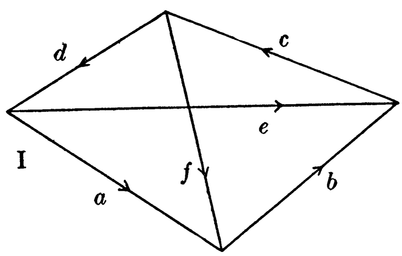


FIG. I

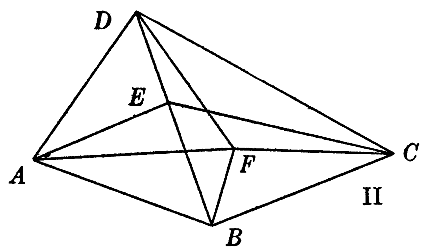


FIG. II

Applying (1) to each of the triangles abe and cde

$$aA + bB - eE = 0,$$
$$cC + dD + eE = 0.$$

Adding these results

$$aA + bB + cC + dD = 0, \tag{2}$$

the equation of all quadrilaterals $ABCD$ pseudo homochiral to a given quadrilateral $abcd$. If f is the other diagonal in I and A, B, C, D is a solution of (2) as above, then applying (1) to each of the triangles bcf and daf, their equations are

$$bB + cC + fF_1 = 0,$$
$$dD + aA - fF_2 = 0.$$

adding and observing (2)

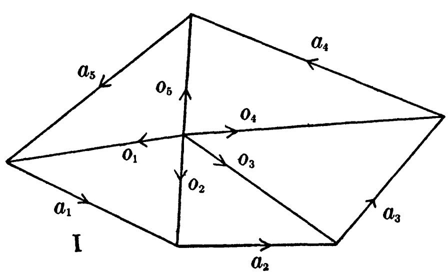
$$f(F_1 - F_2) = 0$$
$$\therefore \quad F_1 \equiv F_2 \equiv F \text{ since } f \neq 0.$$

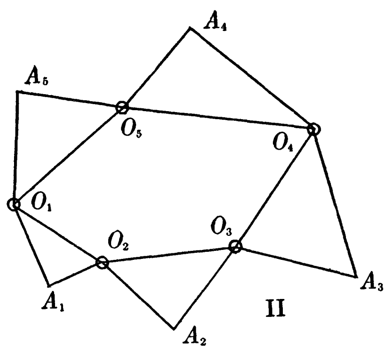
Therefore in any quadrilateral II, pseudo similar to a given quadrilateral I, if triangles be constructed on the opposite sides as bases and similar to the triangles into which a diagonal divides I these triangles will have their vertices common, E and F, and are the points corresponding to the diagonals of I.

3. With this brief preliminary we pass at once to the construction of the general polygon mentioned by Professor Musselman. Let I be a given polygon whose sides in order and direction are the vectors $a_1, a_2, \cdots, a_n$. Select any point arbitrarily in its plane and join to the vertices by the vectors $o_1, o_2, \cdots, o_n$. Fig. I is then divided into n "component" triangles whose bases are the sides of the polygon and each pair of adjacent triangles have a side in common. Apply equation (1) to each of the component triangles in order. In these equations the term o_rO_r occurs twice with opposite signs and on addition there results

$$\sum a_rA_r = 0, \qquad \sum a_r = 0 \tag{3}$$

the equation of all polygons pseudo homochiral to $a_1a_2 \cdots a_n$.





The choice of the O-points is arbitrary and the O-polygon is called the arbitrary polygon and the vertices of the triangles constructed on its sides

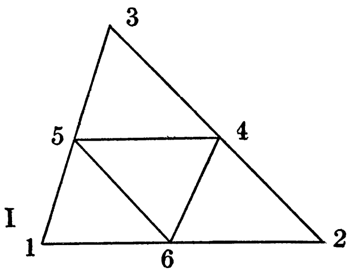
similar to the component triangles of I give a solution of equation (3). This necklace of triangles in II can be deformed in any manner by pin jointing the O-points, and making the sides of the O-polygon elastic it can be stretched or shrunk at will. The so-called "inside or outside" constructions correspond to merely changing the direction signs of the vectors in fig. I. Suppose a certain set of A points as a solution of (3) be fixed in position then the O-points are free to move but any movement of one prescribes the movement of each of the others. For if any O point describes an arc then each of the others must describe a similar arc of a similar curve. To construct a solution of the vector equation (3) all of the A points of the polygon II can be assumed at random except one, then assuming at random one O point and constructing in succession the triangles of the necklace II, by obvious constructions, similar to the component triangles of I, or as we briefly say, completing the necklace, the remaining A point is uniquely determined. Given all the A points but one, A_r, the A_r point is the "generalized" point of mean position of the others.

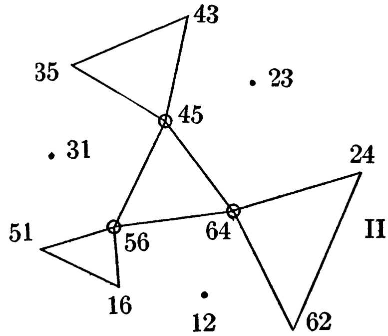
$$A_r = \frac{a_1A_1 + \cdots + a_{r-1}A_{r-1} + a_{r+1}A_{r+1} + \cdots + a_nA_n}{a_1 + \cdots + a_{r-1} + a_{r+1} + \cdots + a_n},$$

or is the "barycentric" center of the A set where the "mass" coordinates are complex numbers, of course including the case when they are real.

Now a word as to the use of term pseudo similarity. Under the old time definition two polygons are similar when they can be divided into the same number of similar triangles arranged in the same order, each to each similarly situated, and two adjacent have a common side. In the above the given polygon I has thus n component triangles, adjacent ones have a *common side*. In the necklace A-polygon it is composed of the same number of triangles arranged in the same order similar each to each and adjacent ones have a *common vertex*. A solution of the linear vector equation (3) is said to be a polygon II pseudo homochiral to the given a-polygon in I. If we change all the signs of the vectors in I the triangles are constructed on the opposite side of the O-polygon above and we get another solution of (3). This is the interpretation of the so-called "inside or outside" construction. Again, if in (3) the a's be replaced by the conjugates of the a's, or what is the same thing geometrically if the polygon I be reflected in any straight line into a polygon I′ then the A-polygon as constructed above is said to be pseudo heterochiral to the polygon I′, or if the A-polygon be reflected in any straight line its reflection is pseudo heterochiral to the polygon I. Some properties of these have been examined by R. L. Echols in his paper referred to.

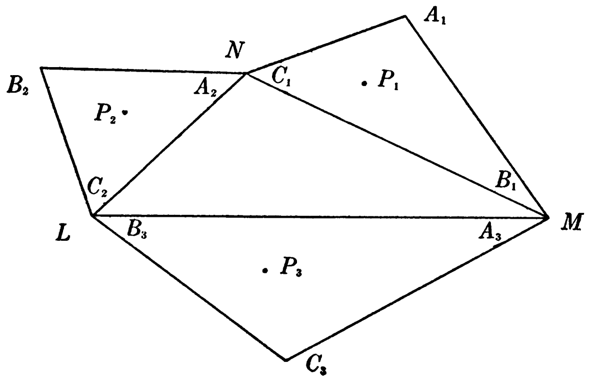
4. Another ornamental figure is obtained by dividing a given triangle into any number of similar triangles by lines parallel to its sides and transforming each of these by equation (1) forming a network of similar triangles having common vertices. Thus for the simplest case using the midtriangle of I the transform is the four triangles in II pin jointed at the common vertices.





However II be distorted the midpoints of 16, 62; 24, 43; 35, 51 give a triangle similar to 123. Dividing the sides of 123 into n equal parts and transforming the component triangles of I into a figure II in the same way an elaborate kaleidoscope pattern of similar triangles with pin jointed vertices is formed in II. The three centroids of the n points into which the n segments of each side is transformed are the corners of a triangle similar to 123 under all deformations of II.

5. We conclude this note, already too long, with a generalization of a popular problem in order to show how simply our algebra proves certain theorems which would otherwise be quite trying. Let x, y, z be the areal coordinates of a point P with respect to a given triangle ABC. Apply (1) to transform ABC into three similar triangles $A_rB_rC_r$, $r=1, 2, 3$ and let P_1, P_2, P_3 be the point x, y, z with respect to each respectively. Fit these three triangles to the sides of any arbitrary triangle LMN as in the figure:



The equations of the three triangles are

$$aA_1 + bN + cM = 0,$$

$$aN + bB_2 + cL = 0,$$

$$aM + bL + cC_3 = 0.$$

Multiply by x, y, z and add, then

$$a(xA_1 + yN + zM) + b(xN + yB_2 + zL) + c(xM + yL + zC_3) = 0,$$

or

$$aP_1 + bP_2 + cP_3 = 0.$$

Therefore the triangle $P_1P_2P_3$ is similar to ABC.

The same result is instantly obtained by transforming the component triangles PAB, PBC, PCA of ABC in I to the simplest necklace polygon II which gives at once $P_1P_2P_3$ similar to ABC. Then complete the triangles having P_1, P_2, P_3 as x, y, z point, giving the fig. 5.

A host of theorems in particular cases are easily read off as class exercises which are attractive to us amateurs; and I endorse what Professor Musselman very properly says, the general cases are worthy of further study, and would add, even by the professional mathematician.

The Reduction in Bitangents of Plane Algebraic Curves Due to Nodes and Cusps

By R. M. Winger, University of Washington

It is a familiar fact, as revealed at once by Plücker's equations, that each node reduces the class of an algebraic plane curve by two, and each cusp by three.[1] Likewise each node absorbs 6 and each cusp 8 points of inflexion (or flexes). That is, each node or cusp has the same effect as every other on the class and number of flexes—and this effect is independent of the order of the curve. It is quite otherwise however with the bitangents. For while the occurrence of nodes or cusps exacts a considerable toll in bitangents, the second node or cusp is not so costly as the first. Again when a curve with nodes acquires a cusp in addition, the cost in bitangents is not so great as when a non-singular curve acquires a cusp. Since the laws governing the absorption of bitangents in the several cases seem not so well known, although implied by Plücker's equations, I propose to derive them here.

Let n denote the order of the curve, d the number of nodes, r the number of cusps and δ the number of bitangents. Then the number $\delta_{d,r}$ of bitangents of a curve with d nodes and r cusps is given by[2]

$$(1)\quad \delta_{d,r} = N - (n+2)(n-3)(2d+3r) + 2d(d-1) + 6dr + 9r(r-1)/2,$$

---

[1] Throughout we shall understand by node a simple crunode or acnode and by cusp the simple or ordinary cusp. Likewise we shall use bitangent for the dual of node as just defined.

[2] For isolated cases this formula, which is hard to remember, is not so convenient as the more familiar $n = \nu(\nu-1) - 2\delta - 3\rho$, in which ν stands for the class and ρ for the number of flexes. Formula (1) can be obtained from the first three of Plücker although he himself derived it independently. See his *Theorie der algebraischen Curven*, p. 207.

where

$$N = n(n - 2)(n^2 - 9)/2$$

and represents the number of bitangents of a non-singular curve $(d=r=0)$.

First, let the curve acquire in succession $1, 2, \cdots d$ nodes but no cusps. Then, substituting $d=1$, $r=0$ in (1) we get

$$\delta_{1,0} = N - 2(n + 2)(n - 3). \tag{2}$$

Thus the first node absorbs $2(n+2)(n-3)$ bitangents. Next, let $d=2$, $r=0$ in (1) and we get

$$\delta_{2,0} = N - 4(n + 2)(n - 3) + 4. \tag{3}$$

Subtracting this from (2), we find that the second node absorbs 4 fewer bitangents than the first.

Generally, if a C^n has d or $d+1$ nodes respectively, the number of bitangents left will be

$$\delta_{d,0} = N - 2d(n + 2)(n - 3) + 2d(d - 1) \tag{4}$$

$$\delta_{d+1,0} = N - 2(d + 1)(n + 2)(n - 3) + 2d(d + 1). \tag{5}$$

Hence the number of new bitangents used in forming the $(d+1)$th node is

$$\delta_{d,0} - \delta_{d+1,0} = 2(n + 2)(n - 3) - 4d,$$

i.e., the $(d+1)$th node costs the curve $4d$ fewer bitangents than the first. Summarizing,

1°. *If a C^n acquire in succession a number of nodes (but no cusps), the first node absorbs $2(n+2)$ $(n-3)$ bitangents, the second 4 fewer and so on, each node absorbing 4 fewer bitangents than the preceding.*

By analogous argument we find

2°. *If a C^n acquire in succession several cusps (without nodes), the first cusp absorbs $3(n+2)$ $(n-3)$ bitangents, the second $3(n+2)$ $(n-3)-9$ and so on, each successive cusp absorbing 9 fewer bitangents than its predecessor.*

Thus the first cusp uses up one and a half times as many bitangents as the first node. But due to the decreased rate of consumption of bitangents by cusps as compared to nodes, it may happen for the lower values of n that the kth node actually absorbs more bitangents than the kth cusp. In the case of the quartic, e.g., the third cusp effects no reduction in bitangents at all, whereas the third node consumes four. And the fourth and fifth cusps of a quintic (without nodes) absorb 15 and 6 bitangents respectively as contrasted with 16 and 12 for the fourth and fifth node there being no cusps.

Suppose now that the curve has d nodes. What is the effect on the number of bitangents of the occurrence of cusps in addition to the nodes? Setting $r=1$ in (1) we get as the number of bitangents of a C^n with d nodes and one cusp

$$\delta_{d,1} = N - (2d + 3)(n + 2)(n - 3) + 2d(d - 1) + 6d.$$

Subtracting this from $\delta_{d,0}$ (equation (4)) we find as the reduction in bitangents due to the cusp

$$\delta_{d,0} - \delta_{d,1} = 3(n + 2)(n - 3) - 6d, \tag{6}$$

a number less by $6d$ than the cost of the first cusp to a curve without nodes. Similarly

$$\delta_{d,1} - \delta_{d,2} = 3(n + 2)(n - 3) - 6d - 9,$$

hence the second cusp absorbs 9 fewer bitangents than the first. And so on. For the number of bitangents required for the $(r+1)$th cusp is

$$\delta_{d,r} - \delta_{d,r+1} = 3(n + 2)(n - 3) - 6d - 9r. \tag{7}$$

Therefore

3°. *If a C^n have d nodes and then acquire cusps (in addition to the nodes), the first cusp absorbs $3(n+2)(n-3)-6d$ bitangents, while each successive cusp absorbs 9 fewer bitangents than its predecessor.*

In the same way we prove

4°. *If a C^n with r cusps acquire nodes, the first node absorbs $2(n+2)(n-3)-6r$ bitangents, while each successive node absorbs 4 fewer bitangents than its predecessor.*

In this and the preceding case the first node (or cusp) consumes fewer bitangents than if the curve had been non-singular, yet the number of bitangents absorbed by each successive node (or cusp) diminishes at the same rate as before, i.e. by 4 and 9 respectively.

One case remains: cusps may arise through specialization of the nodes. From 1° and 2° we see that when a solitary node is changed into a cusp, $(n+2)(n-3)$ additional bitangents are absorbed. Again if a curve have two nodes one of which becomes a cusp, $(n+2)(n-3)-2$ bitangents are lost in the process for this number represents the difference between $\delta_{2,0}$ and $\delta_{1,1}$; while if the second node becomes a cusp, $(n+2)(n-3)-3$ more bitangents disappear. Generally, let the curve have d nodes which undergo a metamorphosis into cusps one after the other. The loss in bitangents due to the appearance of the first cusp is given by

$$\delta_{d,0} - \delta_{d-1,1} = (n + 2)(n - 3) - 2(d - 1), \tag{8}$$

while the second cusp absorbs

$$\delta_{d-1,1} - \delta_{d-2,2} = (n + 2)(n - 3) - 2(d - 1) - 1,$$

or one less than the first. And so on as long as the process continues.[1] For we

[1] It may not be possible for all the nodes to change to cusps. Thus a quintic curve may have 6 nodes or 5 cusps, but of the 6 nodes of a rational quintic only 4 may become cusps, due to the scarcity of flexes.

find that the number of bitangents absorbed by the $(r+1)$th cusp is r fewer than for the first. Or we may say

5°. *If a C^n have d nodes (and no cusps) which change successively into cusps, the first cusp reduces the number of bitangents by $(n+2)\ (n-3)-2(d-1)$, which is equal to the loss in bitangents when a lone node becomes a cusp less twice the number of nodes which remain. And each successive cusp absorbs one bitangent fewer than its predecessor as long as the process continues.*

Rational Curves. Since a rational curve must have the equivalent of $(n-1)(n-2)/2$ double points, including cusps, the number of bitangents of a rational curve of order n with r cusps is, from (1)

$$\delta = 2(n - 2)(n - 3) - r(4n - r - 11)/2. \tag{9}$$

Directly by the use of this formula or as a corollary of theorem 5° we have

6°. *The number of bitangents of a rational curve of order n without cusps is $2(n-2)\ (n-3)$. If the nodes change into cusps successively, the first cusp reduces the number of bitangents by $2n-6$, the second by $2n-7$ and so on, each new cusp absorbing one bitangent fewer than its predecessor.*

In the foregoing theorems we have answered the questions: What is the loss in bitangents when an algebraic plane curve C^n

(a) acquires in succession $1, 2, \cdots d$ nodes (but no cusps)
(b) acquires in succession $1, 2, \cdots r$ cusps (but no nodes)
(c) with d nodes acquires in addition $1, 2, \cdots r$ cusps
(d) with r cusps acquires in addition $1, 2, \cdots d$ nodes
(e) with d nodes lets them change into cusps one after another.

The number of bitangents left after completion of the process in (c) and (d) alike will be $\delta_{d,r}$ as shown in equation (1). The final stage in (e) (theorems 5° and 6°) may be the same as in (b) but in the case of rational curves there will always be nodes (or the equivalent of nodes) left, except when $n=4$.

Compound singularities are to be treated as in all applications of the Plücker formulas—first ascertaining the number of nodes and cusps involved. Additional bitangents may be absorbed by these higher singularities due to the presence of latent or excess double points (nodes and cusps) and double lines (bitangents and flexes). Our formulas give the normal quotas. Thus for a triple point formed when the curve passes again through an ordinary cusp, put $d=2$, $r=1$. But a cusp of the second kind (ramphoid cusp) is equivalent to one node, one cusp, one bitangent and one flex and thus absorbs one more bitangent than an ordinary node and cusp together.

RECENT PUBLICATIONS

EDITED BY R. A. JOHNSON, Brooklyn College of the City of New York

All books for review should be sent directly to the editor of this department, at Brooklyn College, 66 Court Street, Brooklyn, N. Y., and not to any of the other editors or officers of the Association.

REVIEWS

Plane Trigonometry and Four-place Tables. By W. A. Granville, revised by P. F. Smith and J. S. Mikesh. New York, Ginn and Company, 1934. 210 and 43 pages. $1.60.

The volume under review is a less drastic revision of the well known text by Granville than might appear from the cover. Minor changes include addition of an index, amplification of the table of natural functions, and use of the odd-even answer convention; the major change is in the order. The first third of the book is a discussion of the trigonometric functions of any angle; next comes the work on triangle solution; the last third is devoted to addition theorems, identities, and inverse functions. The result is greater emphasis on the function idea and the general angle, and a more unified treatment of numerical work.

There are difficulties with this arrangement as with others. It is a question how well the first 66 pages can be motivated without story problems or triangles; at the other end difficulty arises in connection with the law of tangents for which a geometric proof must be given first and the usual proof postponed. The first proof is made more difficult by unfortunate paging. Another minor criticism is of the section on approximate data where the numerical illustration is confusing.

In quality of exposition we think the revision is successful. The clearness of the original is retained but the step by step style is not overworked. Abundant illustrative examples and problems are found as before and the resulting text should prove as teachable as its predecessor.

E. H. CUTLER

The Mongean Method of Descriptive Geometry. By W. H. Roever. New York, The Macmillan Company, 1933. xiv+151 pages. $3.00.

This small treatise on ordinary or "Mongean" descriptive geometry, a title which at present must be considered as obsolete, has been prepared by Professor Roever "according to the procedure of Gino Loria, professor of mathematics at the University of Genoa and adapted to American needs."

It starts with a foreword by Loria himself and a preface by the author in which he declares that "the subject of descriptive geometry may be properly regarded as belonging to the mathematical disciplines. Notwithstanding this fact it has been almost entirely ignored, or regarded as trivial, by many, if not by most mathematicians in America."

The text itself begins with a valuable introduction with an interesting footnote taken from A. Schoenflies, *Einführung in die Hauptgesetse der Zeichnerischen Darstellungsmethoden*, and this is followed by eleven chapters in which the rudi-

ments of descriptive geometry of point, line and plane are explained. Important chapters which treat of solids and surfaces, their plane sections and interpenetrations, of space curves, etc., where the methods of orthographic projection really show their power and beauty, are missing. A student who wishes to learn something about these more advanced topics would be compelled to consult some other textbook, of which there are many, as for example the excellent Elements of Descriptive Geometry by C. W. MacCord. The reviewer cannot understand why Professor Roever, who is an authority on descriptive geometry, should find it necessary to follow a foreign author in a field which is very elementary and which is very ably treated in a number of American texts. From the book under review one must not conclude that Professor Loria's *Vorlesungen über darstellende Geometrie* in two volumes does not give an excellent account of descriptive geometry and its relation to other mathematical branches. The same is true of other Italian texts, as for instance the *Lezioni di Geometria Descrittiva* which years ago Professor Enriques gave at the University of Bologna, and many French and German treatises on this subject. In all these one finds a very strong mathematical and at the same time practical point of view.

In conclusion it must be stated that Roever's descriptive geometry is very clearly written and may well serve as an elementary introduction for students who do not care for the practical applications but who wish to get at least an inkling of what descriptive geometry is concerned with.

It is very doubtful however whether Professor Roever has succeeded in removing the lack of appreciation for the mathematical value of descriptive geometry by the American mathematician.

The fact is that those ideas and propositions which are important from a mathematical standpoint do not need descriptive geometry for their establishment. It is true, descriptive geometry must make use of geometry, but it is essentially the art of representing on a plane or on a surface figures and forms of three or more dimensions. As such it has, of course, undeniable heuristic value.

ARNOLD EMCH

Tables of Functions with Formulae and Curves. By Eugen Jahnke and Fritz Emde. Second (Revised) Edition. Berlin, G. B. Teubner, 1933. xviii+330 pages. RM 16.

This complete revision of the earlier well-known "Funktionentafeln" includes much additional material in the way of new tables and graphical data. A minor change is the printing of explanatory material in English, as well as in German.

We shall briefly outline the contents. The first few tables are designed to aid in elementary calculations with complex numbers, and in the numerical solution of real cubic equations and certain simple elementary transcendental equations. As the main interest is the higher functions, the elementary functions are given only brief tables, though we find formulas, references to other tables,

and several graphs for them. The Planck's radiation function is tabulated, and graphs of the source functions of heat-flow are given.

Several special definite integrals follow, namely the sine integral, the cosine integral, the logarithmic integral, the gamma function, with its derivatives, and the probability and related integrals, including Fresnel's.

A number of tables associated with elliptic functions and integrals and theta functions are listed. We find here a special table of use in computing the inductance of coils.

Finally there are tables of Legendre's polynomials, Bessel's functions with the related Hankel and Neumann functions, and at the end the zeta function of Riemann. In each case sets of formulas and references to more extended tables, sources and textbooks are given.

The graphs are very clear and informative. Possibly the reader who deciphers the graph on p. 36 with its L for "right angle" and other contractions, or who finds the need for reading fig. 150 on p. 293 with the inverted legend may feel that a little too much ingenuity has been used. The "reliefs" of the functions of a complex variable which show perspective drawings of the surface whose ordinate, at any point in the complex plane is the absolute value of the function at that point, are extremely illuminating as to the location of the zeros and poles, and behavior near the essential singularities. As the authors suggest, these graphs for the elementary functions, elliptic functions, gamma function, and zeta function might well have a place in an elementary course in functions of a complex variable.

Most of the tables are given to four places, which is sufficient for most of the applications, and they are printed in bold faced gothic figures, making for maximum legibility and minimum eye-strain. This revision will undoubtedly prove as invaluable as its predecessor to those engaged in practical computations involving higher functions.

PHILIP FRANKLIN

Tables of the Higher Mathematical Functions, Vol. I. By H. T. Davis. Bloomington, Indiana, The Principia Press, 1933. xiv+377 pages. $6.50

This is the beginning of a series of volumes which, together, will constitute a fairly complete set of tables of the higher mathematical functions. The tables are given to a minimum of ten decimal places (a few to as many as twenty), and as many are based on original computation, and all have been checked for errors, it is evident that Professor Davis and his staff have mapped out an ambitious program. Besides a description of the manner of their computation, a brief history of the functions is given, and extensive bibliographies of earlier tables are included.

The volume before us begins with a chapter on the history of mathematical tables, which is followed by a brief discussion of modern analytical methods of calculation. Interpolation is then discussed, and several tables of coefficients

are given, of use in connection with interpolation itself, as well as those used in finding the derivative of a function from its tabulated values. A general bibliography of mathematical tables completes the introductory matter.

The last half of the book gives the tables proper. There are tables of $\Gamma(x)$ and its logarithm for various real ranges, and a short table of $1/\Gamma(x)$ for certain complex values. In particular, among other values, the tables combined give the values of the logarithm of $|\Gamma(x)|$ for x ranging from -10 to 101, to twelve decimal places. Then follows a series of tables of the psi function, including the values to ten places for x ranging from -10 to 450.

This work should be useful as a master table from which briefer tables for special purposes may be prepared, as well as for those occasions when a few values are wanted with great accuracy.

PHILIP FRANKLIN

The Universe of Science. By H. Levy. New York, The Century Co., 1933. xiii+224 pages. $2.50.

Science may well contemplate from time to time its achievements and its goals. Too infrequently scrutiny is denied the vast social implications of scientific progress. The present book attempts this appraisal. "Science is primarily a movement, a social outgrowth serving social ends," says the author, "and all attempts to isolate any aspect of it, be it even the purest mathematics, from the social movement of which it is an integral part, can lead to nothing but false and dangerous conclusions." The book is divided into five chapters: The Changing Pattern; Unpicking the Threads: The Process of Isolation; The Queen of the Sciences—Mathematics; Scientific Determinism; Science—A Social Venture.

The author regards the methods of science as a system designed to render environment neutral. This he calls the process of isolation. Mathematics he regards as "the method of isolation raised to a fine art" and he uses this discipline to illustrate his principal thesis. The book is stimulating to those who would take a broad view of the social aspects of science. Technical concepts and mathematical formulas are used sparingly, and the chapters may be followed without difficulty by lay readers.

H. T. DAVIS

Mathématiques financières. By J. Dubourdieu. Paris, Librairie Armand Colin, 1932. 219 pages.

In this small volume we find much of the material which is included in what is generally called the theory of investment, although more of the actual applications of the concepts to business and business accounting are given than one customarily finds in American textbooks. The author uses elementary algebra throughout. He discusses stock and bond investments, amortization problems, etc., with practical suggestions as to their application. Careful definitions of all terms used are made and the discussion is illustrated by well selected numerical examples.

The book differs from American texts by its omission of the customary exposition of the theory of life insurance and of the numerical tables of the standard functions of finance.

H. T. DAVIS

The Elements of Euclid. By Isaac Todhunter, with an introduction by Sir Thomas L. Heath. Everyman's Library, No. 891. New York, E. P. Dutton and Co., 1933. $0.70.

It is of considerable importance that Everyman's Library, with its wide sweep of the world's best books, should include the great mathematical classic. It is of equal importance that the particular edition thus chosen should be the one that preserves the spirit of Euclid with such scientific reverence and wisdom as does Todhunter's revision (1862) of Simson's text. As is known to many, Todhunter improved the form in which the arguments had been presented in earlier English texts and standardized more fully the language in which the demonstrations were couched. An American reader, however, will still blink when he reads: "Ratios that are the same to the same ratio, are the same to one another" and a few other theorems given in a complicated form not accepted in American education.

Heath concedes the unsuitable character of the book as a text for beginners, but emphasizes its fascination for any one who "would have the necessary knowledge and judgment to appreciate the highly contentious matters which have to be grappled with in any attempt to set out the essentials of Euclidean geometry as a strictly logical system, and, in particular, the difficulty of making the best selection of unproved postulates or axioms to form the foundation of the subject."

W. D. CAIRNS

MATHEMATICS CLUBS

EDITED BY F. M. WEIDA, The George Washington University, Washington, D.C.

All reports of club activities, suggestions and topics for club programs should be sent to F. M Weida, The George Washington University, Washington, D.C. All manuscripts should be typewritten with double spacing, and with margins at least one inch wide. All club activity manuscripts for the academic year 1933–1934 should be submitted for publication not later than June 1, 1934.

CLUB ACTIVITIES

1932–1933

THE PI MU EPSILON MATHEMATICAL FRATERNITY

Pi Mu Epsilon is an academic fraternity in institutions of university grade. Its primary aim is the advancement of mathematics and scholarship. It is a living, active, working fraternity of scholars in which the members are actively engaged in study and research and the exchange of ideas in the field of mathematical science.

Pi Mu Epsilon of Washington University

The officers for 1932–1933 were: Eugene Stephens, Director; Charles O. Quade, Vice Director; Jessica Young Stephens, Secretary; William M. Yates, Assistant Secretary; William E. Stephens, Treasurer; Henry E. Zeffren, Librarian; Charles Huff, Cecilia Lehmann, Otto H. Schmitt, and S-Marie Vaughn, Student members of the Executive Committee.

There were forty-seven active members during the year 1932–1933. Thirty new members were initiated on April 22, 1933, distributed as follows: from the College of Liberal Arts, nine; from the School of Engineering and Architecture, sixteen; from the School of Graduate Studies, three; from the Southwestern Bell Telephone Company, one; and from the Missouri State Life Insurance Company, one.

The meetings and programs were as follows:

October 5, 1932: "The design of a precision thermostat" by Otto H. Schmitt.

November 8, 1932: "The Lewis charts" by Charles Huff.

December 5, 1932: "Napier's construction of his table of logarithms" by William J. Roa, Jr.; "A short talk on errors" by Alexander S. Langsdorf, Jr.

January 11, 1933: "Selected incidents from the history of mathematics" by Dr. Jesse Osborn; "The auto-gyro" by William E. Stephens; "Report on the Convention at Atlantic City" by Professor William H. Roever.

February 7, 1933: "In pursuit of the equation" by Henry E. Zeffren; "A trip through a steel plant" by Bernard Agruss.

March 6, 1933: "Finite increments as applied to interpolation" by Samuel Rosenbloom.

March 15, 1933: Business meeting; election of new members.

April 22, 1933: Initiation, banquet and social gathering. The speaker of the evening was Professor W. H. Roever and his topic was "The International Mathematical Congress at Zurich, Switzerland, in 1932."

May 13, 1933: Business meeting and social gathering. The Eugene Stephens prize of ten dollars for the paper presented before the chapter during the year that was of most general interest to student members was presented to Henry E. Zeffren. The prize paper was entitled "In pursuit of the equation" and was presented to the chapter on February 7, 1933.

JESSICA YOUNG STEPHENS, *Secretary*

LOCAL MATHEMATICS CLUBS

The Mathematics Club of the University of Iowa

The undergraduate mathematics club of the University of Iowa finished its 1932–1933 academic year feeling that the interest shown by the regular attendance of its members and their support otherwise justified the efforts of the officers in arranging programs for the meetings.

At the last meeting of the club during the year 1931–1932, the following officers for the year 1932–1933 were elected by the active members: J. W. Querry, President; Catherine Mueller, Secretary-Treasurer; Dr. N. B. Conkwright, Faculty Adviser.

Although the club had no special aim toward which it worked, those who planned programs did so with the interest of the undergraduates in mind, hoping to bring to them the interesting applications of the various branches of mathematics. The only requirement for membership was the payment of a small fee for dues to cover the expenditures of the club.

At our first regular meeting, November 3, 1932, Professor Roscoe Woods spoke on "Areas" and developed formulas for the areas of triangles with the combined use of geometry and trigonometry. On December 15, 1932, Professor N. B. Conkwright lectured on "Number theory". Professor L.E. Ward spoke on "Applications of trigonometry and hyperbolic functions to cubic equations" at our meeting on February 2, 1933. Dr. A. T. Craig spoke on "Equation of exchange" on March 2, 1933. Three undergraduates students gave papers at the meeting on March 30, 1933 which were as follows: "The Simson line and its applications in plane geometry" by Catherine Mueller; "The harmonic series in plane geometry" by Eugene Sinn; and "The calculus of variations"

by William Rae. At our final meeting of the year on May 4, 1933, Mr. Arthur Ollivier gave an informal talk on "Magic squares."

CATHERINE MUELLER, *Secretary-Treasurer*

The Saint Norbert College Mathematical Society

This organization is composed of students and faculty members of both the graduate and undergraduate departments of the college. The honorary membership is extended to any person within and without the circles of the college, on the basis of active interest in any field of mathematics and its applications. The club was organized October 10, 1932.

The officers for 1932–1933 were: Lawrence Berner, President; George Claridge, Vice President and Librarian; George J. Kalcik, Secretary-Treasurer; Reverend Dr. L. A. V. DeCleene, O. Praem, Director.

The meetings and programs were as follows:

October 17, 1932: "Teaching problems in Secondary Mathematics" by Lawrence Berner.
November 21, 1932: "A physical demonstration of the Einstein theory" by Mr. George Kalcik.
December 20, 1932: Banquet.
January 14, 1933: "Atomic structure" by Mr. George Claridge.
January 24, 1933: "The coefficient of correlation" by Reverend Dr. DeCleene.
February 13, 1933: Reception of new members.
March 27, 1933: "The voltage generated in a conductor moving in a magnetic field; and maximum, average and effective values" by Mr. Irvin Haus.
May 3, 1933: "Reaction rates in physical chemistry" by Mr.John Fox.
May 22, 1933: "The Bernoulli's" by Mr. George Stoeker.
June 5, 1933: Banquet.

GEORGE J. KALCIK, *Secretary-Treasurer*

The Mathematics Club of the Louisiana State University

The purpose of our club is to stimulate a greater interest in mathematics; to encourage worthy mathematical research; and to afford opportunity for mutual fellowship among its members. All persons who are interested in mathematics or its applications are eligilble for membership in the club.

Our club was recently formed by the graduate students of the mathematics department. The officers were elected and at a meeting called on January 8, 1933, a constitution was adopted. The officers elected were: Ruth Johnson, President; Earl Thomas, Vice President; James A. Ward, Secretary-Treasurer.

The following members were elected to serve on the program committee: Ruth Johnson, Chairman; Dr. Irby C. Nichols and Professor F. A. Rickey.

The following members were elected to serve on the social committee: Mrs. Alta H. Samuels, Chairman; Earl Thomas and Mrs. A. P. Daspit.

The first regular meeting was held on January 17, 1933. The program consisted of two very interesting talks: "The nine-point circle" by Mrs. Samuels; and "A problem in number theory" by Mr. Rickey.

The faculty and members believe that the club will stimulate greater interest and promote mathematics in Louisiana State University.

JAMES A. WARD, *Secretary-Treasurer*

The Mathematics Club of Syracuse University

This club, founded on February 15, 1933, opens its membership to any undergraduate student who is taking or has taken differential calculus in the College of Liberal Arts or to any graduate student taking mathematics.

The club meets once a month and has already created a lively interest among its seventy-five members in the furtherance of creative and vigorous thinking along mathematical lines.

The officers elected for the remainder of the academic year 1932–1933 were: Eileen MacFarland, President; Saul Balmuth, Vice President who is also Chairman of the Program Committee; Beatrice Churchill, Secretary-Treasurer; Franklin Baker, Chairman of the Social Committee; Mrs. May Harwood, Faculty Adviser.

The meetings and programs were as follows:

February 15, 1933: The program consisted of mathematical puzzles and games.

March 8, 1933: "The moon, our nearest neighbor" by Dr. Sidman Poole, Head of the Geography and Geology Department. At this meeting, officers for the academic year 1933–1934 were elected.

May 17, 1933: The events of the year closed with a picnic at Green Lake, Jamesville, N.Y.

BEATRICE CHURCHILL, *Secretary*

The Mathematics Club of St. Xavier College

The mathematics club is a student organization formed for the threefold purpose of discussing topics of general mathematical interest, of furthering this interest, and of uniting those having mathematical knowledge as a common aim.

Meetings are held fortnightly, every third meeting being strictly social. Those eligible for membership are students with mathematics as their major or minor. There were twenty-eight members who contributed actively to the work of the club.

The officers for 1932–1933 were: Eleanor Dunne, President; Emma Genevieve Dum, Vice President; Jean Koepke, Secretary.

The meetings and programs were as follows:

Fall Quarter: "The mathematics of finance." The several topics were discussed by various members of the club.

March 2, 1933: "The determination of the volumes of prismatoids" by Margaret Prendergast.

March 20, 1933: "Mathematical induction with regard to geometry" by Emma Genevieve Dum.

April 6, 1933: Formal reception of new members.

April 20, 1933: Informal initiation and tea.

May 8, 1933: Sophomore meeting. This was an inovation of this year.

May 24, 1933: "Calculating prodigies" by Patricia Dunphy; "Waring's problem with regard to biquadrates" by members of the Theory of Numbers class; "The graphical representation of the cube of a binomial" by Virginia Cheevers; "The nature of determinants" by Ruth Maginsky; "Puzzles of modern geometry" by Edna Bucher. This meeting was an open meeting for friends of the club.

June 1, 1933: Election of officers.

June 12, 1933: Annual picnic.

JEAN KOEPKE, *Secretary*

The Mathematics Club of the University of Toledo

ΔX, the mathematics club of the University of Toledo has completed its fourth year. With a membership of sixty-six the organization has endeavored to augment both the academic and social life of the students who have a common interest in mathematics. Each of the monthly meetings included (a) the presentation and discussion of one or more papers, (b) mathematical recreations or contests, (c) refreshments.

The officers for 1932–1933 were: Arthur Pritchett, President; Dyrexa Chapman, Vice President; Dorothy Jane Pollock, Secretary-Treasurer; Professor Wayne Dancer, Faculty Adviser.

The meetings and programs were as follows:

October 1, 1932: Get-acquainted roast at Close Park.

October 20, 1932: "Maria Agnesi and the witch curve" by Dyrexa Chapman.

November 17, 1932: "Conformal representation" by Arnold Peterson.

December 22, 1932: "Important points in mathematics" by Blanche Fishler. At this Christmas meeting, members exchanged ten-cent gifts and made contributions of food for the needy of

Toledo's tent colony.

January 26, 1933: (Alumni meeting) "Theory and applications of the catenary" by Edwin Jablinski.

February 23, 1933: "The director circle" by Joseph Dence; "Important lines in mathematics" by Fern Welker.

March 30, 1933: (Freshman meeting) "The history of trigonometry" by Dorothy Jane Pollock.

April 27, 1933: "Hyperbolic functions" by Sol Boyk.

May 25, 1933: Annual banquet at Hotel Plaza. "The number *e*" by Mr. Maurice Lemme; Playlet, "Discord in mathematics land."

June 8, 1933: Annual picnic on Raisin River.

DYREXA CHAPMAN, *Vice President*

The Mathematics Club of the College of Saint Teresa

The mathematics club of the College of Saint Teresa, Winona, Minnesota, is combined with the physics club since these two subjects are so closely related by bonds of exactness and of dependency on each other. The club does active work on the campus.

The club met for organization September 21, 1932, at which time Miss Margaret McNamara was chosen as the chairman of the club. She is assisted by Miss Kathryn Hinsenbrock, Miss Kathleen McIllece and Miss Catherine Keeley as the program committee. These are the only officers of the club.

The club had twenty-five active members and is under the sponsorship of the Heads of the Mathematics and Physics departments of the College. All students interested in the activities of the club are eligible for membership; however, anyone having three unexcused absences is deprived of that membership.

The aim of the club is to create and maintain interest in mathematics and physics, by presenting the values of these subjects to the students and by studying various topics not provided for by the regular class periods.

The meetings and programs were as follows:

September 21, 1932: Organization meeting; election of officers.

October 11, 1932: "The aims of the club" by Miss Margaret McNamara; "Suggestions for club work" by the members.

October 26, 1932: "The life and work of Professor Picard" by Miss Louise Suddendorf; "Bunsen Photometer" by Miss Florence Lang; "Women in the field of mathematics" by Sister á Kempis.

November 9, 1932: "The theory of relativity" by Mr. A. C. Bogard.

November 23, 1932: "The science involved in the construction and flying of airships" by Miss Charlotte Bradshaw and Miss Elvira Mars; "Why an arbitrary angle can not be trisected by means of a ruler and compass" by Miss Helen Pieffer.

December 14, 1932: "Dr. Langmuir" by Miss Mary Louise Wiess; "Mathematics" by Miss Kathryn Hinsenbrock.

January 11, 1933: "Contributions of ancient Greece and Rome to mathematics" by Miss Florence Kulick; "Physics of the household" by Miss Sally Guimond.

January 25, 1933: "Theory of numbers and the origin of the number system" by Mr. A. C. Bogard.

February 8, 1933: "The oldest extant mathematics" by Miss Kathryn Hinsenbrock; "Method of determining the wave length of red light" by Miss Florence Lang.

March 8, 1933: "Records of scientific men" by Miss Margaret McNamara.

March 22, 1933: "Practical physics" by Miss Shirely Layde; "Euclid" by Miss Catherine Keeley.

April 12, 1933: A résumé of the year's work.

MARGARET MCNAMARA, *Chairman*

The Mathematics Club of Vanderbilt University and Peabody College

Our mathematics club holds its regular meetings on the third Friday of each month alternately on Vanderbilt and Peabody campuses. The officers are elected at the end of each quarter,

the outgoing officers being the nominating committee. The officers for 1932-1933 were: Fall quarter: Mr. J. W. LaGrone (Vanderbilt), President; Miss Jack Wright (Peabody), Vice President; Miss Dorothy Crook (Vanderbilt), Secretary. Winter quarter: Mr. Albert Sloan (Vanderbilt), President; Miss Roberts (Peabody), Vice President; Miss Mattie Tate Wood (Vanderbilt), Secretary. Spring quarter: Mr. Robert Dunaway (Vanderbilt), President; Mr. Frank Balser (Peabody), Vice President; Miss Alma Foreman (Vanderbilt), Secretary.

There were about fifteen active members in the club.

The aim of the mathematics club is to stimulate interest in mathematics. "All students of Vanderbilt University and Peabody College who have had the calculus or are taking calculus may be members of the club. All instructors in mathematics in both institutions are eligible for membership."

The meetings and programs were as follows:

October 21, 1932: "The dialytic method of solving simultaneous equations" by Professor Wilson L. Miser, Vanderbilt.

November 18, 1932: "Solutions of linear equations" by Mr. Kimbark Peterson.

January 20, 1933: "Seeing the Parthenon mathematically" by Professor F. L. Wren, Peabody.

February 17, 1933: "Some aspects of curved space" by Mr. Frank Balser, Peabody; "Method of plotting the higher plane curves" by Mr. Graham, Vanderbilt.

March 24, 1933: "Descartes' method of finding the slope and normal to a curve" by Miss Dorothy Crook, Vanderbilt; "Mathematics used by the American Indians north of Mexico" by Miss Caty, Peabody.

April 21, 1933: "Einstein's theory" by Dr. Webb, Peabody.

There were two social meetings of the club: On Friday evening, December 9, 1932, we met at the home of Dr. Wren of Peabody. About forty people were present. On Saturday afternoon, April 22, 1933, we met at Percy Warner Park. About thirty people were present.

ALMA FOREMAN, *Secretary*

Phi Chi Mu of Washington and Jefferson College

Phi Chi Mu, honorary fraternity in mathematics, physics, and chemistry, is open to juniors and seniors doing outstanding work in their particular chosen fields. It is limited to twelve active members. The society aims to stimulate interest and activity in the fields of mathematics, chemistry, and physics.

The officers for 1932–1933 were: Samuel P. Delisi, '33, President; Richard H. Parks, '33, Secretary. The faculty advisor is Dr. Clyde S. Atchison, Head of the Mathematics Department.

The meetings and programs were as follows:

November 22, 1932: "The isolation of crystalline pepsin and trypsin" by H. G. Kunz.

December 6, 1932: "Pasteur's experiments in molecular asymmetry" by T. S. Boyd.

February 7, 1933: "The struggle for existence" by G. N. Hoare.

March 14, 1933: "A simple experiment with a sand pendulum defining approximation of Lissajous's curves" by J. H. Lewis.

April 4, 1933: "The changing trend of life" by David Taksa.

May 9, 1933: "Successful tail blastema graft in Triturus Viridescens" by G. B. Ostermann.

May 23, 1933: "Hyperbolic trigonometry" by S. P. Delisi; "Some Physico-Chemical interpretations of life" by B. M. Kagan.

SAMUEL P. DELISI, *President*

The "Irrational Club" of the University of Wyoming

The purpose of our club is to promote good fellowship among those interested in mathematics, and to develop an appreciation of, and a love for the science which is the bond of its members.

The officers for 1932–1933 were: Kenneth Shelver, Positive Square Root, President; Rachel Achenback, Negative Square Root, Vice President; Louise Ross, Keeper of the Log and Bones, Secretary and Treasurer; Dominating Element, Custodians of the Indices, Committee. Our officers

are elected by popular vote each fall term.

There were twenty-four active members.

The meetings and programs were as follows:

October 20, 1932: "Mathematics in Germany" by Miss Neubauer.

November 3, 1932: "The duo decimal system" by Miss Neubauer.

November 17, 1932: "The duo decimal system" by Miss Achenback.

December 27, 1932: "Mathematics at Northwestern" by Professor C. F. Barr.

January 25, 1933: "Technocracy" by Professor O. H. Rechard.

February 9, 1933: "Mathematics in artillery" by Lieutenant Adams.

March 28, 1933: "The solution of a simultaneous system of linear matrix equations in two unknowns" by Miss Achenback.

April 25, 1933: "The slide rule" by Professor Sechrist.

May 9, 1933: "Mathematics and physics" by Dr. Hammond.

LOUISE ROSS, *Secretary and Treasurer*

The Mathematics Club of Wellesley College

The purpose of our club is to further the interest in mathematics through informal meetings in which topics from any field of mathematics shall be discussed.

The officers for 1932–1933 were: Dorothy Reinman, President; Miriam Londy, Vice President; Elizabeth Richardson, Treasurer and Senior Executive; Mary Lind, Junior Executive; Constance Bennett, Secretary. These officers were elected at the last meeting of the club in the year 1931–1932. The executive board of the club which consists of the President, Vice President, Senior and Junior Executives, and the Secretary, appoints two candidates for each office and these are voted on by the members of the club. The Secretary and Junior Executives are juniors and the other offices have to be held by seniors.

Any member of the sophomore (after the first semester), junior, or senior class who is taking elective work in the department of mathematics may become a member of the organization by the payment of yearly dues of seventy-five cents. Any member of the faculty may by vote of the club become a member. During the year 1932–1933 there were forty-five active members.

The meetings and programs were as follows:

October 1932: "The International Mathematical Congress at Zurich" by Miss Marion E. Stark, Professor of Mathematics and Faculty Advisor of our club.

November 1932: "Hyper-complex numbers" by Professor Clara E. Smith, Chairman of the Department of Mathematics.

January 1933: "The mystery at X," a play which used mathematical terms and notations for its humor, written by Professor Marion E. Stark, was presented by members of the club.

February 1933: The meeting was devoted to talks given by members of the club telling of the use of mathematics in the different fields of science and art. Brevard Nesbit spoke on "The applications of mathematics to music"; Emily Bidwell spoke on "The applications of mathematics to chemistry"; Grace Hoyer spoke on "Mathematics in architecture"; Ethel Glass spoke on "Mathematics in economics"; and Edith Witherell spoke on "The applications of mathematics to physics."

March 1933: A debate was given. The subject was: Resolved: That the study of mathematics should have practical applications as its sole aim. The speakers for the affirmative were Elinor Seidel, Miriam Londy, and Betty Richardson. The speakers for the negative were Charlotte Wheaton, Carol Treyz and Mildred Simemdinger.

May 1933: The last meeting of the club for the year. This was a supper meeting at which the officers for the academic year 1933–1934 were elected.

CONSTANCE BENNETT, *Secretary*

PROBLEMS AND SOLUTIONS

EDITED BY OTTO DUNKEL, H. L. OLSON, AND W. F. CHENEY, JR.

ELEMENTARY PROBLEMS

Send all communications about Elementary Problems and Solutions to W. F. Cheney, Jr., Dept. Box 35, Storrs, Connecticut.

The Department of Elementary Problems welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. Problems may be submitted unaccompanied by their solutions.

PROBLEMS FOR SOLUTION

E 100. *Proposed by G. R. Livingston, State Teachers College, San Diego, California.*

In two concentric circles, locate parallel chords in the outer circle which are tangent to the inner circle, by the use of compasses only, finding the ends of the chords and their points of tangency.

E 101. *Proposed by R. S. Underwood, Texas Technological College.*

Prove that the sum of the squares of the integers in the nth row of the Pascal triangle is equal to the nth number in the $(2n-1)$st row.

E 102. *Proposed by Roy MacKay, Albuquerque, New Mexico.*

If P is a point on the Euler line of a triangle whose sides are a, b and c, one-kth of the distance from the circumcenter O, to the orthocenter H, then $OP^2=(9R^2-a^2-b^2-c^2)/k^2$, where R is the circumradius of the triangle.

E 103. *Proposed by Harry Langman, Cooper Union, New York, N.Y.*

Suppose the earth is a sphere of radius four thousand miles. A flexible belt is constructed around the equator, but large enough to encircle a sphere with radius one inch greater. If a vertical pole is placed under the belt at one point, drawing it taut, how high must it be, and how far from this pole does the belt first touch the earth?

E 104. *Proposed by L. S. Johnston, University of Detroit.*

Show that the coordinates of the center (X, Y) of the circle through the points $P_i(x_i, y_i)$ $(i=1, 2, 3)$ are given by

$$X=\frac{D[r^2, y, 1]}{2D[x, y, 1]} \qquad Y=\frac{D[x, r^2, 1]}{2D[x, y, 1]},$$

and that the length t of the tangent to the circle from the origin is given by $t^2=-D[x, y, r^2]/D[x, y, 1]$, where $r_i^2=x_i^2+y_i^2$, and

$$D[x, y, z]=\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}.$$

E 105. *Proposed by W. R. Ransom, Tufts College.*

Laugh this off: $AHAHA+TEHE=TEHAW$. It resulted from substituting a code letter for each digit of a simple example in addition, and it is required to identify the letters and prove the solution unique.

SOLUTIONS

E 70 [1934, 44]. *Proposed by Roy McKay, Albuquerque, N.M.*

Show that the area of a right triangle in terms of the bisector of the right angle, t, and the median to the hypotenuse, m, is given by the formulas,

$$K = 2m^2t/\{(t^2 + 8m^2)^{1/2} \mp t\},$$

where the upper or lower sign is to be used according as t is the bisector of the interior or exterior angle at the right angle vertex.

Solution by Simon Vatriquant, Athênée Royale d'Ixelles, Brussels

If the two perpendicular sides are denoted by a and b, $2K=ab$, and since the hypotenuse is $2m$, $a^2+b^2=4m^2$. The formulae for the lengths of the bisectors of the right angle are

$$\text{(Interior) } t = 2ab(\cos 45°)/(a + b), \text{ and (Exterior) } t = 2ab(\sin 45°)/(a - b),$$

which combine into $t=ab\sqrt{2}/(a\pm b)$. When squared this becomes

$$t^2 = 2a^2b^2/(4m^2 \pm 2ab) = 2K^2/(m^2 \pm K), \text{ so } 2K^2 \mp t^2K - m^2t^2 = 0.$$

The positive roots of this equation are $K=\{(t^4+8m^2t^2)^{1/2}\pm t^2\}/4$, or, rationalizing the numerator, $K=2m^2t/\{(t^2+8m^2)^{1/2}\mp t\}$, as required.

Also solved by L. M. Bauer, W. E. Buker, W. B. Clarke, D. J. Colbert, J. Rosenbaum, E. P. Starke, M. J. Turner, Maud Willey and the proposer.

E 71 [1934, 44]. *Proposed by W. F. Cheney, Jr., Connecticut State College.*

In a simple example of addition, each digit was replaced by a code letter, and the result was

$$\begin{array}{cccccc} N & E & W & T & O & N \\ & K & L & E & I & N \\ \hline K & E & P & L & E & R \end{array}$$

Identify the letters and show that there are just two solutions, which differ only through the interchange of the values determined for O and I.

Solution by B. C. Zimmerman, Orange Walk, British Honduras

From the first two columns at the left it appears that $NE+K+1=KE$, so that $K=9$ and $N=8$. This makes $R=6$.

Since the third column from the left had to provide one to carry into the second column, $W+L\geqq 9$, and neither is <2. If $W+L=9$, there must have been one to carry from the fourth column into the third, so that $TO+EI+1=1LE\geqq$

120. This would make $T+E \geqq 11$, which is impossible for lack of high digits. Consequently $W+L>9$, so that one of them must be 7, and the other is 3, 4 or 5.

Since we have seen that for lack of high digits there can be no carry from the fourth column to the third, nor from the fifth column to the fourth,

$$9 < W + L = 1P = 10, \text{ or } 11, \text{ or } 12. \tag{1}$$

$$2 < L = E + T < 10. \tag{2}$$

$$0 < I + O = E - 1 < 5, \text{ since } E < 6. \tag{3}$$

If $E=2$, then from (1), $P=0$ or 1, but from (3) these values would belong to I and O. Consequently, using (2) and (3), $2<E<L=4$, or 5, or 7. Now if $L=4$, then $E=3$, and from (2) $T=1$. But this would make $W=7$ and $P=1$ also, which is impossible. Therefore $4<L=5$ or 7. Hence $W=7$ or 5, and $P=2$. This leaves the values 0, 1, 3 and 4 to be assigned in some order to E, I, O and T. If $E=3$, then by (3) $I+O=2$, and either I or O is 2, which conflicts with the known fact that $P=2$. Hence $E=4$. Then by (3), I and O must be 0 and 3, one way or the other, and the only value left for T is 1. By (2) we see that $L=5$ so $W=7$, and the two solutions to the problem are

$$\begin{array}{r} 847108 \\ 95438 \\ \hline 942546 \end{array} \quad \text{and} \quad \begin{array}{r} 847138 \\ 95408 \\ \hline 942546 \end{array}$$

Also solved by L. M. Bauer, W. E. Buker, D. J. Colbert, Mary L. Constable, Elizabeth Giles, R. A. Johnson, Jr., Erna Jonas, C. F. Lewis, Theodore Lindquist, Roy MacKay, W. R. Ransom, Helen Rock, E. P. Starke, C. W. Trigg, Simon Vatriquant and the proposer.

E 72 [1934, 45]. *Proposed by J. M. West, Pennsylvania State College.*

Given that $A+B+C=180°$, prove that

$$\sin A \cos^2 A \sin (B - C) + \sin B \cos^2 B \sin (C - A) + \sin C \cos^2 C \sin (A - B) = 0.$$

Solution by E. P. Starke, Rutgers University

In the first term put

$$\begin{aligned} \sin A \sin (B - C) &= \sin (B + C) \sin (B - C) \\ &= \tfrac{1}{2} \cos 2C - \tfrac{1}{2} \cos 2B = \cos^2 C - \cos^2 B, \end{aligned}$$

so that this term becomes $\cos^2 C \cos^2 A - \cos^2 A \cos^2 B$. Analogous transformations of the other two terms of the given expression give two similar expressions which, added to this, obviously cancel to zero.

Also solved by Filiberto Amoroso, Leon Battig, L. M. Bauer, Ruth S. Berkow, Clyde Bridger, I. W. Burr, D. J. Colbert, Erna Jonas, Roy MacKay, Mrs. Yetta V. Maizlish, C. W. Munshower, C. B. Read, C. W. Trigg, Simon Vatriquant and the proposer.

E 73 [1934, 45]. *Proposed by W. F. Cheney, Jr., Connecticut State College.*

Show that there is just one right triangle whose three sides are relatively prime integers between 2000 and 3000.

Solution by J. Rosenbaum, Milford, Conn.

It is well known that if the three sides of a right triangle are relatively prime integers, they are given by the expressions, $2pq$, p^2-q^2, and p^2+q^2, with p and q relatively prime integers, one even, and $q<p$.

Since 2000 is less than each leg of the right triangle in this problem, $2000\sqrt{2}<$the hypothenuse<3000. Consequently

$$2828 < p^2 + q^2 < 3000. \tag{1}$$

Furthermore,

$$2000 < p^2 - q^2, \tag{2}$$

whence by addition

$$4828 < 2p^2,\ 50 \leqq p. \tag{3}$$

Again by hypothesis

$$2000 < 2pq, \tag{4}$$

which by subtraction from (1) gives

$$(p - q)^2 < 1000,\ p - q \leqq 31 \tag{5}$$

and by (3)

$$19 \leqq q. \tag{6}$$

Substituting in (1) gives

$$p^2 < 2639,\ p \leqq 51. \tag{7}$$

Dividing into (4) gives

$$2000/51 < 2q,\ 20 \leqq q, \tag{8}$$

which, with (1), gives

$$p^2 < 2600,\ p \leqq 50, \tag{9}$$

which with (3) makes $p=50$. Substituting (9) in (1) now gives $q^2<500$, or $q\leqq 22$, which, taken with (8) and the consideration that since p is even, q must be odd, makes $q=21$.

Since it has been shown that the only possible values for p and q are 50 and 21 respectively, there can be but one triangle which satisfies the given conditions. Substitution shows its sides to be 2100, 2059 and 2941, satisfying all the requirements, and hence constituting the unique solution.

Also solved by H. T. R. Aude, W. E. Buker, D. J. Colbert, Roy MacKay,

W. R. Ransom, A. A. Rood, E. P. Starke, C. W. Trigg, Simon Vatriquant and the proposer.

E 74 [1934, 45]. *Proposed by J. E. Trevor, Cornell University.*

A vertical sheet of horizontal rays of light falls upon the outside of a horizontal reflecting circular cylinder, the axis of which meets the incident sheet at an arbitrary angle. The reflected rays form an illuminated curve on a dark screen parallel to the incident sheet. Find the equation of this curve.

Editor's Note. Inasmuch as no complete and full solution to this problem has been as yet received, more time is being allowed on it, in the hope that others will try it and either send in complete solutions or voice the opinion that it is too difficult for an elementary problem.

Advanced Problems

Send all communications about Advanced Problems to Otto Dunkel, Washington University, St. Louis, Mo. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide.

Problems containing results believed to be new, or extensions of old results are especially sought. The editorial work would be greatly facilitated, if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known textbooks or results found in readily accessible sources, will not be proposed as problems for solution in this department. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

PROBLEMS FOR SOLUTION

3686. *Proposed by Elmer Schuyler, Brooklyn, New York.*

Construct the triangle ABC, given the centers of the escribed circles corresponding to the sides BC and AB and the point of contact of BC with the inscribed circle.

3687. *Proposed by Melvin Dresher, Yale University.*

If $S(i, j)$ denotes the sum of the divisors common both to i and j, show that:

$$\begin{vmatrix} S(1,1) & S(1,2) & \cdots & S(1,n) \\ S(2,1) & S(2,2) & \cdots & S(2,n) \\ \cdot & \cdot & \cdot & \cdot \\ S(n,1) & S(n,2) & \cdots & S(n,n) \end{vmatrix} = n!.$$

3688. *Proposed by W. D. Baten, University of Michigan.*

Show that

$$I = \sum_{r=0}^{n-1} (n-1)^{(r)} n^{(-r-1)} = 4^{n-1} B(n, n),$$

where

$$a^{(s)} = a(a-1)(a-2) \cdots (a-s+1), \qquad a^{(0)} = 1,$$

and

$$a^{(-s)} = \frac{1}{a(a+1)(a+2)\cdots(a+s-1)},$$

and $B(n, n)$ is the familiar beta function.

3689. *Proposed by Maud Willey, Long Beach, Miss.*

If the points $A_i (i=1, 2, 3, 4)$ are the vertices of a tetrahedron T, O is any point, B_i is the trace of line A_iO on the face of T opposite A_i, and C_i is the harmonic conjugate of O with respect to A_i and B_i; then each edge of T is coplanar with a line through two of the points C_i. If O is the centroid of T, it is also the centroid of the tetrahedron with the vertices C_i.

3690. *Proposed by W. M. Whyburn, University of California at Los Angeles.*

Solve the functional equation

$$f(x)f(-x) = c^2 = [f(0)]^2,$$

subject to the single restriction that $f(x)$ be a single valued, positive, real function of the real variable x.

3691. *Proposed by E. P. Starke, Rutgers University.*

Show that

$$\sum_{j=k}^{[n/2]} \binom{j}{k}\binom{n}{2j} = 2^{n-2k-1}\left[\binom{n-k}{k} + \binom{n-k-1}{k-1}\right],$$

where n is a positive integer, k is a positive integer or zero, $[n/2]$ is the greatest integer in $n/2$, and $k \leqq [n/2]$.

3692. *Proposed by E. P. Starke, Rutgers University.*

Show that there are four distinct sets of integers which satisfy the equations

$$x_1 + x_2 + x_3 = 54, \qquad x_1^2 + x_2^2 + x_3^2 = 1406.$$

Develop a general method of attack for similar problems in which 54 and 1406 are replaced by a and b.

SOLUTIONS

3617 [1933, 363]. *Proposed by W. H. Rasche, Virginia Polytechnic Institute.*

Given the center and the principal axes of an ellipse, and a line A anywhere in the plane of the ellipse, it is required by a ruler-compass construction and without drawing the ellipse to construct those normals to the ellipse that are parallel to the line A.

I. *Solution by E. P. Starke, Rutgers University.*

Let $G'G = 2a$ be the major axis and O its mid-point; let the minor auxiliary circle cut $G'G$ in K' and K, $K'K = 2b$. Draw OR perpendicular to the given line A, and let OR cut in R the perpendicular at G to $G'G$. Let S be the fourth vertex

of the rectangle $KGRS$; and draw OS. Let OS', perpendicular to OS, cut the major auxiliary circle in S' and the minor auxiliary circle in U'. Let $S'T'$, perpendicular to $G'G$ at T', cut in P the line through U' parallel to $G'G$. Then P is one of the required points of the ellipse, and the perpendicular from P to OR is one of the required normals. The other solution is obtained by taking P' so that PP' has O for its mid-point.

Proof. If the major auxiliary circle is rotated about $G'G$ through an angle $\cos^{-1} b/a$, its orthogonal projection on the original plane is the given ellipse. The orthogonal diameters of the circle OS and OS' project into the conjugate diameters OR and OP of the ellipse. For, if OR and KS meet in L, $KL/KS = KL/GR = b/a$. Also $T'P/T'S' = b/a$, and therefore P is on the ellipse. Since the tangent at P is parallel to the diameter OR, which is conjugate to OP, the perpendicular from P to OR is the normal at P.

II. *Solution by Ethel I. Moody, Pennsylvania State College.*

Let the principal axes of the ellipse be PL and MN and its center be O. From P draw a perpendicular PX to the line A. The tangents to the ellipse at M and P are perpendicular to the axes at these points. Let Y be the intersection of these tangents. The flat pencils $P(MLNY)$ and $M(YLNP)$ are projective. Sections of these pencils by two lines through N are perspective ranges. By the use of these perspective ranges, the line MT of the pencil $M(YLNP)$ which corresponds to PX in the projectivity can be determined. The point of intersection Z of MT and PX is a point of the ellipse, and the harmonic conjugate of the point at infinity on PX with respect to P and Z is the mid-point R of the segment PZ. Therefore, RO is the polar line of this point at infinity. The sections of the two projective pencils $P(MLNY)$ and $M(YLNP)$ by the line RO are two superposed projective ranges, the self-corresponding points of which are the points in which RO intersects the ellipse. These self-corresponding points D and D' can be found by the use of any convenient circle. The lines through D and D' and parallel to PX are tangents to the ellipse at these points, and the perpendiculars to these tangents at D and D' are the normals to the ellipse which are parallel to the line A.

Solved also by W. B. Campbell, L. S. Johnston, R. MacKay, A. Pelletier, and S. Vatriquant.

A Note by the Editors. The solution [1923, 341] given by T. M. Blackslee of problem 2971 [1922, 179] leads to a construction which is probably the one which would occur to most solvers in some of its modified forms. Construct the foci F and F'. Through the focus F draw a line Q_1Q_2 parallel to the given line A; and let the circle with center F' and radius $2a$ cut this line in Q_1 and Q_2. Then the perpendicular bisectors of FQ_1 and FQ_2 cut $F'Q_1$ and $F'Q_2$, respectively, in the desired points P_1 and P_2 of the ellipse.

Another construction, which may not be so obvious, may be derived from the ellipse construction in problem 519 [1917, 327], solution [1919, 415]. Let

$OA = a$ be the major semi-axis, and take K on OA so that $KA = b$. Draw OL perpendicular to the given direction A cutting in L the circle with OK as a diameter. Let AL cut the circle again in M. On OM take $OM' = OK$, and let L' be the foot of the perpendicular from M' on OL. Then $M'L'$ is one of the required normals; and, if $M'L'$ is produced to A' so that $L'A' = LA$, A' is the corresponding point on the ellipse.

The projective method of construction above may be varied. Let $YUU'Y'$ be the rectangle whose sides are tangent to the ellipse at M, L, N, P. Tangents to the ellipse cut projective ranges of points on YU and $Y'U'$ for which Y, U, M, ∞ and Y', U', ∞, N are corresponding points. Project the last set of four points on YU by rays perpendicular to the given line A thus obtaining the points Y'', U'', ∞, N'' on YU. Using the circle construction of the above solution we find the two self-corresponding points of the two projective ranges on YU. If V is one of these two points, let $Y'V$ cut MN in W. Draw YW cutting $Y'U'$ in V'. Then the intersection Z of PW and VV' is one of the required points on the ellipse.

The simple proofs of these constructions are left to the reader.

3619 [1933, 363]. *Proposed by V. F. Ivanoff, Berkeley, California.*

A plane cuts the edges OA, OB, OC of a parallelopiped in points A', B', C', respectively, and the diagonal OP in a point P'. Prove that

$$\frac{OA}{OA'} + \frac{OB}{OB'} + \frac{OC}{OC'} = \frac{OP}{OP'}.$$

Solution by C. Eugene Buell, Graduate Student, Washington University.

For convenience, we take the axes of coordinates along the edges $OA = a$, $OB = b$, $OC = c$. The equation of the cutting plane is then

$$\frac{x}{a'} + \frac{y}{b'} + \frac{z}{c'} = 1,$$

where $a' = OA'$, etc. The equations of the diagonal OP are

$$\frac{x}{a} = \frac{y}{b} = \frac{z}{c}.$$

The coordinates of P' are given by

$$x' = a/\lambda, \qquad y' = b/\lambda, \qquad z' = c/\lambda,$$

$$\lambda = \frac{a}{a'} + \frac{b}{b'} + \frac{c}{c'}.$$

Hence

$$\frac{OP}{OP'} = \frac{a}{x'} = \lambda = \frac{a}{a'} + \frac{b}{b'} + \frac{c}{c'},$$

which is the desired result.

Solved also by A. D. Bradley, J. W. Clawson, L. S. Johnstón, A. Pelletier, H. D. Ruderman, W. P. Udinski, and Maud Willey.

3620 [1933, 363]. *Proposed by S. A. Corey, Des Moines, Iowa.*

Find solutions of the functional equation

$$a^4u^2 + u^2 + 2a^2u^2 + a^4 + 2a^2 = v^2$$

in which u and v are to be determined as rational functions of a.

Solution by Morgan Ward, California Institute of Technology

The functional equation may be written in the equivalent form

$$[v - (a^2 + 1)u][v + (a^2 + 1)u] = a^4 + 2a^2. \tag{1}$$

Since u and v are to be rational functions of a, both parentheses on the left of (1) are rational in a. Hence we may set

$$v \pm (a^2 + 1)u = \frac{P(a)}{Q(a)}, \qquad v \mp (a^2 + 1)u = (a^4 + 2a^2)\frac{Q(a)}{P(a)},$$

where $P(a)$ and $Q(a)$ are any two polynomials in a. For definiteness, we assume that they have no common zeros, and that the leading coefficient of $Q(a)$ is unity. On solving these two equations for u and v, we find that

$$u = \frac{P^2(a) - Q^2(a)(a^4 + 2a^2)}{2(a^2 + 1)P(a)Q(a)}, \qquad v = \frac{P^2(a) + Q^2(a)(a^4 + 2a^2)}{2P(a)Q(a)}. \tag{2}$$

We have dropped the double sign with u, since with any solution of (1) we may obviously associate another differing from it only in sign.

From its mode of derivation, (2) gives all the rational solutions of (1), and affords a convenient canonical representation of them; for it is easily shown that for a particular rational solution (u, v) of (1), the polynomials $P(a)$ and $Q(a)$ are uniquely determined, except for a common factor, which can be suppressed.

It is of some interest to determine all solutions of the functional equation which are polynomials in a. We readily find from the equations (2) that necessary and sufficient conditions that u and v be polynomials in a are that

$$Q(a) = 1; \qquad a^4 + 2a^2 \equiv 0 \bmod P(a); \qquad P^2(a) + 1 \equiv 0 \bmod (a^2 + 1).$$

These conditions lead to the following twelve possible values for $P(a)$:

$$\pm\, i, \quad \pm\, a, \quad \pm\, ia^2, \quad \pm\, i(a^2 + 2), \quad \pm\, a(a^2 + 2), \quad \pm\, ia^2(a^2 + 2),$$

where as usual $i = \sqrt{-1}$. The equations (2) then yield the corresponding polynomial solutions. Only three of these are found to be essentially distinct; namely,

$$u = i, \quad v = i; \quad u = \frac{i}{2}(a^2 + 1), \quad v = \frac{i}{2}(a^4 + 2a^2 - 1); \quad \text{and}$$

$$u = \frac{a}{2}, \quad v = \frac{a}{2}(a^2 + 3).$$

Thus there is essentially only one solution of the functional equation for which u, v are polynomials in a with real coefficients.

As a slight generalization, a similar argument shows that the equations (2) represent the most general solution of the functional equation in which u and v are to be determined as meromorphic functions of a provided that $P(a)$ and $Q(a)$ are taken to be entire functions of a with no common zeros.

The functional equation

$$\phi^2(a)u^2 + \psi(a) = v^2$$

where $\phi(a)$, $\psi(a)$ are any meromorphic functions of a may obviously be treated similarly.

Solved also by E. P. Starke and S. Vatriquant.

3621 [1933, 364]. *Proposed by A. S. Levens, University of Minnesota.*

Extend the graphical method for the solution of real roots of a quadratic, as given in Dickson's *First Course in the theory of equations*, p. 29, to permit the reading of complex roots.

I. *Solution by Roy MacKay, Albuquerque High School.*

The equation considered in the reference is in the form $x^2 - ax + b = 0$, where a and b are real and constructible; and the construction given for the case of real roots is as follows: The points $Q(a, b)$ and $B(0, 1)$ having been located with respect to rectangular axes, a circle is constructed with the diameter BQ. If this circle cuts the x-axis in the points M and N, then the lengths of OM and ON are the roots of the given equation.

If the circle does not cut the x-axis, $a^2 < 4b$ and the roots are complex. In this case let H and K be the projections of the center of the above circle upon the x and y axes, respectively. Describe a circle with center K and radius $OK = \frac{1}{2}(b+1)$, and let it cut in L the line through B parallel to the x-axis. With H as center describe a circle with radius BL cutting the y-axis in E and F, E lying upon the positive side. Then the roots are $x_1 = OH + iOE$ and $x_2 = OH + i\,OF$. For, from the construction, $BL^2 = b$; and, since $a^2 < 4b$, $\frac{1}{2}a < BL$. Thus the points E and F are real in this case. Also $x_1 + x_2 = 2\ OH = a$, and $x_1x_2 = OH^2 + OE^2 = HE^2 = BL^2 = b$. Therefore x_1 and x_2 are the roots.

II. *Solution by J. Shaylor Woodruff, Abington High School, Pa.*

If the circle in Dickson's graphical method does not cut the x-axis, in which case the roots are complex, draw a "shadow" circle, i.e., one which is tangent to the Dickson circle and with its center vertically below the center of that circle and at a distance below the x-axis equal to the radius of that circle. This "shadow" circle cuts the x-axis in the points M and N, and the complex roots are $OA \pm i\,AM$, where A is the projection of the two centers on the x-axis. For

the equation of this circle is

$$(x - \tfrac{1}{2}a)^2 + [y + \tfrac{1}{2}\{a^2 + (b - 1)^2\}^{1/2}]^2 = \tfrac{1}{4}(b + 1)^2;$$

and its intercepts on the x-axis are

$$\tfrac{1}{2}a \pm \tfrac{1}{2}(4b - a^2)^{1/2} = OA \pm AM.$$

But for the complex roots of the given equation we must multiply AM by i.

Solved also by H. Karnow, E. P. Starke, and the proposer.

3622 [1933, 364]. *Proposed by Otto Dunkel, Washington University.*

A set of circles pass through a fixed point A and each circle of the set is tangent to a fixed circle I which does not contain A. Each circle of the set cuts the two tangents from A to I in a pair of corresponding points. Prove by synthetic geometry that the envelope of the straight lines joining the pairs of corresponding points is a circle.

This is the converse of the theorem in 3416 [1930, 157] an analytic proof of which was given [1930, 559].

Solution by A. S. Householder, Washburn College

It is convenient to use the notion of directed angle as defined by R. A. Johnson in his *Modern Geometry*, in order that a single statement may suffice for the different positions of the circles of the family. The directed angle ABC, represented by $\angle ABC$, is the smallest positive angle through which the line AB must be rotated about B to come into coincidence with the line BC. The theorem is a simple consequence of the geometry of inversion.

Let AT and AT' be tangent to the circle I at T and T'. Then with the circle of center A and radius AT as the circle of inversion, the circle I is self-inverse, and TT' intersects AI in the point J which is the inverse of I. Any circle X containing A and tangent to I at P, has as its inverse the line tangent to I at Q on AP. Hence this line intersects the self-inverse lines AT and AT' at points S and S', inverse to the points R and R' of intersection of X with the same lines. The theorem will be proved if it can be shown that $\angle R'RJ = \angle JRT$, since in this case the circle with center J, having RR' as a tangent will also have AT and hence AT' as tangents, and is thus fixed. But this is true, for noting again that R and S, R' and S', I and J are inverse in pairs, and that T is self-inverse it follows that

$$\angle R'RJ = \angle T'AI + \angle ISS'$$

$$\angle JRT = \angle IAT + \angle TSI = \angle T'AI + \angle ISS' = \angle R'RJ.$$

A Note by the Editors. The tangent SS' to I cuts the two tangents AT and AT' in two projective ranges of points S and S': and hence the ranges of inverse points R and R' are projective. From this it follows that the envelope of RR' is a conic tangent to AT and AT'. For one set of positions of SS', I is the inscribed circle of triangle ASS'; and for these positions the external angle at S'

is twice the external angle at I of the triangle AIS. But from the inversion we see that the first external angle is equal to $\angle R'RT$ and the second is equal to $\angle JRT$. This suffices to show that the whole envelope is a circle with center J tangent to AT and AT'.

3624 [1933, 427]. *Proposed by N. A. Court, University of Oklahoma.*

If the lines joining the vertices of a tetrahedron to its circumcenter meet the respective opposite faces in the centroids of these faces, each edge of the tetrahedron is equal to its opposite edge.

I. *Solution by L. M. Bauer, Menaul School, Albuquerque, N.M.*

In a tetrahedron $ABCD$ with circumcenter O, let M, R, N, P, T, and K be the midpoints of edges BC, AC, AB, CD, BD, and AD, respectively. Also, let the centroids of the respective faces opposite vertices A, B, C, and D be G_1, G_2, G_3, and G_4.

Since G_4 is on AM and G_2 is on CK, DG_4 and BG_2 are respectively in planes ADM and BCK, which intersect along MK. O is on both DG_4 and BG_2; therefore, it is on MK, and MOK is a straight line. Similarly, ROT is a straight line. Also MT and RK are equal and parallel to PC; hence $MRKT$ is a parallelogram whose diagonals meet in O. It now follows that $RO=TO$ and $MO=KO$. Since AC and BD are chords of the same sphere whose mid-points R and T are equidistant from its center O, we have $AC=BD$. Similar equalities hold for the remaining opposite edges.

II. *Solution by P. W. Allen Raine, University of Virginia.*

Let $\mathbf{t}$, $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ be the (vector) coordinates of the vertices of the given tetrahedron (T), and let (T') be a tetrahedron with vertices $\mathbf{t}'$, $\mathbf{u}'$, $\mathbf{v}'$, $\mathbf{w}'$ given by

$$t' = \frac{u+v+w-t}{2}, \qquad u' = \frac{v+w+t-u}{2},$$

$$v' = \frac{w+t+u-v}{2}, \qquad w' = \frac{t+u+v-w}{2}.$$

Then, clearly,

$$t-u' = u-t', \qquad t-v' = v-t', \qquad t-w' = w-t'.$$

The tetrahedra (T) and (T') therefore form a parallelopiped (P), whose diagonals intersect in the point

$$M = \frac{t+t'}{2} = \frac{u+u'}{2} = \frac{v+v'}{2} = \frac{w+w'}{2}.$$

Moreover, since the centroid G_t of the face opposite t satisfies

$$G_t = \frac{u+v+w}{3},$$

we get

$$t' = \frac{3G_t - t}{2}.$$

Therefore t' is collinear with G_t and t. And, since, by hypothesis the circumcenter C of (T) is collinear with G_t and t, the three points C, t, t' are collinear. In exactly the same way we conclude C, u, u' are collinear. Consequently, since the diagonals t, t' and u, u' of (P) have only the point M in common, $C=M$; whence, since

$$\boldsymbol{M} - \boldsymbol{t} = \boldsymbol{t}' - \boldsymbol{M}, \text{ etc.},$$

the circumcenter M of (T) is also the circumcenter of (T'), and the parellelopiped (P) is rectangular. Hence the diagonals t, u and v', w' of the face t, u, v', w' of (P) are equal. And, since

$$\boldsymbol{v}' - \boldsymbol{w}' = \boldsymbol{w} - \boldsymbol{v},$$

the edges t, u and w, v of (T) are equal.

A Note by the Editors. The faces ABC and DCB have equal corresponding edges; they are therefore congruent and their circumcircles are equal. Hence their planes are equidistant from O, and O is also the center of the inscribed sphere. The point of contact of the inscribed sphere with any face is the circumcenter of that face.

Conversely, if the opposite edges of the tetrahedron $ABCD$ are equal, the circumcenter, incenter, and centroid coincide. For the faces ABC and DCB are congruent; and, if M is the mid-point of BC, the triangle MAD is isosceles and MK, where K is the mid-point of AD, is perpendicular to AD. Similarly, it is perpendicular to BC. Hence the mid-point of MK is equidistant from A, B, C, D; it is therefore the circumcenter O. It can be easily shown that, in the isosceles triangle MAD, the line AO, which bisects MK in O, cuts the side MD in a point G_1 which divides the latter in the ratio 1:2. Hence O is also the centroid; and, by the above proof, it is also the incenter. The solution [1932, 495] of problem 3482 [1931, 170] is related to this problem. For that solution shows that, if the circumcenter and the incenter of a tetrahedron coincide, the point of coincidence is also the centroid.

Solved also by Frank Ayres, Jr., A. D. Bradley, J. W. Clawson, T. C. Esty, R. MacKay, A. Pelletier, J. Rosenbaum, T. L. Smith, F. Underwood, and S. Vatriquant.

A solution of 3523 by A. H. Wilson has been received after the printing of a solution [1932, 558].

NEWS AND NOTICES

Readers are invited to contribute to the general interest of this department by sending items to Proessor J. H. Weaver, Ohio State University, Columbus, Ohio

At the fifteenth annual meeting of the National Council of Teachers of Mathematics held at Cleveland, Ohio, February 23 and 24, 1934, the following papers were presented: The Future of Geometry in the High School, by Ralph Beatley; The Future of Geometry in the High School, by Roland R. Smith; A "Panel" Discussion on the Present Crisis in Secondary Mathematics, by ten members of the Board of Directors; The Problem of Ability Grouping, Administrative Phases, by C. M. Stokes; Remedial Work in Arithmetic, by Genevieve Skehan; The Problem of Individual Differences, by Clara E. Murphy; An Experiment in Teaching Graphs, by Mrs. W. E. Pitcher; What Can We Do to Meet the Challenge of the Present Situation in Secondary Mathematics?, by William D. Reeve; A Report of the Policy Committee, by J. O. Hassler; Mathematics and Music, by Professor Carl A. Garabedian.

At this meeting the following officers were elected for the year 1934–35: President, J. O. Hassler, Professor of Mathematics, University of Oklahoma, Vice-President, Allen R. Congdon, University of Nebraska; Members of the Board of Directors: William Betz, Rochester, New York; H. C. Christofferson, Oxford, Ohio; Edith Woolsey, Minneapolis, Minnesota; Martha Hildebrandt, Maywood, Illinois.

In the mathematics section of the Basic Science exhibits at the Century of Progress Fair are certain models of projections of hyperspace regular figures made by P. S. Donchian, of Hartford, Conn. The set includes the six regular figures of 4-dimensional space and the hypercube series extended to n-dimensions.

Professor A. H. Compton of the University of Chicago has been elected President of the American Physical Society.

Professor P. A. M. Dirac, of the University of Cambridge, will be Visiting Professor of Mathematical Physics at the Institute for Advanced Study, Princeton, New Jersey, for the academic year, 1934–35.

Dr. Irving Langmuir, Associate Director of the Research Laboratory of the General Electric Company, has been appointed honorary Chancellor of Union College for the present academic year. He will deliver the principal address at the annual commencement of the college on June 11.

Dr. E. B. Wilson, Professor of Vital Statistics in the School of Public Health, Harvard University, delivered the principal address in connection with the annual Sigma Xi day held on February 22 at the University of Rochester. Dr. Wilson's topic was "Factors in Mental Ability."

Professor S. W. Hunton, who for fifty-one years has been head of the department of mathematics at Mount Allison University, Sackville, New Brunswick, will retire at the close of the present academic year.

Professor J. F. Ritt of Columbia University, has been granted leave of absence for the Winter Session of the year 1934–35.

Dr. H. H. Alden of the Ohio State University has been appointed head of the department of mathematics at the New Mexico Military Institute, Roswell, New Mexico.

Associate Professor H. F. Mac Neish, head of the department of mathematics at Brooklyn College, has been promoted to a full professorship.

Assistant Professor Max Morris has been promoted to an associate professorship at the Case School of Applied Science.

Assistant Professor C. O. Oakley of Brown University has been appointed professor of mathematics at Haverford College for 1934–35.

Dr. C. Grace Shover, of the Ohio State University, has been awarded the Emmy Noether Fellowship at Bryn Mawr College for the year 1934–35.

Professor F. W. Hanawalt, of the College of Puget Sound, died in November 1933. He was a charter member and a life member of the Association.

Professor A. S. Hathaway died at his home in Boerne, Texas, on March 11, 1934, at the age of 78.

Professor E. N. Johnson, for the past thirty years head of the department of mathematics at Butler University, died April 24, 1934. He was a charter member of the Association.

Sir Thomas Muir, late Superintendent-General of Education, Cape Colony, South Africa, died March 21, 1934, at the age of eighty-nine. He was a charter member of the Mathematical Association. By deed of gift his library is now the property of the State and in the keeping of the Librarian of the South African Public Library at Capetown. It is known as the Muir Mathematical Library.

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DIRECTORY

EDITORIAL CORRESPONDENCE should be addressed to the EDITOR-IN-CHIEF, W. B CARVER, White Hall, Cornell University, Ithaca, N.Y.

BOOKS FOR REVIEW should be addressed to R. A. JOHNSON, Brooklyn College, 66 Court Street, Brooklyn, N.Y.

BUSINESS CORRESPONDENCE should be addressed to the SECRETARY-TREASURER of the Association, W. D. CAIRNS, 33 Peters Hall, Oberlin, Ohio.

CHANGE OF ADDRESS: Members should send notice of any change of address to the SECRETARY-TREASURER, W. D. CAIRNS, Oberlin, Ohio, before the 10th of each month.

MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Eighteenth Summer Meeting of the Association, Williamstown, Mass., Sept. 3-4, 1934.

The following is a list of the sections of the Association, with dates of those section meetings which have been scheduled for 1934 and reported to the Secretary.

ALLEGHENY MOUNTAIN, Pittsburgh, Pa., Feb. 10; Washington, Pa., May 5.
ILLINOIS, Jacksonville, May 4-5.
INDIANA, La Fayette, May 11-12.
IOWA, Des Moines, April 20-21.
KANSAS, Topeka, Mar. 17.
KENTUCKY, May.
LOUISIANA-MISSISSIPPI, Jackson, Miss., Mar. 23-24.
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Baltimore, Md., Dec. 8.
MICHIGAN, Ann Arbor, Mar. 17.
MINNESOTA, Northfield, May 12.
MISSOURI.
NEBRASKA, Crete, Apr. 27.
OHIO, Columbus, Apr. 5.
OKLAHOMA, Oklahoma City, Feb. 9.
PHILADELPHIA, Philadelphia, Dec. 1.
ROCKY MOUNTAIN, Colorado Springs, Apr. 20-21.
SOUTHEASTERN, University, Ala., Mar. 30-31.
SOUTHERN CALIFORNIA, Riverside, Mar. 3.
TEXAS, College Station, May 5.
WISCONSIN, Oshkosh, May 5.

AFFILIATED ORGANIZATIONS: THE NEW ENGLAND ASSOCIATION OF TEACHERS OF MATHEMATICS. THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS.

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THE TWENTIETH ANNUAL MEETING OF THE KANSAS SECTION

The twentieth annual meeting of the Kansas Section of the Mathematical Association of America was held at Topeka, in the High School Building, Saturday, March 17, 1934.

The morning session was a joint meeting with the Kansas Association of Mathematics Teachers. After the luncheon, with its pleasant social features, the two groups met in separate session. Dean R. W. Babcock, Chairman of the Section, presided at the joint session and at the Section meeting in the afternoon.

Eighty-four persons attended the meeting, including the following thirty-three members of the Association: Mary N. Arnoldy, R. W. Babcock, Wealthy Babcock, Lois E. Bell, Florence Black, E. E. Colyer, R. D. Daugherty, Lucy T. Dougherty, W. H. Garrett, F. C. German, W. A. Harshbarger, A. J. Hoare, A. S. Householder, Emma Hyde, W. C. Janes, H. E. Jordan, C. F. Lewis, W. H. Lyons, Anna Marm, U. G. Mitchell, Thirza A. Mossman, F. W. Owens, Helen B. Owens, O. J. Peterson, C. B. Read, B. L. Remick, D. H. Richert, J. A. G. Shirk, G. W. Smith, W. T. Stratton, H. W. Taylor, J. J. Wheeler, A. E. White.

During the past year, there has been one death: Professor W. H. Andrews, Kansas State College, Manhattan, a charter member of the Section, who died May 26, 1933.

A pleasant episode of the afternoon session, was the surprise visit of Doctor F. W. Owens, and his wife, Doctor Helen B. Owens, of Pennsylvania State College, who were passing through Kansas on their way to a six months' rest in California. They were known to many of the members, and we were happy to greet them and wish them a most happy vacation.

The following officers were chosen for the coming year: Chairman, G. W. Smith, University of Kansas; Vice-chairman, W. T. Stratton, Kansas State College; Secretary-Treasurer, Lucy T. Doughtery, Kansas City Junior College.

The following eight papers were presented:

1. "Graphical method for Horner's process" by Professor C. B. Read, University of Wichita.

2. "Klein's method of introducing the concept of logarithms" by Professor D. H. Richert, Bethel College.

3. "India's contribution to mathematics" by Charles Saldanha, St. Mary's College, by invitation.

4. "Horizontal functions" by Professor R. W. Hart, State Teachers College, Pittsburg, by invitation.

5. "Vector geometry" by Professor A. S. Householder, Washburn College.

6. "Some focal properties of a pencil of conics" by Professor W. H. Lyons, Kansas State College.

7. "Reality of the double tangent contact parameters of the rational symmetric quartic curve" by Sister Mary Nicholas Arnoldy, Marymount College.

8. "Derivation of the equation of a ruled surface in series form" by Gilbert Ulmer, University of Kansas, by invitation.

Abstracts of these papers follow:

1. Two experiments, conducted at the University of Wichita, were made regarding the relative merits of Horner's method vs. a graphic method for approximating the roots of an equation. The test used required an answer free from error, the score being the time consumed. In both experiments, Professor Read stated a very definite advantage existed in favor of the graphic method, both in the mean time required to solve the problem and in the time required for presentation of the subject matter to the class.

2. Klein proceeds from the principle that the proper source from which to bring in new functions is the quadrature of known curves. According to this principle the logarithm of x is defined by the equation

$$S = \int_1^x \frac{du}{u} = \log x,$$

where S is the area under the hyperbola $uv=1$, between the ordinates corresponding to $u=1$, and $u=x$.

This leaves open the question as to the "base" of the system. Professor Richert supplies the proof that it follows from Klein's definition of logarithm that the number e is the required base, where the term "base" means the number whose logarithm is l, and where e is defined in the usual way.

3. The invention of the symbol "0" is India's contribution to mathematics. The Reverend Charles Saldanha showed how this invention makes our number-system simple and elastic. The invention of zero laid the foundation for certain parts of elementary mathematics (exclusive of geometry) and has done more for the progress of science and culture than any other mathematical invention. The Hindu mind has a fascination for figures that recalls the Greek passion for geometry. A modern Hindu genius, Ramanujan, who died at the early age of thirty-two, has, in the words of Professor G. H. Hardy of Oxford, "left behind him work which will take mathematicians fifty years to estimate."

4. A brief explanation of horizontal functions as employed by Professor H. J. Ettlinger at the University of Texas was given by R. W. Hart. Horizontal functions were defined, and the application of them to the Lebesgue integral was explained.

5. Since Cartesian coordinates can be developed as a special application of vector theory, the vector is equally general, and often simpler and more concise, as an instrument for the study of elementary geometry. Professor Householder attempts, therefore, to outline an undergraduate course in the theory of vectors, with applications to geometry rather than, as is usual, to mechanics.

6. A particular conic of the system is determined as the locus of the center of a circle that passes through a fixed point F and is tangent to another circle whose

center is at F'. Then F and F' will be the foci of that particular conic. Any four points on this conic are taken as the base points of the pencil. Then an analytic expression is obtained involving the distances from these four points to the focus F. By choosing the four points properly and by changing to polar coordinates a graph of the focal curve is obtained. Certain of these special cases are considered.

7. Sister Mary Nicholas found the octavic giving the double tangent parameters. See the writer's dissertation "The Reality of the Double Tangents of the Rational Symmetric Quartic Curve." (Catholic University of America.) This factored into two quartics, the one giving the parameters of the horizontal double tangents, the other those of the slanting double tangents. The first quartic was reduced and the conditions set down that the four roots be all real, all imaginary, or two real and two imaginary. Regions in a Cartesian plane were found where these conditions are satisfied. The second quartic contains only second and fourth degree terms of the parameter, hence the reality of the contact parameters can easily be determined.

8. Wilczynski obtained a canonical development for the equation of a ruled surface in the neighborhood of a general point on the surface by the use of a system of partial differential equations. Stouffer simplified the process by the use of a system of ordinary differential equations, and obtained a development similar to that of Wilczynski with an associated tetrahedron of reference whose geometrical significance was easily determined. In this paper Gilbert Ulmer obtains the canonical development of Stouffer for the equation of a ruled surface without the introduction of differential equations, and locates the associated tetrahedron of reference geometrically.

LUCY T. DOUGHERTY, *Secretary*

THE MARCH MEETING OF THE SOUTHEASTERN SECTION

The twelfth annual meeting of the Southeastern Section of the Mathematical Association of America was held at the University of Alabama, University, Alabama, on Friday and Saturday, March 30–31, 1934. Sessions were held in the afternoon and evening of the 30th, and on the morning of the 31st. The chairman of the Section, Professor D. F. Barrow, presided, except Friday evening.

The attendance was one hundred twenty-eight including the following twenty-nine members of the Association from eighteen institutions: Berd R. Allen, D. F. Barrow, R. V. Blair, Iris Callaway, J. B. Coleman, Orpha Ann Culmer, Forrest Cumming, U. P. Davis, B. F. Dostal, Arnold Dresden, T. R. Eagles, H. K. Fulmer, L. D. Hampton, R. A. Hefner, G. W. Hess, Ruby Hightower, Rosa L. Jackson, F. W. Kokomoor, F. A. Lewis, J. F. Messick, W. A. Moore, W. P. Ott, Z. M. Pirenian, B. P. Reinsch, A. J. Robinson, H. A. Robinson, Sarah H. Rodgers, T. M. Simpson, R. P. Stephens.

On the evening of the 30th, a dinner was held in honor of the visiting speaker, Professor Arnold Dresden. At this time Professor W. P. Ott presided.

At the business session on the 31st the following officers were chosen for 1934–35: Chairman, W. P. Ott, University of Alabama; Vice-Chairman, F. W. Kokomoor, University of Florida; Secretary-Treasurer, H. A. Robinson, Agnes Scott College. The next meeting was tentatively scheduled for March, 1935, at Agnes Scott College. A resolution was passed relative to the great loss sustained by the Southeastern Section in the passing of two of its former chairmen, Professors A. B. Morton and C. D. Killebrew.

The following twenty-one papers were read:

1. "Preparation for freshman mathematics" by Professor J. B. Coleman, University of South Carolina.

2. "A new course in freshman mathematics" by Dean R. P. Stephens, University of Georgia.

3. "Modern algebra" by Professor T. M. Simpson, University of Florida.

4. "A discussion of the existence of Nasik magic squares" by Dr. Ralph Hefner, Georgia School of Technology.

5. "Some particular methods of making mathematics more meaningful and practical to students" by Professor B. P. Reinsch, Southern College.

6. "Computation of certain coefficients in solutions of Bessel's equation" by Dean Floyd Field, Georgia School of Technology.

7. "Transfinite numbers" by Miss Martha N. Watson, Auburn, Alabama, by invitation.

8. "A simple solution of a reduced quartic" by Professor G. W. Hess, Howard College.

9. "The acnodal cubic and its inflections" by R. V. Blair, Vanderbilt University.

10. "The Mathematical Association of America and American Mathematics" by Professor Arnold Dresden, Swarthmore College.

11. "A graphical technique in the method of least squares" by Dr. H. H. Germond, University of Florida, by invitation.

12. "A note on a lemma by Graves" by J. D. Mancill, University of Alabama, by invitation.

13. "Pseudo-isosceles triangles" by Professor D. F. Barrow, University of Georgia.

14. "Some aspects of the logical foundations of mathematics" by Professor Arnold Dresden.

15. "Finite plane Euclidean geometry" by A. N. McPherson, Vanderbilt University, by invitation.

16. "Some properties of two associated sets of polynomials" by H. S. Thurston, University of Alabama, by invitation.

17. "Remarks on quantum mechanics and generalized numerology" by B. F. Dostal, University of Florida.

18. "Two theorems concerning squared integers" by Dean E. A. Bailey, LaGrange College.

19. "An extension of the method of partial fractions" by Professor H. K. Fulmer, Georgia School of Technology.

20. "An invariant set of eight functions permuted according to the permutations of the simple group of order 168" by Professor F. A. Lewis, University of Alabama.

21. "A composition of combination numbers" by Dr. D. C. Harkin, Alabama Polytechnic Institute, by invitation, by title.

In the absence of the author, paper number six was read by Professor Fulmer. Abstracts of the papers follow, the numbers corresponding to the numbers in the list of titles:

1. A comparison of those students who took college algebra immediately after completing the high school course, with those who allowed an interval of one or more years to intervene shows that the first group had less than half as many casualties and twice the number of high grades.

2. To meet the needs of the new curriculum, relative to general preparation for good citizenship, the junior colleges of the University System of Georgia will offer a unified course consisting of a few topics in algebra, statistics, investment, insurance, and the right triangle in trigonometry.

3. This paper discussed points of view in modern algebra by comparing the two outstanding books in English, Bôcher's "Introduction to Higher Algebra" and Dickson's "Modern Algebraic Theories," published 20 years apart, with a recent German text "Moderne Algebra" by van der Waerden. Particular stress was laid on the fundamental notions of homomorphism and isomorphism.

4. By dividing the squares into four cases, the existence of Nasik squares, where $n \geqq 4$ and $n \neq 2(2m-1)$, was shown. The fourth case $n = 2(2m-1)$ does not exist.

5. Topics were used to illustrate two methods of analysis and presentation: (1) the explanation of new ideas and processes in terms of the old and familiar; and (2) the systematic association of the abstract with the concrete, the graphical with the analytical.

6. This paper established the equivalence of coefficients resulting from two solutions of Bessel's equation.

7. A survey of the theory of transfinite numbers as defined by Cantor was given. Some interesting properties of transfinite cardinal and ordinal numbers were mentioned.

8. A solution of the reduced quartic which called for the use of only one root of the auxiliary cubic was given.

9. Parametric equations of the point acnodal cubic were developed and the three points of inflection found. The parametric equations of the curve in terms of the line coordinates and the inflectional tangents were derived. Some dual properties of the point curve and the corresponding line curve were mentioned.

10. The most effective contribution which the Association can make to the development of mathematics in this country lies in bringing about wider recognition of the significance of mathematics for the training and liberal education of the people. Why is mathematics important, not only for the future scientist

or engineer, but also for those who never expect to use mathematics in a technical way? How can the Association meet the arguments of those who wish to abolish the required study of mathematics in the schools?

11. The values of the parameters, a_i, which make $f(x, a_i)$ a best fit, reduce in certain cases to definite integrals or areas which may be measured with suitable instruments. The graphical technique of routine procedure was discussed.

12. After appropriate modification of certain hypotheses, a lemma by Graves on positively strong extremaloids was extended to certain curves which do not possess necessarily the extremal property on arcs between corners. Application was made to the case where an arc of the minimizing curve on the boundary surface was allowed to have corners.

13. A construction with ruler and compasses of the pseudo-isosceles triangle was given and a special example where the angles were 12°, 36° and 132° was presented.

14. This paper dealt with the role of logic in the foundations of mathematics, the trend towards the use of formal logic, the paradoxes which have arisen and various attempts to deal with them. The work of Peano, Whitehead, Russell, Brouwer and Hilbert was briefly considered.

15. A finite plane Euclidean geometry was set up by using as coordinates of points the marks of a Galois field. A close similarity was shown to exist between this geometry and the plane geometry with which we are familiar.

16. By means of DeMoivre's theorem, Weber derived two sets of polynomials which satisfy a certain recurrent relation. The polynomials were obtained as expansions of determinants of order n and $n-1$ respectively. Properties of these functions were given.

17. Professor Dostal gave a review of recent developments in Quantum Mechanics, the latest results of which represent a return to Einstein's Kinematic Methods, especially the work of Lanczos, Born, and of Madelung and Flügge. The work of the latter two investigators seems to promise an explanation of the whole range of Quantum Phenomena in terms of forces operating through a 4th space-dimension. This would link up Quantum Physics with the 5-dimensional Kaluza-Klein world, and make it possible to foresee the general nature of some of the steps necessary to effect the union of Quantum Mechanics with Electromagnetism and Gravitation.

18. Two theorems concerning squared integers were demonstrated.

19. The traditional method of partial fractions was extended to transcendental functions. The discussion here was restricted to types for which the resulting partial fractions could be readily integrated.

20. This paper exhibited a collineation group and an invariant set of eight functions which are permuted by the collineations of the group according to the permutations of the simple group of order 168.

21. Vieta's results, different from DeMoivre's, for sec mx are reached by a certain matric composition of binomial coefficients.

H. A. ROBINSON, *Secretary*

EXPONENTIAL NUMBERS

By E. T. BELL, California Institute of Technology

1. In some calculations which I was asked to check, a discrepancy was easily traced to the mathematical handbook* that had been used, where the MacLaurin expansion for $e^{\sin x}$ is given incorrectly. This suggested the desirability of having some readily applicable numerical check on the tedious algebra involved in expanding functions of the type $e^{f(x)}$ in MacLaurin series, when the expansion exists, and this in turn led to the definition of *exponential integers* and the investigation of their simpler arithmetical properties, with some of which this note is concerned. The simplest of the congruence properties developed in §6 are ample for checks on the series in the handbooks.

2. Let $e^{f(x)-f(0)}$ admit a MacLaurin expansion, and let both this series and the power series expansion

$$f(x) = c_0 + c_1 x + \cdots + c_n x^n + \cdots$$

be absolutely convergent and differentiable term by term for $0<|x|<k$. The coefficients $a_n (n \geqq 0)$ in

$$e^{-f(0)}e^{f(x)} = a_0 + a_1 x + a_2 \frac{x^2}{2!} + \cdots + a_n \frac{x^n}{n!} + \cdots$$

will be called *the exponential numbers associated with* $c_0, c_1, \cdots, c_n, \cdots$, or, using the symbolic or umbral notation, we shall say that a is associated with c. When all the $a_n (n=0, 1, \cdots)$ associated with c are integers, we shall refer to them as *exponential integers*. It will be seen presently that a sufficient condition that the a_n be exponential integers is that $r!c_r$ be an integer for all integers $r>0$.

To avoid questions of convergence we shall give (§4) an independent definition of exponential numbers and integers, in which no infinite process is involved. However, as the coefficients first presented themselves as above, we have given the preliminary definition. Some examples in §7 illustrate the general theorems of §§3–6.

3. For some $0<|x|<k$ let both of the expansions

$$(3.1) \qquad f(x) = \sum_{n=0} c_n x^n, \qquad e^{-f(0)}e^{f(x)} = \sum_{n=0} \alpha_n \frac{x^n}{n!}$$

be absolutely convergent and termwise differentiable. Write

$$(3.2) \qquad c_n \equiv a_n/n!,$$

and pass to the umbral notation. Then we have

$$(3.3) \qquad f(x) = e^{ax}, \qquad e^{-a_0}e^{e^{ax}} = e^{ax}.$$

* W. Láska, *Sammlung von Formeln*, Braunschweig, 1888–1894. This handbook is still quite popular with some physicists. The error mentioned is only one of several that reappear, without acknowledgement to Láska, in later handbooks.

By the usual rules of the umbral calculus[1] we may proceed to differentiate the second of these with respect to x, under the hypotheses on (3.1); the umbrae a, α are treated as ordinaries until symbolic formulas are translated back into ordinary notation, when a^n, α^n $(n=0, 1, \cdots)$ are replaced by a_n, α_n. It is to be noted that if ϕ, ψ are umbrae, and p, q ordinaries$\neq 0$,

$$(p\phi + q\psi)^0 = p^0\phi^0q^0\psi^0 = \phi^0\psi^0 = \phi_0\psi_0.$$

Thus we find

$$ae^{ax}e^{\alpha x} = \alpha e^{\alpha x}, \qquad ae^{(a+\alpha)x} = \alpha e^{\alpha x},$$

and hence, by equating coefficients of x^n,

$$a(a+\alpha)^n = \alpha^{n+1}(n \geqq 0), \qquad \alpha_0 = 1. \tag{3.4}$$

In ordinary notation, (3.4) is

$$\sum_{j=0}^{n} \binom{n}{j} a_{j+1}\alpha_{n-j} = \alpha_{n+1}(n \geqq 0), \qquad \alpha_0 = 1. \tag{3.5}$$

The special cases when $f(x)$ is an even, an odd function of x give recurrences which may be derived from (3.5). It is more interesting however to obtain them independently.

For some $0<|x|<k_0$ let both of the expansions

$$h(x) = \sum_{n=0} b_{2n}x^{2n}, \qquad e^{-h(0)}e^{h(x)} = \sum_{n=0} \eta_{2n}\frac{x^{2n}}{(2n)!} \tag{3.6}$$

be absolutely convergent and termwise differentiable, and similarly for $0<|x|<k_1$ and

$$g(x) = \sum_{n=0} d_{2n+1}x^{2n+1}, \qquad e^{g(x)} = \sum_{n=0} \omega_n\frac{x^n}{n!}. \tag{3.7}$$

Write

$$b_{2n} \equiv r_{2n}/(2n)!, \qquad d_{2n+1} \equiv s_{2n+1}/(2n+1)!. \tag{3.8}$$

Then, from (3.6), (3.7), we have

$$h(x) = \cosh rx, \qquad e^{-r_0}e^{\cosh rx} = \cosh \eta x, \tag{3.9}$$

$$g(x) = \sinh sx, \qquad e^{\sinh sx} = e^{\omega x}. \tag{3.10}$$

Differentiation of these with respect to x gives

$$r \sinh rx \cosh \eta x = \eta \sinh \eta x, \qquad s \cosh sx \cdot e^{\omega x} = \omega e^{\omega x};$$

whence

[1] Due to Blissard. There is a sufficient account of this calculus in E. Lucas' *Théorie des Nombres*, Chap. 13. In my *Algebraic Arithmetic*, 1927, I have developed this calculus further.

$$r[\sinh(r+\eta)x + \sinh(r-\eta)x] = 2\eta \sinh \eta x,$$

$$s[e^{(\omega+s)x} + e^{(\omega-s)x}] = 2\omega e^{\omega x};$$

which give the recurrences

$$(3.11)\qquad r[(r+\eta)^{2n+1} + (r-\eta)^{2n+1}] = 2\eta^{2n+2}(n \geqq 0), \qquad \eta_0 = 1,$$

$$(3.12)\qquad s[(\omega+s)^n + (\omega-s)^n] = 2\omega^{n+1}(n \geqq 0), \qquad \omega_0 = 1;$$

or, in ordinary notation,

$$(3.13)\qquad \sum_{j=0}^{n}\binom{2n+1}{2j} r_{2n+2-2j}\eta_{2j} = \eta_{2n+2}(n \geqq 0), \qquad \eta_0 = 1,$$

$$(3.14)\qquad \sum_{j=0}^{[n/2]}\binom{n}{2j} s_{2j+1}\omega_{n-2j} = \omega_{n+1}(n \geqq 0), \qquad \omega_0 = 1.$$

Instead of using the umbral calculus we might have obtained the recurrences (3.5), (3.13), (3.14) by Leibniz' theorem applied to the first derivatives of the generating functions for α, η, ω.

4. The arithmetical properties of the coefficients α, η, ω are implied by the recurrences and are independent of the origin of the α_n, η_n, ω_n as coefficients in power series. Accordingly we lay down the following *definitions of the exponential numbers* α_n, η_n, ω_n $(n \geqq 0)$ *of the first, second and third kinds respectively, associated respectively with* c_n, b_{2n}, d_{2n+1}:

$$(4.1)\qquad a(a+\alpha)^n = \alpha^{n+1}(n \geqq 0), \qquad \alpha_0 = 1,$$

$$a_n \equiv n!c_n;$$

$$(4.2)\qquad r[(r+\eta)^{2n+1} + (r-\eta)^{2n+1}] = 2\eta^{2n+2}(n \geqq 0), \qquad \eta_0 = 1,$$

$$r_{2n} \equiv (2n)!b_{2n};$$

$$(4.3)\qquad s[(\omega+s)^n + (\omega-s)^n] = 2\omega^{n+1}(n \geqq 0), \qquad \omega_0 = 1,$$

$$s_{2n+1} \equiv (2n+1)!d_{2n+1}.$$

The equivalents of (4.1), (4.2), (4.3) in ordinary notations are (3.5), (3.13), (3.14). As already indicated, (4.2), (4.3) are special cases of (4.1).

When the exponential numbers are integers, we shall call them *exponential integers.*

A sufficient condition that the $\alpha_n(n \geqq 0)$ be integers is that all $j!c_j(j>0)$ be integers, by (4.1). The corresponding condition for $\eta_{2n}(n \geqq 0)$ is that all $(2j)!b_{2j}(j>0)$ be integers; and for $\omega_n(n \geqq 0)$, that all $(2j+1)!b_{2j+1}(j \geqq 0)$ be integers.

To extend the recurrences (4.1)–(4.3), let

$$P(x) \equiv h_0 + h_1 x + \cdots + h_m x^m, \qquad h_m \neq 0,$$

be any polynomial of degree m in x. Then, by (4.1), we have

$$a\sum_{n=0}^{m} h_n(a+\alpha)^n = \alpha\sum_{n=0}^{m} h_n\alpha^n,$$

and therefore

$$aP(a+\alpha) = \alpha P(\alpha). \tag{4.4}$$

In the same way, from (4.2), we find

$$r[(r+\eta)P((r+\eta)^2) + (r-\eta)P((r-\eta)^2)] = 2\eta^2 P(\eta^2), \tag{4.5}$$

and from (4.3),

$$s[P(\omega+s) + P(\omega-s)] = 2\omega P(\omega). \tag{4.6}$$

Finally, $P(x)$ may be replaced by a power series, provided the series in (4.4)–(4.6) are then convergent.[1]

The generating function of ω being odd, the numbers ω satisfy a bilinear recurrence,[2]

$$(\omega' - \omega'')^{2m} = 0(m > 0), \qquad \omega' \equiv \omega \equiv \omega'', \tag{4.7}$$

or, in ordinary notation,

$$2\sum_{j=0}^{m-1}(-1)^j\binom{2m}{j}\omega_j\omega_{2m-j} + (-1)^m\omega_m^2 = 0(m > 0). \tag{4.8}$$

This can be proved from (4.1), or as suggested in the footnote.

5. The simplest arithmetical properties of exponential integers follow from theorems concerning the residues of binomial coefficients to a prime modulus.[3] We shall require the following.

Let p be prime. Then

$$\binom{p}{r} \equiv 0 \bmod p,\ 0 < r < p;\binom{p}{0} \equiv \binom{p}{p} \equiv 1 \bmod p. \tag{5.1}$$

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[1] It is sometimes said that a result such as (4.4) is a generalization of the result such as (4.1) from which it is derived. This is incorrect: (4.1) and (4.4) express the same fact; they are logically equivalent, for it is obvious that each implies the other. Nevertheless (4.4) is sometimes more suggestive than (4.1).

[2] Such recurrences are most readily obtained by the symbolic method. Here

$$1 = e^{g(x)}e^{-g(x)},\ = e^{g(x)+g(-x)}$$

since $g(x)$ is odd; hence

$$1 = e^{(\omega'-\omega'')x},$$

where $\omega'^m \equiv \omega_m$, $\omega''^m \equiv \omega_m$. Hence $(\omega'-\omega'')^0 = 1$, $(\omega'-\omega'')^n = 0$ $(n>0)$. If n is odd, the last is a trivial identity. But if $n = 2m$, we have (4.8). The infinite formal processes used in this derivation are independent of questions of convergence, and have been validated in my book cited.

[3] See Lucas, loc. cit., §228; several are due to Lucas. See also Dickson, *History of the Theory of Numbers*, vol. 1, Chap. 9. The convenient symbolic expression (5.4) for the residue of $(\lambda+\mu)^N$ modulo p was given in my paper on Anharmonic Polynomials, *Transactions of the American Mathematical Society*, vol. 34 (1922), p. 109.

It will be convenient to use the customary extension of the notation $\binom{m}{n}$ to negative values of m,

$$\binom{-m}{0} = 1, \qquad \binom{-m}{n} = (-m)(-m-1)\cdots(-m-n+1)/n!,$$

for $m>0$, $n>0$. The next residue theorem may then be written

$$\binom{p-j}{h} \equiv \binom{-j}{h} \bmod p, \qquad 0 \leqq h < p-j < p. \tag{5.2}$$

The third residue theorem is

$$\begin{gathered}\binom{m}{n} \equiv \binom{m_1}{n_1}\binom{m_1'}{n_1'} \bmod p,\\ m>0,\ n\geqq 0,\ m = m_1 p + m_1',\ n = n_1 p + n_1',\\ 0 \leqq m_1',\ n_1' < p;\ \binom{r}{s} = 0 \text{ if } s > r.\end{gathered} \tag{5.3}$$

A similar reduction may be applied to $\binom{m_1}{n_1}$, provided at least one of m_1, n_1, exceeds p, and so on.

To pass to the general case, we apply (5.3) to $(\lambda+\mu)^N$, where λ, μ are either ordinaries or umbrae (the latter includes the former as a special case), and N is an integer>0. Let

$$N = g_n p^n + g_{n-1}p^{n-1} + \cdots + g_1 p + g_0,$$
$$0 \leqq g_j < p(j = 0, \cdots, n-1),\ g_n \neq 0,\ g_n < p,$$

be the expression of N in the scale of p. Then we find easily the following congruences,

$$(\lambda \pm \mu)^N \equiv \sum_{j=0}^{n} (\lambda^{p^j} \pm \mu^{p^j})^{g_j} \bmod p, \tag{5.4}$$

(the upper or the lower signs being taken throughout), in which, if λ, μ are umbrae, all the indicated binomial expansions and subsequent multiplications are to be performed as in common algebra before exponents are lowered.

6. *Let α, η, ω in* §4 *be exponential integers, and let p be prime.* Then, applying (5.1) to §4, we get

$$a_1\alpha_p + a_{p+1} \equiv \alpha_{p+1} \bmod p, \tag{6.1}$$

$$r_{p+1} \equiv \eta_{p+1} \bmod p,\ p > 2,$$

$$s_1\omega_p \equiv \omega_{p+1} \bmod p,\ p > 2. \tag{6.3}$$

It is easily seen that the second and third of these are included in the first, and similarly for all following triads of congruences in this section.

From the recurrences in §4 written in ordinary notation, we get the following by (5.2),

$$(6.4)\qquad a_{p-k+1} + \sum_{j=0}^{p-k-1} \binom{-k}{j} a_{j+1}\alpha_{p-k-j} \equiv \alpha_{p-k+1} \bmod p,$$

$$0 < p - k < p;$$

$$(6.5)\qquad \sum_{j=0}^{(p-1)/2-k} \binom{-2k}{2j} r_{p+1-2k-2j}\eta_{2j} \equiv \eta_{p+1-2k} \bmod p,$$

$$p > 2,\ 0 < p - 2k < p;$$

$$(6.6)\qquad \sum_{j=0}^{[(p-2k)/2]} \binom{-2k}{2j} s_{2j+1}\omega_{p-2k-2j} \equiv \omega_{p-2k+1} \bmod p,$$

$$p > 2,\ 0 < p - 2k < p;$$

$$s_{p-2h} + \sum_{j=0}^{(p-3)/2-h} \binom{-2h-1}{2j} s_{2j+1}\omega_{p-2h-1-2j} \equiv \omega_{p-2h} \bmod p,$$

$$p > 2,\ 0 \leqq h \leqq (p-3)/2.$$

The corresponding congruence from (4.8) gives a theorem on quadratic residues.

The following very special cases of (5.4) applied to §4 will suffice.

$$(6.7)\qquad a(a^p + \alpha^p)(a + \alpha)^h \equiv \alpha^{p+h+1} \bmod p,$$

$$(6.8)\qquad r[(r^p + \eta^p)(r + \eta)^{2h} + (r^p - \eta^p)(r - \eta)^{2h}] \equiv 2\eta^{p+2h+1} \bmod p,\ p > 2,$$

$$(6.9)\qquad s[(\omega^p + s^p)(\omega + s)^h + (\omega^p - s^p)(\omega - s)^h] \equiv 2\omega^{p+h+1} \bmod p;$$

(6.7), (6.9) are valid for $0 \leqq h < p$, (6.8) for $0 \leqq 2h < p$. For $h = 0$, these become (6.1)–(6.3).

7. The coefficients in the MacLaurin expansions in §3 can be calculated successively by the recurrences in §4. If desired, the recurrences can be solved to give the coefficients explicitly as determinants; the results, however, are useless for computation and appear to have no value for the deduction of arithmetical theorems. Herschel's theorem in the calculus of finite differences is sometimes useful in furnishing manageable explicit forms from which something can be inferred. If $\phi(e^t)$ has a MacLaurin expansion, Herschel's theorem[1] gives the expansion in the form

$$\phi(e^t) = \phi(1) + \phi(E)0\cdot t + \phi(E)0^2\cdot\frac{t^2}{2!} + \cdots \equiv \phi(E)e^{0\cdot t},$$

where $E \equiv 1 + \Delta$ is the usual operator in finite differences.

The simplest example is (ϵ umbral)

$$e^{-1}e^{e^x} = e^{\epsilon x},\ \epsilon_0 = 1,\ \epsilon_n \equiv \left(1 + \frac{\Delta}{1!} + \frac{\Delta^2}{2!} + \cdots + \frac{\Delta^n}{n!}\right)0^n.$$

[1] See Boole, *Finite Differences*, (reprinted by Stechert), p. 24. The so-called "differences of nothing" required in numerical work are tabulated for a short range by Steffensen, *Interpolation* (Williams and Wilkins, 1927), p. 55.

But (a well known formula),

$$\frac{\Delta^n 0^m}{n!} = \frac{1}{n!}\sum_{r=0}^{n-1}(-1)^r\binom{n}{r}(n-r)^m;$$

whence,

$$\epsilon_n = \sum_{s=1}^{n}\frac{1}{(s-1)!}\left[\sum_{r=0}^{s-1}(-1)^r\binom{s-1}{r}(s-r)^{n-1}\right], \quad n > 0.$$

The ϵ's have interesting connections with the numbers of Bernoulli and Stirling but these need not be discussed now. Using Steffensen's table (loc. cit.) for $\Delta^n 0^m/n!$ we get

$$\epsilon_1 = 1,\ \epsilon_2 = 2,\ \epsilon_3 = 5,\ \epsilon_4 = 15,\ \epsilon_5 = 52,\ \epsilon_6 = 203,\ \epsilon_7 = 877,$$

$$\epsilon_8 = 4140,\ \epsilon_9 = 21147,\ \epsilon_{10} = 115975.$$

Referring to (4.1), we see that here $a_n = 1 (n \geqq 0)$, and hence the congruence (6.1) becomes, for $\alpha \equiv \epsilon$,

$$\epsilon_p + 1 \equiv \epsilon_{p+1} \bmod p.$$

The cases $k = 1, 2$ of (6.4) give

$$1 + \sum_{j=0}^{p-2}(-)^j\epsilon_{p-1-j} \equiv \epsilon_p \bmod p,$$

$$1 + \sum_{j=0}^{p-3}(-1)^j(j+1)\epsilon_{p-2-j} \equiv \epsilon_{p-1} \bmod p, \qquad p > 2;$$

while (6.7) with $h = 1$ gives

$$2 + \epsilon_p + \epsilon_{p+1} \equiv \epsilon_{p+2} \bmod p,$$

and with $h = 2$,

$$5 + \epsilon_p + 2\epsilon_{p+1} + \epsilon_{p+2} \equiv \epsilon_{p+3} \bmod p, \qquad p > 2,$$

and so on. These are verified by the above numerical values. Incidentally, the check renders probable the accuracy of Steffensen's table.

Further simple illustrations of expansions of the type (3.1) giving sequences of exponential integers defined by recurrences of the type (3.4) or (4.1) are given by $e^{f(x)}$ where $f(x) = \tan x$ or $\arctan x$.

In (4.2) take $r_{2n} = (-1)^n$, and therefore $b_{2n} = (-1)^n/(2n)!$. Hence, in (3.6), $h(x) = \cos x$. The coefficients can be calculated by (4.2) or (3.13). Write $\eta \equiv \kappa$ in this case. Then

$$\sum_{j=0}^{n}(-1)^j\binom{2n+1}{2j}\kappa_{2j} = (-1)^{n+1}\kappa_{2n+2},$$

and we find

$$\kappa_0 = 1,\ \kappa_2 = -1,\ \kappa_4 = 4,\ \kappa_6 = -31,\ \kappa_8 = 379,\ \kappa_{10} = -6556,$$

which are sufficient for verifying the congruences. Thus (6.2) becomes

$$\kappa_{p+1} \equiv (-1)^{(p+1)/2} \bmod p, \qquad p > 2,$$

which is checked, and similarly for (6.5) with $k=1$,

$$\sum_{j=0}^{(p-3)/2} (-1)^j(2j+1)\kappa_{2j} \equiv (-1)^{(p-1)/2}\kappa_{p-1} \bmod p, \qquad p > 2,$$

and for (6.8) with $h=1$,

$$\kappa_{p+3} \equiv 2[(-1)^{(p+3)/2} - \kappa_{p+1}] \bmod p, \qquad p > 2.$$

Combining the first and third of these we get the simpler result

$$\kappa_{p+3} \equiv 4(-1)^{(p+3)/2} \bmod p, \qquad p > 2.$$

As an example of the remaining type, take $s_{2n+1} = (-1)^n$ in (4.3), and hence $g(x) = \sin x$ in (3.7). Write $\omega \equiv \sigma$ for this choice, and calculate the σ by (3.14). Then

$$\sigma_0 = 1,\ \sigma_1 = 1,\ \sigma_2 = 1,\ \sigma_3 = 0,\ \sigma_4 = -3,\ \sigma_5 = -8,\ \sigma_6 = -3,\ \sigma_7 = 56,$$

$$\sigma_8 = 217,\ \sigma_9 = 64,\ \sigma_{10} = -2951,\ \sigma_{11} = -12672,\ \sigma_{12} = 5973.$$

The congruences are, from (6.3),

$$\sigma_p \equiv \sigma_{p+1} \bmod p, \qquad p > 2;$$

from the first of (6.6), with $k=1$,

$$\sum_{j=0}^{[(p-2)/2]} (-1)^j(2j+1)\sigma_{p-2-2j} \equiv \sigma_{p-1} \bmod p, \qquad p > 2,$$

and from the second, with $h=1$,

$$(-1)^{(p-3)/2} + \sum_{j=0}^{(p-5)/2} (-1)^j(j+1)(2j+1)\sigma_{p-3-2j} \equiv \sigma_{p-2} \bmod p, \quad p > 4;$$

while from (6.9) with $h=1$ we get

$$\sigma_{p+1} + (-1)^{(p+1)/2} \equiv \sigma_{p+2} \bmod p, \qquad p > 2.$$

All of the congruences are checked by the above numerical values.

From the well known expansions for $e^{x\cos\phi}$, $e^{x\sin\phi}$ in series of Bessel coefficients, we find immediately the following independent expressions for the integers κ, σ:

$$\kappa_{2t} = e^{-1}(-1)^t \sum_{s=1}^{\infty} i^s J_s(-i)s^{2t}(t > 0),$$

$$\sigma_{2t} = 2(-1)^t \sum_{s=0}^{\infty} (-1)^t J_{2s}(-i)(2s)^{2t}(t > 0),$$

$$\sigma_{2t+1} = 2i(-1)^t \sum_{s=1}^{\infty} (-1)^t J_{2s-1}(-i)(2s-1)^{2t+1}(t \geqq 0),$$

where $i \equiv (-1)^{1/2}$.

ON FERMAT'S LAST THEOREM[1]

By GLENN JAMES, University of California at Los Angeles

1. Introduction. We consider the equation

$$x^n + y^n = z^n, \tag{1}$$

where n is an odd prime integer, x, y, z are positive integers relatively prime in pairs and $y>x$. There is no loss of generality in this last restriction, since if $x=y$, z cannot be an integer. In this paper we confine ourselves to the so-called "first case," namely that in which $x, y, z \not\equiv 0 \bmod n$. Certain intervals, in which $z-y$ must lie if x, y, z satisfy (1), are easily established by analytic methods. For instance[2]

$$2[1-(1/2)^{1/n}]x^n/z^{n-1} > z-y > x^n/(nz^{n-1}). \tag{2}$$

This paper is a first step in an attempt to prove Fermat's Last Theorem by excluding $z-y$ from some such interval by means of number theory properties of x, y, z and n. Our central theorem states that (1) fails unless

$$z - y \geqq (cn+1)^n, \quad c \geqq 2.$$

This theorem, incidentally, completes the various attempts to prove that x, y, and z are composite.[3]

Another by-product is a proof of the cubic case, which seems to be new and suggests another possible point of attack on the general problem.

2. Preliminary Considerations. It was proved by P. Barlow[4] that if (1) holds when $x, y, z \not\equiv 0 \bmod n$, then

$$x+y=r^n, \qquad z-x=s^n, \qquad z-y=t^n \tag{3}$$

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[1] Presented to the American Mathematical Society, August 31, 1932.

[2] To establish these limits one writes (1) in the form

$$x^n + [z-(z-y)]^n = z^n \text{ or } (x^n/z^n) + [1-(z-y)/z]^n = 1.$$

Putting v for x^n/z^n and αv for $(z-y)/z$ we get $v+(1-\alpha v)^n=1$. Whence $\alpha=(1/v)[1-(1-v)^{1/n}]$ $=j(v)$. Now, $1/2>v>0$, $j(0)=1/n$ and $j'(v)>0$, whence $2[1-(1/2)^{1/n}]>\alpha>1/n$. From this (2) readily follows.

[3] H. F. Talbot, Trans. Royal Society, Edinburgh, vol. 21 (1857), pp. 403–6, and others proved that x, y, z are composite unless $z-y$ is unity. We have removed this troublesome case.

[4] Jour. Nat. Phil. Chem. and Arts., vol. 27 (1810), p. 193.

where r, s and t are integers.

Legendre noted that[1]

(4) $$x + y - z \equiv 0 \bmod \left[n(x + y)^{1/n}(z - x)^{1/n}(z - y)^{1/n}\right].$$

Obviously

(5) $$z - y < z - x \text{ and } z < x + y,$$

whence

(6) $$z - y < z - x < x + y.$$

Also

(7) $$z < y + x/n,$$

as can be seen by substituting $y + x/n$ for z in (1). Finally we can write

(8) $$(z - x)^{1/n} + (z - y)^{1/n} - (x + y)^{1/n} = cn,$$

where c is zero or even since exactly two of x, y and z are odd. This relation is obtained as follows: apply Fermat's First Theorem to conditions (3), obtaining

$$-(x + y) \equiv -(x + y)^{1/n},\ (z - x) \equiv (z - x)^{1/n},\ (z - y) \equiv (z - y)^{1/n}, \bmod n.$$

The sum of these three congruences and that which results from doubling both members of (4) can be written in the algebraic form (8).

3. We now prove that c of (8) is positive. We first show by a reductio ad absurdum argument that, if (1) and (8) are simultaneously satisfied, then c is not zero. Upon the basis of this result we prove that c can not be negative.

Divide equation (1) by z^n. Put $c = 0$ in (8). Then divide through by $z^{1/n}$. The results are, respectively,

(9) $$(x/z)^n + (y/z)^n = 1$$

and

(10) $$(1 - x/z)^{1/n} + (1 - y/z)^{1/n} - (x/z + y/z)^{1/n} = 0.$$

Now we substitute

(11) $$Y^n = 1 - z,/x \qquad Y^n = 1 - y/z,$$

obtaining

(12) $$(1 - X^n)^n + (1 - Y^n)^n = 1$$

and

(13) $$X^n + Y^n + (X + Y)^n - 2 = 0.$$

These curves are symmetrical with respect to the line $Y = X$. Moreover we are interested only in positive values of the variables. Hence we need consider only

[1] *Théorie des nombres*, ed. 2, 1808, second supplement, Sept., 1825, pp. 1–40.

the segment of these curves in the region $1>X>Y>0$. It is easy to show that for any n equation (12) defines Y as a single-valued, continuous, monotonically decreasing function of X over the interval $(1-2^{-1/n})^{1/n}<X<1$; the derivative dY/dX exists at each point and is given by the expression $-[X(1-X^n)/Y(1-Y^n)]^{n-1}$; and the line $Y=mX$, $0<m<1$, cuts the curve once and only once in the region. Similarly, for any n, equation (13) defines Y as a single-valued, continuous, monotonically decreasing function of X over the interval $1/(1+2^{n-1})^{1/n}<X<1$; the derivative dY/dX exists at each point and is given by the expression $-[X^{n-1}+(X+Y)^{n-1}]/[Y^{n-1}+(X+Y)^{n-1}]$; and the line $Y=mX$, $0<m<1$, cuts the curve once and only once in the region.

LEMMA. Within the region $1>X>Y>0$, and, for a fixed n, the locus of (13) separates the locus of (12) from the origin.[1]

Proof: The slopes of (13) and (12) at the point (1, 0) are -2 and $-n^{(n-1)/n}$ respectively. The former curve cuts the line $Y=X$ at the point whose coordinates are $(1/(1+2^{n-1})^{1/n}$, which approaches 1/2 decreasing monotonically as n increases. The latter curve cuts this line in the point whose coordinates are $(1-1/2^{1/n})^{1/n}$, which approaches 1, increasing monotonically as n increases. When $n=2$ the intercept of (13) on $Y=X$ exceeds that of (12). When $n\geqq 3$ the reverse is true. Their values when $n=3$ are respectively $.585-$ and $.591-$. The case in which $n=3$ lends itself to ordinary treatment, whereas the proximity of the curves in this case makes it difficult to include in the general discussion. Hence we treat the former case separately.

Making the substitution $Y=mX$ in (12) and (13) with $n=3$, eliminating X between the two results and multiplying by the common denominator we obtain

$$[(1+m)^3-(1-m^3)]^3+[(1+m)^3+(1-m^3)]^3 - [(1+m)^3+(1+m^3)]^3=0,$$

which reduces to

$$3m^3(1+m)^3=0; \tag{15}$$

whence there are no admissable values of m.

Curves (13) and (12) when $n=5$, cut the line $Y=X$ in points whose coordinates are respectively $.5674+$ and $.6643+$. To prove that (13) separates (12) from the origin, in the region considered, for $n\geqq 5$ we prove that neither cuts the parabola,

$$Y=1-X^2 \tag{16}$$

in that region. This suffices since this parabola cuts $Y=X$ between the intersections of (13) and (12) with that line. We first show that for a fixed X and n, the ordinate of (12) exceeds the ordinate of (16), that is

[1] Separates in the sense that every straight line through the origin crossing the region cuts (12) farther from the origin than it does (13).

$$[1 - [1 - (1 - X^n)^n]^{1/n}]^{1/n} > 1 - X^2 \tag{17}$$

or

$$1 - [1 - (1 - X^n)^n]^{1/n} > (1 - X^2)^n.$$

Expanding the bracket on the left and dividing by the right member we obtain

$$\begin{aligned}(1/n)[1 + X^2 + X^4 + \cdots + X^n - 1/(1 + X)]^n \\ + (n - 1)(1 - X^n)^{2n}/[2n^2(1 - X^2)^n] + \cdots > 1.\end{aligned} \tag{18}$$

It suffices to show that

$$1 + X^2 + X^4 + \cdots + X^{n-3} > n^{1/n} \tag{19}$$

since the other terms on the left are positive. The right member of (19) is a decreasing function of n while the left is an increasing function of both n and X. The least possible n is 5 and the least[1] possible X is .6645+. For these values, (19) becomes

$$1 + (.6635)^2 = 1.44 + > 5^{1/5} = 1.38 - .$$

To prove that (13) does not cut (16) we first show that the slope of the former at the point where it cuts the line $Y = mX$ is numerically less than the slope of the latter at the point where it cuts this line. These slopes are respectively

$$-\frac{1 + (1 + m)^{n-1}}{m^{n-1} + (1 + m)^{n-1}} \text{ and } m - \sqrt{4 + m^2}.$$

Since the former is a numerically decreasing function of n, it suffices to show that the inequality holds true if n be given the value 4. Noting that

$$-m + \sqrt{4 + m^2} > -m + 2$$

we see that it suffices to show that

$$[m^3 + (1 + m)^3](2 - m) \geqq 1 + (1 + m)^3.$$

This reduces to the true inequality

$$1 \geqq m^3.$$

The curves (13) and (16) both pass through the point whose coordinates are (1, 0), and for both Y is a continuous, single valued, and monotonically decreasing function of X, and the curves intersect in at most a finite number of points. In the light of these considerations we see that there must be a point of intersection *next* to (1, 0) on curve (16) say, if such a point exists anywhere in the region. Between the point (1, 0) and the *next* point of intersection, if such exist, curve (13) separates (16) from the origin by virtue of the above inequality

[1] X^n is least when x/z is greatest, see (11). This occurs when $x = y$, that is when, from (1), $2x^n = z^n$ or $x/z = 1/2^{1/n}$. X is least when $(1 - 1/2)^{1/n}$ is least, which occurs when $n = 5$.

between the slopes. Consequently if they intersect the numerical value of the slope of (13) must exceed that of (16) at this point of intersection, which contradicts what we have just proved about the slopes.

THEOREM I. *If x, y, z, n satisfy the equation*

$$x^n + y^n = z^n, \qquad z > y > x > 0, \qquad x, y, z \not\equiv 0 \bmod n, \qquad n > 2, \tag{20}$$

then the parameter c, defined by the relation

$$cn = (z - x)^{1/n} + (z - y)^{1/n} - (x + y)^{1/n} \tag{21}$$

is greater than zero.[1]

Proof. Write (21) in the form

$$(1 - x/z)^{1/n} + (1 - y/z)^{1/n} - (x/z + y/z)^{1/n} = cn/z^{1/n} = c', \tag{22}$$

where $c' \neq 0$ according to our lemma.

As in the proof of the lemma, we make the substitutions (11) and get

$$X^n + Y^n + (X + Y - C')^n - 2 = 0, \tag{23}$$

where c' is obviously greater than $-2^{1/n}$ since X and Y are positive. Now look upon z as fixed. Then if, for a given n, X and Y, c' is negative, the point at which the line $Y = mX$ cuts (23) lies nearer the origin than does the point at which this line cuts (13). For if we substitute $Y = mX$ in (23) we obtain, with c' negative,

$$X^n(1 + m^n) + [X(1 - m) + |c'|]^n - 2 = 0, \qquad |c'|^n < 2, \quad X > 0. \tag{24}$$

Obviously, m and n being fixed, X is a continuous, single valued, decreasing function of $|c'|$, whence X is a maximum when $c' = 0$. Thus (23) separates (13) from the origin, and therefore (23) can not intersect (12) when $c' < 0$.

4. THEOREM II. *If x, y, z, n* satisfy

$$x^n + y^n = z^n, \qquad z > y > x > 0,$$

where n is an odd prime, x, y, z, are integers, relatively prime in pairs and prime to n, then $z - y$ is equal to or greater than $(cn+1)^n$, where

$$c = \frac{(z - y)^{1/n} + (z - x)^{1/n} - (x + y)^{1/n}}{n} \geqq 2.$$

Proof. Returning to (8) we write that relation in the form,

$$(z-y)^{/1/n} = cn - (z - x)^{1/n} + (x + y)^{1/n}. \tag{25}$$

Whence, by Theorem I and (6),

$$\begin{aligned} z - y &> (cn)^n \\ &\geqq (cn + 1)^n \geqq (2n + 1)^n, \end{aligned}$$

since c is even.

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[1] In case $n = 2$ it can be shown, easily, that $c < 0$.

Corollary. Fermat's Equation, (1) *cannot be satisfied by integers* x, y, z, n, *where* x, y, $z \not\equiv 0$ *mod* n, *unless the smallest of* x, y, z *exceeds* $n(2n+1)^n$.

This is a consequence of the above theorem and (7).

Since the first case of Fermat's Last Theorem has been proved for $n \leqq 14{,}000$, this corollary makes it humanly impossible to find an exception to this case of the theorem. A somewhat larger but very much more complicated lower limit can be obtained by combining (6), (3), (2) and this corollary.[1]

5. The Cubic Case. The fact that "c" of (8) is positive leads to a contradiction in case $n=3$. According to Theorem I, we can write

$$(z-x)^{1/3}+(z-y)^{1/3}>(x+y)^{1/3}. \tag{a}$$

Cubing this inequality and collecting terms we get

$$3\left[(z-x)^{2/3}(z-y)^{1/3}+(z-x)^{1/3}(z-y)^{2/3}\right]>2(x+y-z). \tag{b}$$

However by (4)

$$x+y-z=Kn(z-x)^{1/n}(z-y)^{1/n}(x+y)^{1/n}, \tag{c}$$

whence

$$2(x+y-z) \geqq 6(z-x)^{1/3}(z-y)^{1/3}(x+y)^{1/3} \tag{d}$$

$$\geqq 3\left[(z-x)^{2/3}(z-y)^{1/3}+(z-x)^{1/3}(z-y)^{2/3}\right] \tag{e}$$

because of (6). Inequalities (b) and (e) are inconsistent.

ON $N+2$ MUTUALLY ORTHOGONAL HYPERSPHERES IN EUCLIDEAN N-SPACE

By G. E. RAYNOR, Lehigh University

1. *Introduction.* In an interesting paper[2] N. A. Court has developed many properties of the circles, spheres and tetrahedra related to the configuration of five mutually orthogonal spheres.[3] Such a system of spheres is amazingly prolific in interesting relations and is well worth study on its own account. But in addition to its intrinsic interest it is of importance in other connections. It

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[1] There seems to have been very little done toward determining lower limits for x, y, and z. See Legendre's *Théorie des Nombres*, 3rd ed. (1830), vol. II, pp. 361–8. For the cases $n=7$, 11, 13, 17, he proved that x, y, and z must be very large. See also P. R. Heyl, *A Superior Limit to n in Fermat's Equation for a Given Value of z*, The Mathematical Gazette, vol. 11 (1922–23), p. 368.

[2] *On Five Mutually Orthogonal Spheres*, Annals of Mathematics, vol. 30 (1929), p. 613.

[3] See also the paper by Court, *On Four Mutually Orthogonal Circles*, Annals of Mathematics, vol. 29 (1928), p. 369.

forms the basis for the so-called pentaspherical system of coordinates and this in turn constitutes a natural groundwork for the study of the interesting surfaces known as the cyclides.[1]

In this paper we shall derive a number of the relations which seem most fundamental in the study of a system of $n+2$ mutually orthogonal hyperspheres in n-space. We shall use a system of non-homogeneous rectangular coordinates in Euclidean n-space. Some of our theorems will be direct generalizations of Court's, while others are believed to be new. It seems that the subject is quite capable of further development and undoubtedly the reader, if interested, can find many additional results.

2. *Preliminary considerations.* Let an arbitrary point in our space of n dimensions be designated by the coordinates

$$x^{(1)}, x^{(2)}, \cdots, x^{(n)}.$$

Also let $S_1, S_2, \cdots, S_{n+2}$ be $n+2$ proper hyperspheres in this space with centers at the points $P_1, P_2, \cdots, P_{n+2}$ and of radii $r_1, r_2, \cdots, r_{n+2}$. Let the coordinates of P_i be $x_i^{(1)}, x_i^{(2)}, \cdots, x_i^{(n)}$. Then the equation of S_i may be written

$$S_i = (x^{(1)} - x_i^{(1)})^2 + (x^{(2)} - x_i^{(2)})^2 + \cdots + (x^{(n)} - x_i^{(n)})^2 - r_i^2 = 0. \tag{1}$$

For purposes of abbreviation it will be convenient to write this equation in the form

$$S_i = \sum (x - x_i)^2 - r_i^2 = 0, \tag{2}$$

the $\sum$ thus indicating a summation with regard to the suppressed superscripts. Equation (2) may also be written

$$S_i = \sum x^2 - 2\sum x_i x + \sum x_i^2 - r_i^2 = 0. \tag{3}$$

If λ be an arbitrary parameter the equation

$$S_i + \lambda S_j = 0 \tag{4}$$

represents a pencil of hyperspheres passing through the points, real or imaginary, common to S_i and S_j. For $\lambda = -1$ equation (4) reduces to

$$2\sum (x_i - x_j)x - \sum x_i^2 + \sum x_j^2 + r_i^2 - r_j^2 = 0. \tag{5}$$

This is the equation of a hyperplane, the *radical* hyperplane of S_i and S_j.

In what follows we shall have occasion to make use of the idea of the polar hyperplane[2] of a point P_j with respect to a hypersphere S_i. From (3) it follows that the polar hyperplane of P_j with respect to S_i is

$$\sum xx_j - \sum (x + x_j)x_i + \sum x_i^2 - r_i^2 = 0,$$

which by a slight rearrangement of terms may be written

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[1] For an elementary exposition of pentaspherical coordinates and a brief treatment of the cyclides, see Woods, *Higher Geometry*, Chap. XVI.

[2] For a brief exposition of this idea, see Woods, loc. cit., pp. 392, 393.

$$\sum (x_i - x_j)x + \sum x_i x_j - \sum x_i^2 + r_i^2 = 0. \tag{6}$$

3. *Orthogonal hyperspheres.* Let $P(x^{(1)}, x^{(2)}, \cdots, x^{(n)})$ be a point common to S_i and S_j. The direction numbers of the normal to S_i at P are

$$x^{(1)} - x_i^{(1)},\ x^{(2)} - x_i^{(2)},\ \cdots,\ x^{(n)} - x_i^{(n)}$$

and of the normals to S_j

$$x^{(1)} - x_j^{(1)},\ x^{(2)} - x_j^{(2)},\ \cdots,\ x^{n} - x_j^{(n)}.$$

Hence a necessary and sufficient condition that S_i and S_j be orthogonal at P is

$$\sum (x - x_i)(x - x_j) = 0, \tag{7}$$

or expanding

$$\sum x^2 - \sum (x_i + x_j)x + \sum x_i x_j = 0. \tag{8}$$

By (3) the equation of S_j is

$$S_j = \sum x^2 - 2\sum x_j x + \sum x_j^2 - r_j^2 = 0. \tag{9}$$

Subtracting twice (8) from the sum of (3) and (9) we obtain

$$\sum (x_i - x_j)^2 = r_i^2 + r_j^2. \tag{10}$$

Since (7) is also obviously a consequence of (3), (9) and (10) it follows that (10) is also a necessary and sufficient condition that S_i and S_j be orthogonal. It, of course, merely expresses the condition that the square of the distance between the centers of S_i and S_j is equal to the sum of the squares of their radii. Since (10) is independent of the original point P it follows, as is well known, that if two hyperspheres be orthogonal at one of their points of intersection the same is true at all such points. This is of course to be expected by analogy with the situation in two and three dimensions.

For later convenience let us write

$$\sum x_i^2 - r_i^2 = 2c_i. \tag{11}$$

Then (10) may be written

$$\sum x_i x_j = c_i + c_j, \qquad i \neq j. \tag{12}$$

Also the radical hyperplane (5) becomes in this notation

$$\sum (x_i - x_j)x = c_i - c_j. \tag{13}$$

From the $n(n+1)/2$ distinct equations (12) we can see how a system of $n+2$ mutually orthogonal hyperspheres $S_1, S_2, \cdots, S_{n+2}$ may be built up. Let us choose the center $(x_1^{(1)}, x_1^{(2)}, \cdots, x_1^{(n)})$ and the radius r_1 of S_1 arbitrarily. By (11) c_1 will then also be determined. Now for S_2, by (12), we must have

$$\sum x_1 x_2 = c_1 + c_2. \tag{14}$$

If we choose the center $(x_2^{(1)}, x_2^{(2)}, \cdots, x_2^{(n)})$ arbitrarily (14) will determine c_2, and then by (11) r_2 can be found. S_2 will now be fixed.

For S_3 by (12) we must have

$$\begin{aligned} \sum x_1x_3 &= c_1 + c_3 \\ \sum x_2x_3 &= c_2 + c_3. \end{aligned} \tag{15}$$

Subtracting the second of these from the first we have

$$\sum (x_1 - x_2)x_3 = c_1 - c_2. \tag{16}$$

This by comparison with (13) shows that P_3 must lie on the radical hyperplane $S_1 - S_2 = 0$ of S_1 and S_2. Taking then, for the coordinates of P_3, a set of numbers which satisfy (16) we can, from either of (15), determine c_3 and then by (11) r_3. Notice that if (16) and either one of (15) be satisfied by P_3 and c_3 the other one of (15) will also be satisfied.

For S_4 we must have

$$\begin{aligned} \sum x_1x_4 &= c_1 + c_4 \\ \sum x_2x_4 &= c_2 + c_4 \\ \sum x_3x_4 &= c_3 + c_4. \end{aligned} \tag{17}$$

Subtracting the second and third of these equations from the first we have

$$\begin{aligned} \sum (x_1 - x_2)x_4 &= c_1 - c_2 \\ \sum (x_1 - x_3)x_4 &= c_1 - c_3. \end{aligned} \tag{18}$$

These equations show that P_4 lies on the intersection of the radical hyperplanes

$$\begin{aligned} S_1 - S_2 &= 0 \\ S_1 - S_3 &= 0. \end{aligned} \tag{19}$$

Furthermore, the radical hyperplane of S_2 and S_3, namely

$$S_2 - S_3 = 0$$

passes through the intersection of these two hyperplanes since $S_2 - S_3$ is merely a linear combination of equations (19). Now equations (18) will have a finite solution provided *not* every determinant of order two in the matrix

$$\left\| \begin{matrix} x_1^{(1)} - x_2^{(1)} & \cdots & x_1^{(n)} - x_2^{(n)} \\ x_1^{(1)} - x_3^{(1)} & \cdots & x_1^{(n)} - x_3^{(n)} \end{matrix} \right\|$$

vanishes. This condition will be satisfied provided P_3 does not lie on the line joining P_1 and P_2. Hence, provided P_3 has been so chosen, we can determine a finite set of coordinates for P_4 from (18), then c_4 from (7) and finally r_4 from (11).

In general, for S_k we have

$$
(20) \qquad \begin{aligned}
\sum x_1x_k &= c_1 + c_k \\
\sum x_2x_k &= c_2 + c_k \\
&\cdots\cdots \\
\sum x_ix_k &= c_i + c_k \\
&\cdots\cdots \\
\sum x_{k-1}x_k &= c_{k-1} + c_k.
\end{aligned}
$$

Subtracting each of these equations after the first, from the first we get

$$
(21) \qquad \begin{aligned}
\sum (x_1 - x_2)x_k &= c_1 - c_2 \\
&\cdots\cdots \\
\sum (x_1 - x_i)x_k &= c_1 - c_i \\
&\cdots\cdots \\
\sum (x_1 - x_{k-1})x_k &= c_1 - c_{k-1}
\end{aligned}
$$

which show that P_k must lie on the intersection of the hyperplanes

$$
(22) \qquad \begin{aligned}
S_1 - S_2 &= 0 \\
&\cdots\cdots \\
S_1 - S_i &= 0 \\
&\cdots\cdots \\
S_1 - S_{k-1} &= 0.
\end{aligned}
$$

Now these $k-2$ hyperplanes will intersect in a finite $n-k+2$ flat[1] provided not every determinant of order $k-2$ of the matrix

$$
\left\| \begin{matrix}
x_1^{(1)} - x_2^{(1)} & \cdots & x_1^{(n)} - x_2^{(n)} \\
\cdots & \cdots & \cdots \\
x_1^{(1)} - x_i^{(1)} & \cdots & x_1^{(n)} - x_i^{(n)} \\
\cdots & \cdots & \cdots \\
x_1^{(1)} - x_{k-1}^{(1)} & \cdots & x_1^{(n)} - x_{k-1}^{(n)}
\end{matrix} \right\|
$$

vanishes. This condition will be satisfied provided the $k-1$ points $P_1, P_2, \cdots, P_{k-1}$ do not all lie in the same $k-3$ flat. That this proviso can be satisfied is evident from the step by step process by which this set of points has been found. Having chosen P_k we then obtain c_k and r_k as before. We notice incidently that the $(k-1)(k-2)/2$ hyperplanes, obtained by taking all possible pairs of the $k-1$ hyperspheres $S_1, S_2, \cdots, S_{k-1}$, all pass through the $n-k+2$ flat determined by (22), since any equation such as

[1] See Woods, loc. cit., p. 388.

$$S_i - S_j = 0$$

is a linear combination of

$$S_1 - S_i = 0 \text{ and } S_1 - S_j = 0.$$

Finally we see that for $k = n+2$ there will be exactly n equations (21). These will therefore determine the n coordinates of P_{n+2} uniquely and hence from (20) and (11) c_{n+2} and r_{n+2} will also be determined. This process thus terminates at the $(n+2)$th step, S_{n+2} is uniquely determined from the previous $n+1$ hyperspheres, and we see that not more than $n+2$ mutually orthogonal hyperspheres can exist in n-space.

4. *Orthocentric groups of points.* From (12) we have

$$\begin{aligned} \sum x_i x_k &= c_i + c_k \\ \sum x_j x_k &= c_j + c_k. \end{aligned} \tag{23}$$

Hence subtracting we get

$$\sum (x_i - x_j) x_k = c_i - c_j, \tag{24}$$

where k is any of the integers, $1, 2, \cdots, n+2$ except i or j. Now by (12) the polar hyperplane (6) of P_j with respect to the sphere S_i reduces to

$$\sum (x_i - x_j) x = c_i - c_j. \tag{25}$$

Hence by (24) all of the n points P_k $(k \neq i, j)$ lie on this polar hyperplane. Thus it follows that of the $n+1$ points P_j $(j \neq i)$ any one is the pole with respect to the hypersphere S_i of the hyperplane determined by the other n. The figure made up of these $n+1$ points and $n+1$ hyperplanes we will call an $(n+1)$-point. It is a generalization of the triangle and tetrahedron in two and three dimensions respectively. We may state the above result as

THEOREM 1. *If $n+2$ hyperspheres are mutually orthogonal, each is the conjugate sphere of the $(n+1)$-point determined by the centers of the remaining $n+1$ spheres.*

This is a generalization of Court's theorem 2(b).[1]

Writing the equation similar to (24) corresponding to the point P_m we have

$$\sum (x_i - x_j) x_m = c_i - c_j, \qquad m \neq i, j. \tag{26}$$

Subtracting (26) from (24) we obtain

$$\sum (x_i - x_j)(x_k - x_m) = 0, \qquad i \neq k, m; j \neq k, m. \tag{27}$$

Thus

[1] This and all future references to Court apply to the paper cited in our first footnote.

THEOREM 2. *If $n+2$ hyperspheres are mutually orthogonal, the line joining any pair of centers is orthogonal to the line joining any other pair, neither point of one pair being a point of the other pair.*[1]

This is a generalization of Court's theorem 2(a) which says that the centers of five mutually orthogonal spheres form an orthocentric group of points. Now an orthocentric group of points is such that any four of them form a tetrahedron whose altitudes meet in a point, the orthocenter. The figures called $(n+1)$-points above have an analogous property. As we have seen, n vertices of this $(n+1)$-point determine a hyperplane, which we shall designate as *opposite* to the remaining vertex. Also, let us call a line through this vertex and orthogonal to the opposite hyperplane an altitude. Now consider the $(n+1)$-point determined by the points $P_j(j=1, 2, \cdots, n+2$; except $i)$. Designate this $(n+1)$-point by T_i. Equation (25) shows that the line joining P_i to any vertex P_j of T_i is orthogonal to the opposite hyperplane since (25) is precisely the equation of this hyperplane. Hence the $n+1$ altitudes of T_i all pass through P_i, which may therefore be designated the orthocenter of T_i. Thus we have

THEOREM 3. *If $n+2$ hyperspheres are mutually orthogonal their centers form an orthocentric group of points.*

5. *Hypercircles of intersection.* Let us designate the line joining the centers P_i and P_j by the symbol l_{ij}. Then the equations of l_{ij} may be written

$$x^{(p)} = x_i^{(p)} + \lambda(x_i^{(p)} - x_j^{(p)}), \qquad p = 1, 2, \cdots, n. \tag{28}$$

The point P_{ij} where this line meets the radical hyperplane of S_i and S_j is then given by (13) and (28) as follows. Substituting (28) in (13) we have

$$\sum (x_i - x_j)x_i + \lambda \sum (x_i - x_j)^2 = c_i - c_j,$$

or

$$\sum x_i^2 - \sum x_i x_j + \lambda \sum (x_i - x_j)^2 = c_i - c_j,$$

which by (10), (11) and (12) becomes

$$r_i^2 + 2c_i - c_i - c_j + \lambda(r_i^2 + r_j^2) = c_i - c_j$$

from which we get

$$\lambda = -\frac{r_i^2}{r_i^2 + r_j^2}.$$

Substituting this value of λ in (28) we find

$$x_{ij} = \frac{x_i^{(p)} r_j^2 + x_j^{(p)} r_i^2}{r_i^2 + r_j^2}, \qquad p = 1, 2, \cdots, n. \tag{29}$$

Let $(x^{(1)}, x^{(2)}, \cdots, x^{(n)})$ be any point P, real or imaginary, common to S_i and S_j. Then the square of the distance d between P and P_{ij} is given by

[1] The complementary theorem for the case in which the two pairs have a point in common is given below, theorem 5.

$$
\begin{aligned}
d^2 &= \sum (x - x_{ij})^2 \\
&= \sum \left(x - \frac{x_i r_j^2 + x_j r_i^2}{r_i^2 + r_j^2} \right)^2 \\
&= \frac{r_i^4 \sum (x - x_j)^2 + r_j^4 \sum (x - x_i)^2 + 2r_i^2 r_j^2 \sum (x - x_i)(x - x_j)}{(r_i^2 + r_j^2)^2}.
\end{aligned}
$$

By (2) and (7) this reduces to

$$
\begin{aligned}
d^2 &= \frac{r_i^4 r_j^2 + r_j^4 r_i^2}{(r_i^2 + r_j^2)^2} \\
&= \frac{r_i^2 r_j^2}{r_i^2 + r_j^2}.
\end{aligned} \tag{30}
$$

This shows that the points common to S_i and S_j are at a constant distance from P_{ij} and lie on what we shall call a *hypercircle*.[1] Let the radius of this hypercircle be h_{ij}. Then (30) may be written

$$h_{ij}^2 = \frac{r_i^2 r_j^2}{r_i^2 + r_j^2}, \tag{31}$$

and from this we have at once

$$\frac{1}{h_{ij}^2} = \frac{1}{r_i^2} + \frac{1}{r_j^2}. \tag{32}$$

This relation we state as

THEOREM 4. *If two hyperspheres are orthogonal the reciprocal of the square of the radius of their common hypercircle is equal to the sum of the reciprocals of the squares of their radii.*

The cosine of the angle between l_{ij} and l_{ik} is given by the equation

$$\cos (l_{ij}, l_{ik}) = \frac{\sum (x_i - x_j)(x_i - x_k)}{[\sum (x_i - x_j)^2]^{1/2} [\sum (x_i - x_k)^2]^{1/2}}. \tag{33}$$

Now

$$\sum (x_i - x_j)(x_i - x_k) = r_i^2, \tag{34}$$

for the left side of this equation is equal to

$$\sum x_i^2 - \sum x_i x_k - \sum x_j x_i + \sum x_j x_k$$

and this by (11) and (12) is equal to

$$r_i^2 + 2c_i - c_i - c_k - c_j - c_i + c_j + c_k = r_i^2.$$

[1] The use of the word hypercircle is only for convenience in this connection. This hypercircle is, of course, a hypersphere in space of $n-1$ dimensions.

Hence, by (10) and (34), (33) becomes

$$\cos (l_{ij}, l_{ik}) = \frac{r_i^2}{(r_i^2 + r_j^2)^{1/2}(r_i^2 + r_k^2)^{1/2}}. \tag{35}$$

With the help of (31) this may be put into another form. If r_i and r_j are real we may write

$$h_{ij} = \frac{r_i r_j}{(r_i^2 + r_j^2)^{1/2}}. \tag{36}$$

However, we shall see later that one of the radii $r_1, r_2, \cdots, r_{n+2}$ must be a pure imaginary, so that (36) needs slightly more justification. Since all the centers P_i have been chosen to be real it follows from (34) that the square of any radius, such as r_i^2, is real. Also from (10) it follows that a sum such as $r_i^2+r_j^2$ must be positive, and hence that not more than one square such as r_i^2 can be negative. Thus in (31) the denominator is positive while the numerator may be either positive or negative. If positive, we may write (36) at once since then h_{ij} represents a real distance and may be taken as positive. If the numerator $r_i^2 r_j^2$ is negative either r_i^2 or r_j^2 is negative. Let

$$r_i^2 = -R_i^2, \qquad R_i > 0.$$

Then we define r_i by the equation

$$r_i = iR_i, \qquad i = \sqrt{-1}.$$

In this case we take

$$h_{ij} = \frac{iR_i r_j}{(r_i^2 + r_j^2)^{1/2}}$$

and h_{ij} will then be a pure imaginary of the form ai where $a>0$. Thus in either case we take h_{ij} as given by (36).

We have also

$$h_{ik} = \frac{r_i r_k}{(r_i^2 + r_k^2)^{1/2}}.$$

Hence

$$h_{ij}h_{ik} = \frac{r_i^2 r_j r_k}{(r_i^2 + r_j^2)^{1/2}(r_i^2 + r_k^2)^{1/2}}$$

and this combined with (35) gives

$$\cos (l_{ij}, l_{ik}) = \frac{h_{ij}h_{ik}}{r_j r_k}.$$

Thus we have

THEOREM 5. *If three hyperspheres are mutually orthogonal the cosine of the angle between the lines joining the centers of two of them to the center of the third is equal to the product of the radii of the hypercircles in which the first two intersect the third divided by the product of the radii of the first two.*

6. *On the measure of the $(n+1)$-point associated with S_i.* Let us define the symbol V_i by the equation

$$V_i = \frac{1}{n!}\begin{vmatrix} x_{i+1}^{(1)} & x_{i+1}^{(2)} & \cdots & x_{i+1}^{(n)} & 1 \\ x_{i+2}^{(1)} & x_{i+2}^{(2)} & \cdots & x_{i+2}^{(n)} & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ x_{i+n+1}^{(1)} & x_{i+n+1}^{(2)} & \cdots & x_{i+n+1}^{(n)} & 1 \end{vmatrix} \tag{37}$$

where any subscript $i+j$ is to be reduced modulo $n+2$ if $i+j>n+2$. Then by analogy with two and three dimensions we say that the absolute value of V_i is the *measure* of the $(n+1)$-point determined by the $n+1$ centers, P_{i+1}, P_{i+2}, $\cdots$, P_{i+n+1}.

We shall call the $(n+1)$-point determined by these points the associated central $(n+1)$-point of the hypersphere S_i. We now seek to express the product V_iV_j in terms of the r_i. Let us suppose at first that $i\neq j$ and that

$$j = i + k. \tag{38}$$

Now in the determinant for V_i the element $x_j^{(p)}$ will occur in the kth row. Subtract the elements of this row from each of the corresponding elements of the other rows and then expand by minors with respect to the last column. We then have for V_i the following expression in which the matrix represents a determinant of order n whose pth column is the one written.

$$V_i = \frac{(-1)^{k+n+1}}{n!}\left\|\begin{matrix} x_{i+1}^{(p)} - x_j^{(p)} \\ x_{i+2}^{(p)} - x_j^{(p)} \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ x_{j-1}^{(p)} - x_j^{(p)} \\ x_{j+1}^{(p)} - x_j^{(p)} \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ x_{i+n+1}^{(p)} - x_j^{(p)} \end{matrix}\right\| \qquad p = 1, 2, \cdots, n. \tag{39}$$

Similarly in the determinant for V_j the element $x_i^{(p)}$ will occur in the mth row, where, since $j>i$, $j+m=i+n+2$. Hence, by (38)

$$m = n + 2 - k. \tag{40}$$

Now treating this determinant in a manner similar to that of the last paragraph we obtain

$$(41)\qquad V_j = \frac{(-1)^{m+n+1}}{n!}\left\|\begin{matrix} x_{j+1}^{(p)} - x_i^{(p)} \\ x_{j+2}^{(p)} - x_i^{(p)} \\ \cdots\cdots \\ x_{i-1}^{(p)} - x_i^{(p)} \\ x_{i+1}^{(p)} - x_i^{(p)} \\ \cdots\cdots \\ x_{j+n+1}^{(p)} - x_i^{(p)} \end{matrix}\right\| \qquad p = 1, 2, \cdots, n.$$

We now form the product of V_i and V_j and in multiplying the determinants combine rows with rows, the pth column of the product being formed by combining the pth row in V_i with the rows of V_j. Keeping in mind equations (27), (34) and (40) we find

$$(42)\qquad V_iV_j = \frac{(-1)^n}{(n!)^2}\left|\begin{array}{cccc|cccc} 0 & 0 & \cdots & 0 & r_{i+1}^2 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & r_{i+2}^2 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & r_{j+n+1}^2 \\ \hline r_{j+1}^2 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & r_{j+2}^2 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdots & r_{i+n+1}^2 & 0 & 0 & \cdots & 0 \end{array}\right|.$$

The block of zeros in the upper left hand corner of this determinant consists of $k-1$ rows and columns. If we now expand this determinant remembering that it is of the nth order we obtain, after making some obvious simplifications of the exponent of (-1)

$$(43)\qquad V_iV_j = \frac{(-1)^{k(n-k)+1}}{(n!)^2} r_{j+1}^2 r_{j+2}^2 \cdots r_{i+n+1}^2 r_{i+1}^2 \cdots r_{j+n}^2 r_{j+n+1}^2.$$

Of the $n+2$ radii r_h each occurs in this product except r_i and r_j. Let the symbol P be defined by the equation

$$(44)\qquad P = -\frac{1}{(n!)^2}\prod_{h=1}^{n+2} r_h^2.$$

Then (34) may be written

$$(45)\qquad V_iV_j = (-1)^{k(n-k)}\frac{P}{r_i^2 r_j^2}.$$

To obtain V_i^2 we may proceed as follows. From (45) we have

$$V_iV_{i+1} = (-1)^{n-1}\frac{P}{r_i^2r_{i+1}^2}$$

and

$$V_{i-1}V_i = (-1)^{n-1}\frac{P}{r_{i-1}^2r_i^2}\cdot$$

Multiplying these two equations we have

$$V_{i-1}V_{i+1}V_i^2 = \frac{P^2}{r_{i-1}^2r_{i+1}^2r_i^4}\cdot \tag{46}$$

But by (45)

$$V_{i-1}V_{i+1} = \frac{P}{r_{i-1}^2r_{i+1}^2}\cdot$$

Setting this in (46) and simplifying we have

$$V_i^2 = \frac{P}{r_i^4}\cdot \tag{47}$$

We may now combine (45) and (47) as

THEOREM 6. *The product V_iV_j, i and j equal or not, may be obtained by dividing $(-1)^{k(n-k)}P$ by the product $r_i^2r_j^2$.*

It may be of interest to note that the factor $(-1)^{k(n-k)}$ may be written, in view of (38) and a slight additional modification, in the form $(-1)^{(i+j)(n-i-j)}$. Also, if n be odd, $k(n-k)$ will be even whether k be odd or even, and hence for odd spaces the factor $(-1)^{k(n-k)}$ may be dropped from all equations in which it appears.

If we write (47) in the form

$$r_i^4V_i^2 = P \tag{48}$$

we have

THEOREM 7. *Given $n+2$ mutually orthogonal hyperspheres, the product of the fourth power of the radius of any one of them by the square of the measure of its associated central $(n+1)$-point is constant.*

A result slightly more general than this may be obtained as follows. By multiplying both sides of (45) by V_j we get

$$V_iV_j^2 = (-1)^{k(n-k)}\frac{P}{r_i^2r_j^2}V_j. \tag{49}$$

But by (47)

$$V_j^2 = \frac{P}{r_j^4}.$$

If we substitute this in (49), multiply both sides of that equation by $i(n-i)$ and make use of the relation $j=k+i$ we obtain

$$(- 1)^{i(n-i)} r_i^2 V_i = (- 1)^{j(n-j)} r_j^2 V_j. \tag{50}$$

We thus have

THEOREM 8. *The value of the product* $(-1)^{i(n-i)} r_i^2 V_i$ *is independent of the index* i.

As remarked in another connection, if n be *odd* the exponent $i(n-i)$ is even whether i be odd or even and hence powers of (-1) in (50) may be dropped.

If the $n+2$ equations obtained from (48) by letting i take on all its values from 1 to $n+2$ be multiplied together we obtain, by making use of (44), after a slight reduction, the formula

$$\prod_{i=1}^{n+2} \frac{r_i^{2n}}{V_i^2} = (- 1)^{n+2}(n!)^{2(n+2)}.$$

This is a generalization suggested in a remark by Court.[1]

7. *Further relations among the radii* r_i *and* h_{ij}. Let us now return to equation (37). In the determinant subtract the last row from each of the other rows and then expand by minors with respect to the last column. Next multiply V_i by itself, and in multiplying the determinants combine rows with rows. Then making use of (10) and (34) we obtain

$$V_i^2 = \frac{1}{(n!)^2} \begin{vmatrix} r_{i+1}^2 + r_{i+n+1}^2 & r_{i+n+1}^2 & \cdots & r_{i+n+1}^2 \\ r_{i+n+1}^2 & r_{i+2}^2 + r_{i+n+1}^2 & \cdots & r_{i+n+1}^2 \\ \cdot & \cdot & \cdot & \cdot \\ r_{i+n+1}^2 & r_{i+n+1}^2 & \cdots & r_{i+n}^2 + r_{i+n+1}^2 \end{vmatrix}. \tag{51}$$

In this determinant, which is of the nth order, every element except those in the principal diagonal is r_{i+n+1}^2 while the jth element in the principal diagonal is $r_{i+j}^2 + r_{i+n+1}^2$. Now let us subtract the last column from each of the other columns. Then remove from the jth row $(j=1, 2, \cdots, n)$ the factor r_{i+j}^2 and from the last column the factor r_{i+n+1}^2. Then add the first $n-1$ rows to the last row. Every element in the last row will now be zero except the nth one which will be

$$\sum_{j=1}^{n+1} \frac{1}{r_{i+j}^2}.$$

We thus obtain

[1] Loc. cit. p. 617.

$$V_i^2 = \frac{1}{(n!)^2}\prod_{j=1}^{n+1} r_{i+j}^2 \sum_{j=1}^{n+1} \frac{1}{r_{i+j}^2}. \tag{52}$$

Now it is evident that by a slight change in notation equation (44) may be written

$$P = -\frac{1}{(n!)^2}\prod_{j=0}^{n+1} r_{i+j}^2. \tag{44'}$$

Equation (47) will then become

$$V_i^2 = -\frac{1}{(n!)^2}\frac{1}{r_i^2}\prod_{j=1}^{n+1} r_{i+j}^2. \tag{47'}$$

Combining this with (52) we get

$$\frac{1}{(n!)^2}\prod_{j=1}^{n+1} r_{i+j}^2 \sum_{j=0}^{n+1} \frac{1}{r_{i+j}^2} = 0,$$

or since none of the r_{i+j} is zero

$$\sum_{j=0}^{n+1} \frac{1}{r_{i+j}^2} = 0.$$

But again by a change of notation, this is equivalent to

$$\sum_{i=1}^{n+2} \frac{1}{r_i^2} = 0. \tag{53}$$

We state this result as

THEOREM 9. *If $n+2$ hyperspheres are mutually orthogonal the sum of the reciprocals of the squares of their radii is zero.*[1]

For an application of (53) let us return to equation (32) which gives a relation between the r_i and the radii of the $(n+2)(n+1)/2$ hypercircles in which our $n+2$ hyperplanes taken in pairs intersect. If we add the $(n+2)(n+1)/2$ distinct equations (32) we have

$$\frac{1}{2}\sum_{i,j=1}^{n+2} \frac{1}{h_{ij}^2} = (n+1)\sum_{i=1}^{n+2} \frac{1}{r_i^2} \tag{54}$$

where on the left $h_{ij}=h_{ji}$ and in no term is $i=j$. By (53) the right side of (54) is zero and hence we have the following generalization of one of Court's theorems.[2]

[1] Since all the r_i^2 are real, equation (53) gives the result anticipated in 5, that one of the r_i must be a pure imaginary.

[2] Loc. cit. p. 615, §7 part (b).

THEOREM 10. *The sum of the reciprocals of the squares of the radii of the* $(n+2)(n+1)/2$ *hypercircles in which* $n+2$ *mutually orthogonal hyperspheres cut each other is zero.*

We may, however, derive a theorem considerably stronger than this. Let us arrange our $n+2$ hyperspheres in a circular permutation, say

$$S_i,\ S_{i+1},\ S_{i+2},\ \cdots,\ S_i. \tag{55}$$

Now consider the $n+2$ hypercircles obtained by taking the intersections of adjacent pairs in this permutation. If we add the $n+2$ equations (32) corresponding to this set of hypercircles we see that each r_i will occur exactly twice and the sum on the right will be

$$2\sum_{i=1}^{n+2}\frac{1}{r_i^2},$$

which by (53) is zero. Now the two senses in which the circular permutation (55) may be taken lead to the same set of hypercircles, and hence we are led by this process to $(n+1)!/2$ distinct sets of hypercircles. Thus we have

THEOREM 11. *If* $n+2$ *hyperspheres are mutually orthogonal, then from the* $(n+2)(n+1)/2$ *hypercircles in which they intersect there may be selected* $(n+1)!/2$ *sets of* $n+2$ *hypercircles each, such that the sum of the reciprocals of the squares of the radii of the* $n+2$ *hypercircles in each set is zero.*

Other theorems of a similar nature may be derived but we shall not pursue the subject further.

8. *The one dimensional case.* In closing it may be of interest to note that many of the above theorems can be interpreted for the case $n=1$. Our hyperspheres now become point pairs whose radii are half the lengths of the segments determined by these points, and whose centers are the mid-points of those segments. Also, from the orthogonality condition (10) it will be found that two orthogonal point pairs form a harmonic set of points, the two points of one pair being harmonic conjugates with respect to the other pair.

QUESTIONS, DISCUSSIONS, AND NOTES

EDITED BY R. E. GILMAN, Brown University, Providence, Rhode Island

The department of Questions and Discussions in the Monthly is open to all forms of activity in collegiate mathematics, including the teaching of mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.

A TRIGONOMETRIC INTERPOLATION FORMULA

By PAUL HERGET, Cincinnati Observatory

From the simple properties of the sine and cosine we are able to derive an interesting equation which should prove very useful in the construction of

tables of these functions. Suppose $\sin x$ is tabulated for equal intervals of the argument. Then the second difference opposite a particular $\sin x_0$ is

$$\sin(x_0 + h) + \sin(x_0 - h) - 2\sin x_0 = A \sin x_0$$

where h is the interval of the argument, and $A = -2\,(1-\cos h)$. Similarly the fourth difference is $A^2 \sin x_0$, etc.

Now using Everett's interpolation formula, we have

$$\sin x = m\left[1 + \frac{(m^2 - 1^2)A}{6} + \frac{(m^2 - 1^2)(m^2 - 2^2)A^2}{120} + \cdots\right]\sin x_0$$
$$+ n\left[1 + \frac{(n^2 - 1^2)A}{6} + \frac{(m^2 - 1^2)(m^2 - 2^2)A^2}{120} + \cdots\right]\sin(x_0 + h),$$

where $n = (x - x_0)/h$, $m = 1 - n$. The ratio test shows that the series in brackets converges, except for $A = -4$. These same relations are true for the cosine.

Digressing for a moment, if we set $x_0 = 0$, we have the following general formula for the sine of any fractional part of an angle:

$$\sin nX = n\left[1 + \frac{(n^2 - 1^2)A}{6} + \frac{(n^2 - 1^2)(n^2 - 2^2)A^2}{120} + \cdots\right]\sin X,$$

where $A = -2(1 - \cos x)$, and n is less than unity.

In the process of constructing a complete table by interpolating between the values of the function which are known at equal, but larger, intervals of the argument, the coefficients remain unchanged from one interval to the next for the same value of n, and may be used over and over. Furthermore, the interpolated value of the function is obtained directly by the sum of two multiplications, and there is no cumulative error in the process, as there is in schemes of numerical integration.

To illustrate, suppose we wish to construct a table in which the sine and cosine are given for every hundredth of a radian, and we know the values of the functions for every tenth of a radian. Then $h = .1$, $A = -.009\,991\,669$, and $n = .1$, $.2, \cdots, .9$ successively. The values of the coefficients of $\sin x_0$ and $\sin(x_0 + h)$ are, by pairs:

.900 285 217	.100 165 192
.800 480 407	.200 320 367
.700 595 549	.300 455 511
.600 640 632	.400 560 609
.500 625 652	.500 625 652

To illustrate the interpolation in several intervals we take, for example,

$x_0 = .6$	$\sin .6 = .564\ 642\ 473$	$\cos .6 = .825\ 335\ 615$
$x_0 + h = .7$	$\sin .7 = .644\ 217\ 687$	$\cos .7 = .764\ 842\ 187$
$x_0 + 2h = .8$	$\sin .8 = .717\ 356\ 091$	$\cos .8 = .696\ 706\ 709$

Then, for example,

$$\sin .61 = .900\,285\,217 \sin .6 + .100\,165\,192 \sin .7 = .5728\,6746$$

$$\cos .68 = .200\,320\,367 \cos .6 + .800\,480\,407 \cos .7 = .7775\,7272$$

$$\sin .73 = .700\,595\,549 \sin .7 + .300\,455\,511 \sin .8 = .6668\,6963$$

The complete computation for these two intervals gives:

x	$\sin x$	$\cos x$	x	$\sin x$	$\cos x$
.60	.5646 4247	.8253 3561	.70	.6442 1769	.7648 4219
.61	.5728 6746	.8196 4802	.71	.6518 3377	.7583 6188
.62	.5810 3516	.8138 7846	.72	.6593 8467	.7518 0573
.63	.5891 4476	.8080 2751	.73	.6668 6963	.7451 7440
.64	.5971 9544	.8020 9576	.74	.6742 8791	.7384 6856
.65	.6051 8641	.7960 8380	.75	.6816 3876	.7316 8887
.66	.6131 1685	.7899 9223	.76	.6889 2144	.7248 3601
.67	.6209 8599	.7838 2167	.77	.6961 3524	.7179 1067
.68	.6287 9302	.7775 7272	.78	.7032 7942	.7109 1354
.69	.6365 3718	.7712 4601	.79	.7103 5327	.7038 4532

A Generalization of the Orthopole Theorem

By R. Goormaghtigh, Bruges, Belgium

Theorem. Let T be one of the in- or excenters of a triangle $B_1B_2B_3$; $C_1C_2C_3$ a triangle having its vertices on TB_1, TB_2, TB_3; $A_1A_2A_3$ the medial triangle of $B_1B_2B_3$; D_1, D_2, D_3 the projections of A_1, A_2, A_3 on B_2B_3, B_3B_1, B_1B_2 respectively: then the perpendiculars drawn from D_1, D_2, D_3 on A_2A_3, A_3A_1, A_1A_2 are concurrent.

Proof. Let Γ_1, Γ_2, Γ_3 be the circles having C_1, C_2, C_3 as centers and tangent respectively to B_1B_3 and B_1B_2, B_2B_1 and B_2B_3, B_3B_2 and B_3B_1, the contact points being F_{21} and F_{31}, F_{32} and F_{12}; F_{13} and F_{23}.

Since A_1, A_2, A_3 are the mid-points of C_2C_3, C_3C_1, C_1C_2, the points D_1, D_2, D_3 are the mid-points of $F_{21}F_{31}$, $F_{32}F_{12}$, $F_{13}F_{23}$ and therefore belong to the radical axes of Γ_2 and Γ_3, Γ_3 and Γ_1, and Γ_1 and Γ_2.

Hence the perpendiculars drawn from D_1, D_2, D_3 on A_2A_3, A_3A_1, A_1A_2 are the radical axes of Γ_2 and Γ_3, Γ_3 and Γ_1, Γ_1 and Γ_2, and are therefore concurrent at a point P, radical center of Γ_1, Γ_2, Γ_3.

The Orthopole-Theorem. The preceding theorem contains as a special case the orthopole-theorem, which will be obtained when B_1, B_2, B_3 are on a straight line Δ and T is the point at infinity in the direction perpendicular to Δ:

If D_1, D_2, D_3 are the projections of the vertices of a triangle $A_1A_2A_3$ on a straight line Δ, the perpendiculars drawn from D_1, D_2, D_3 on A_2A_3, A_3A_1, A_1A_2 are concurrent at a point P.

The point P is the orthopole of Δ for the triangle $A_1A_2A_3$.

Analogue in Space. Analogous theorems will be easily obtained in the case of the tetrahedron; for instance:

If the vertices of a tetrahedron $A_1A_2A_3A_4$ are projected at D_1, D_2, D_3, D_4 on a plane, the perpendiculars drawn from the centers of the circles $D_2D_3D_4$, $D_3D_4D_1$, $D_4D_1D_2$, $D_1D_2D_3$ on the planes $A_2A_3A_4$, $A_3A_4A_1$, $A_4A_1A_2$, $A_1A_2A_3$ are concurrent.

Conic Sections from Whose Equations the XY-Term May be Eliminated by a Rotation of Axes Involving no Surd Numbers

By D. C. Duncan, University of California

The removal of the xy-term from the equation of the real conic, $Ax^2+Bxy+Cy^2+Dx+Ex+F=0$, by rotating the axes through an angle θ determined by the usual relation, $\tan 2\theta=B/(A-C)$, in general involves "two-story" radicals in the formulas of transformation. In introducing the process to the student, specially devised equations are invariably used which do not involve additional algebraic difficulties in irrationalities. The standard procedure of finding 2θ by the relation $\tan 2\theta=B/(A-C)$, then $\cos 2\theta$, from which $\sin\theta$ and $\cos\theta$ are available by $\sqrt{(1-\cos 2\theta)/2}$ and $\sqrt{(1+\cos 2\theta)/2}$, when applied, for example, to $12x^2+24xy+5y^2+Dx+Ey+F=0$, leads most agreeably to $\sin\theta=3/5$ and $\cos\theta=4/5$. In fact, the illustrative examples in texts usually involve the ratio $\pm 24/7$ for $B/(A-C)$ for the very reason that no surds are involved in the reductions. In this note I wish to call attention to the comparatively rare occurrence of absolute values of the ratio $B/(A-C)$ which involve only rational operations in the subsequent reduction, and to indicate a means of finding all values of the ratio $B/(A-C)$ from which these desirable equations may be formed "for purposes of illustration."

Suppose that $\sin\theta$ and $\cos\theta$ are to be rational; then they must be of the form $(m^2-n^2)/(m^2+n^2)$ and $2mn/(m^2+n^2)$, or vice versa. We then have

$$2mn/(m^2+n^2)=\sqrt{(1+\cos 2\theta)/2},\qquad (m^2-n^2)/(m^2+n^2)=\sqrt{(1-\cos 2\theta)/2},$$

from which one obtains

$$\cos 2\theta=[8m^2n^2-(m^2+n^2)^2]/(m^2+n^2)^2,$$

and hence

$$\tan 2\theta=4mn(m^2-n^2)/[8m^2n^2-(m^2+n^2)^2]=\pm B/(A-C),$$

a necessary condition that no surds appear in the calculations. This condition is readily noted to be also sufficient in that these steps are immediately reversible and lead back to $\sin\theta$ and $\cos\theta$.

Accordingly one may build special equations at pleasure whose reduction to standard forms by rotation and translation involve only rational numbers[1] by choosing B to be of the form $\pm 4mn(m^2-n^2)$ and $A-C$ of the form $\pm[8m^2n^2-(m^2+n^2)^2]$. There are only 7 absolute values of these ratios in which the

[1] All the different ratios will be obtained without duplication by letting m and n range over all pairs of positive relatively prime odd integers. (Editor.)

members are less than 2,000, namely, 24/7, 120/119, 240/161, 336/527, 840/41, 840/1081, and 720/1519.

RECENT PUBLICATIONS

EDITED BY R. A. JOHNSON, Brooklyn College of the City of New York

All books for review should be sent directly to the editor of this department, at Brooklyn College, 66 Court Street, Brooklyn, N. Y., and not to any of the other editors or officers of the Association.

REVIEWS

Non-Euclidean Geometry and *Galois and the Theory of Groups.* Text by Lillian R. Lieber, drawings by H. G. Lieber. Lancaster, Pennsylvania, The Science Press, 1932 and 1933. 34 pages and 60 pages respectively.

The purpose of these little books is to furnish an easy introduction to the subjects treated in them. Each of them contains a fairly detailed account of a small part of the subject, made palatable to the beginner. To expedite reading they are written out with separate phrases on separate lines; this gives somewhat the appearance of free verse although that is not the authors' intention. There are illustrations, but these are only vaguely connected with the text and are fantastic pictures[1] of the goings-on in a mysterious land called "Mathesis" whose inhabitants take great interest in mathematical matters. At the end of each book is a list of "morals" based upon the subject matter; some of these are somewhat reminiscent of the Duchess in *Alice in Wonderland* (without being as funny).

The first booklet begins with a brief discussion of the assumption leading to the three classical geometries: parabolic (Euclidean), elliptic, hyperbolic. The author associates the second with Riemann, thereby helping to perpetuate a common error. Then follows a good discussion of elementary hyperbolic geometry, thought of as the geometry on a "pseudo-sphere." The outlines of the proofs are given so that the student gets a good idea of how things are done. By contrast some statements are made about geometry on a sphere; the author does not mention the distinction between this and elliptic non-euclidean geometry. Then comes a short discussion of postulate systems and finally the "morals" mentioned above.

The second book is more ambitious. First there is an explanation of the important ideas from group theory and then a description, by means of an example, of the construction of the Galois group of an equation. Unfortunately, the author fails to emphasize the fundamental character of the theorem on p. 31, this being the theorem upon which the real significance of the Galois group depends. We then find the Galois criterion for solvability by radicals and a brief application to the general equation with arbitrary coefficients. Finally, there is an applica-

[1] The illustrations, indeed, seem to show clearly the influence of H. W. van Loon. (EDITOR.)

tion to the problems of construction by ruler and compasses. Into this a vicious circle has crept (on p. 47) which can be avoided only by a clearer statement of the Galois criterion (or rather of the special case of it that is needed here).

The reviewer found these little books interesting but doubts whether they give the beginner a proper introduction to the subject. Their main fault lies in wrong emphasis, which is likely to give the beginner a distorted picture. The first is on the whole the better one; its point of view is somewhat restricted but that can perhaps not be helped in so small a book. In the second book the reader will frequently miss the point entirely because the author has failed to point out the fundamental significance of the things she is doing. It must be admitted, of course, that the properties of groups are rather diverse and the meaning and properties of the Galois group of an equation are somewhat subtle. So it may very well be that this book also represents the best that can be done in so few pages.

H. W. BRINKMANN

An Introduction to Statistical Analysis. By C. H. Richardson. New York, Harcourt, Brace, and Company, 1934. xi+285 pages. $3.00.

The scope of this book is indicated by the title; as the author says, it is not an exhaustive treatise, but an introduction containing the minimum essentials. The content divides naturally into three nearly equal parts. The first contains introductory material and a discussion of the various statistical constants. The second is concerned with relations between two variables—trend lines, simple correlation, and curve fitting. The last deals with probability, the point binomial and the normal curve, and an introduction to the theory of sampling. No mathematics beyond the ordinary secondary school course in algebra is presupposed.

The effect of such a limitation is too often that the student acquires a certain facility in substituting numbers in formulae and in operating a computing machine, but gets little knowledge of the real meaning of these formulae and little insight into the value of his results. Fortunately the author seems to be well aware of this danger and endeavors to combat it by proving his formulae and carefully explaining their limitations. An evident enthusiasm for the subject and an excellent style combine to make the book an interesting one, there is a good collection of problems, and the publishers are to be complimented on the appearance of the book.

Certain weaknesses seem inevitable on account of the small amount of mathematical preparation assumed. The reviewer cannot escape a feeling that students with this minimum preparation will find the algebraic work heavy going at times, not from lack of information but from lack of training. On the other hand, the better prepared student is apt to be slowed up by diffuse explanations intended for the less proficient. On the whole, however, this book is to be commended to the attention of those conducting elementary courses in mathematical statistics.

W. A. WILSON

MATHEMATICS CLUBS

EDITED BY F. M. WEIDA, The George Washington University, Washington, D.C.

All reports of club activities, suggestions and topics for club programs, and material of interest to clubs should be sent to F. M. Weida, The George Washington University, Washington, D.C. All manuscript should be typewritten, with double spacing, and with margins at least one inch wide.

CLUB ACTIVITIES

1933–1934

THE PI MU EPSILON MATHEMATICAL FRATERNITY

Pi Mu Epsilon is an academic fraternity in institutions of university grade. Its primary aim is the advancement of mathematics and scholarship. It is a living, active, working fraternity of scholars in which the members are actively engaged in study and research and the exchange of ideas in the field of mathematical science.

Pi Mu Epsilon of Duke University

Greetings and best wishes to all chapters from the Duke Alpha chapter of North Carolina of Pi Mu Epsilon. We are very glad to be able to report a most successful year under the direction of the following officers who were elected in May 1933: James V. Bernado, Director; Blaine Harkness, Vice Director; John C. Lennox, Secretary; Eleanor Markham, Treasurer; and Lloyd P. Julian, Librarian.

All officers are elected at the next-to-the-last meeting of each school year. The officers are nominated by a faculty committee and are elected by popular vote of the active members of the club.

Membership includes faculty members of the department of mathematics, graduate and undergraduate students of Duke University who have excelled in mathematics. Initiation to the organization is held twice annually, once in January and again in April. The membership for this year was forty-five.

The club held regular meetings on the first Wednesday of each month. The February meeting was turned over to the annual banquet; the May meeting, to the annual picnic. The prize given to the undergraduate presenting the best talk during monthly meetings was awarded to Eugene Grabbe.

The meetings and programs were as follows:

November 9, 1933: "Gyroscopic principles and their applications" by Dr. J. C. Mouzon, Professor of Physics.

December 6, 1933: "Definition" by Dr. Gilbert, Department of Philosophy. At this meeting twelve new members were initiated.

January 10, 1934: "Evolutes and involutes and an application to the cycloidal pendulum" by James V. Bernado.

February 2, 1934: Dr. Arnold Dresden, President of The Mathematical Association of America, gave a lecture in the afternoon on "General aspects of the calculus of variations." In the evening the annual banquet was held. The evening program was as follows: A short address by J. V. Bernado; "Curriculum problems" by Dr. Helen Barton of the North Carolina University for Women; A short talk by Eugene Grabbe. The assembly then adjourned to a lecture room where Dr. Arnold Dresden talked on "Continuity and convergence."

March 7, 1934: "Theory of errors and methods of least squares" by J. C. Lennox. At this meeting seven new members were initiated.

April 4, 1934: "Fractional integration" by Eugene Grabbe. Election of officers for 1934–1935.

May 2, 1934: Annual Picnic at Crystal Lake.

J. C. LENNOX, *Secretary*

LOCAL MATHEMATICS CLUBS

The Mathematics Club of Brown University

The staff of our mathematics club consisted of the following two committees: *Committee on Program*: Professor A. A. Bennett and Professor C. O. Oakley, faculty representatives; Robert Devereux Eddy, '35; Rhoda Madden, '35; Richard Hart Morse, '34; Martha Mildred Wicks, '36. *Committee on Arrangements*: Mr. D. J. Colbert, faculty representative; Shirley Marie Olive Battey, '36; Henry Brainard Fancher, '35; Evelyn Heath Lawrence, '34.

The meetings and programs were as follows:

November 13, 1933: "History of the parallel postulate" by Rhoda Madden, '35; "Elliptic geometry on a sphere" by Richard Hart Morse, '34; "A spherical image of hyperbolic space" by Robert Devereux Eddy, '35.

December 11, 1933: "Binomial coefficients" by Eleanor Thornton Ide, '34; "Factorials and some algebraic extensions" by Chester Hall Page, '34; "Stirling's formula and the gamma function" by Arthur Peck Young, '34.

January 15, 1934: "Bessel" by Esther Lillian Holmes, '34; "Dedekind" by Lela Mae Kirkbride, '34; "Monge" by Evelyn Heath Lawrence, '34.

February 26, 1934: "Transfinite ordinals" by Walter Francis Wong King, '34; "Nonenumerable sets" by Elizabeth Rita Blanchard, '35; "Space-filling curves" by Maurice Eugene Marks, '34.

March 19, 1934: "The integrallogarithm" by Charles Robert Wilks, '34; "Logarithmic tables and slide rules" by Nicholas Voci, '34.

April 16, 1934: "Recent slide rule conquests" by William Fitch Cheney, Jr., Professor of Mathematics, Connecticut State College.

A. A. BENNETT, *Member, Program Committee*

The Mathematics Club of the University of Colorado

The Mathematics Club of the University of Colorado had an average attendance of about twenty members. Meetings were held every second week in the Student Memorial Building. Any one who is interested in mathematics is eligible to become a member. Dr. Aubrey John Kempner, one of the Editors of THE AMERICAN MATHEMATICAL MONTHLY, is the faculty sponsor.

The officers of the club for the academic year 1933–1934 were: Bryson Reinhardt, President; Patricia Tobin, Vice President; Virginia Sink, Secretary: and Hans Hansen, Treasurer.

The meetings and programs were as follows:

October 12, 1933: "Mathematical oddities" by Dr. Glen Wakeham; election of officers.

October 26, 1933: "Mathematical puzzles" by Mr. Jack Britton, Instructor of Engineering Mathematics.

November 9, 1933: "Some of the problems asked the Tau Beta Pledges" by Mr. Clifford Sniveley, Instructor of Engineering Mathematics; "The origin of analytic geometry" by Patricia Tobin.

November 23, 1933: "The differential equation and the discriminant" by Professor G. H. Light.

January 11, 1934: "Square roots" by Bryson Reinhardt; "Some remarks on the theory of relativity" by C. S. Merril.

January 25, 1934: "Illusions and hallucinations" by Mr. Wallace Swan.

February 8, 1934: "The trisection of an angle" by C. S. Merril; "Regular polygons" by Dr. A. J. Kempner.

March 1, 1934: "Geometry of paper folding" by Dr. A. J. Kempner.

VIRGINIA SINK, *Secretary*

The Mathematics Club of Lafayette College

Our club, the William S. Hall Mathematics Club, is a student organization which was started at the beginning of the academic year 1933–1934. The purpose of the club is to consider mathematical problems not generally discussed in regular courses and to promote interest generally. Membership is open to those students who have evidenced an interest in mathematics and who welcome this opportunity to further their knowledge of the subject.

The club was named "The William S. Hall Mathematics Club" in honor of Dr. William S. Hall, who is retiring this year from active work after completing fifty years in the department.

We had a total membership of forty-five of which seven were members of the faculty.

The officers for 1933–1934 were: Leon T. Johnson, President; Donald MacDougall, Secretary; Dr. W. M. Smith, Faculty Adviser; Professor John Cawley and Jack F. Sassaman, Program Committee.

At the April meeting, the club had the good fortune in being able to secure Professor James A. Shohat of the University of Pennsylvania as a guest speaker.

The meetings and programs were as follows:

October 19, 1933: "Non-Euclidean geometries" by Dr. W. M. Smith; Election of officers.

November 23, 1933: "Arc to chord ratio" by Professor Benner.

January 12, 1934: "The solution of the cubic" by Mr. Beesley, Mr. MacDougall, and Mr. Transue.

March 1, 1934: "Hyperbolic functions" by Mr. Elkins and Mr. Froberg.

March 23, 1934: "Geometric and algebraic curiosities" by Mr. Hazeltine, Mr. Wolf, and Mr. Carpenter; Election of officers for 1934–1935.

April 23, 1934: "On certain interesting numbers and their relation to periodic decimal fractions" by Professor James A. Shohat of the University of Pennsylvania.

DONALD MACDOUGALL, *Secretary*

THE KAPPA MU EPSILON MATHEMATICAL FRATERNITY

Kappa Mu Epsilon of Mississippi State College

The Mississippi Beta Chapter of State College represents one of several local clubs which united to form the Kappa Mu Epsilon Mathematical Fraternity for undergraduates. The State men purposely delayed filing credentials for a charter so that their sisters from the Mississippi State College for Women might hold the charter for the Alpha Chapter. Annual joint sessions held alternately on each campus have become a tradition of most gratifying promise between the Alpha and Beta Chapters.

We have three regular meetings each year for the purpose of hearing papers, copies of which are filed with the secretary. Frequent called meetings are held for the transaction of business. We had thirty-one active members and sixteen initiates during the year.

The officers for 1933–1934 were: E. P. Coleman, President; C. B. House, Vice President; W. S. McCormick, Treasurer; J. W. Hammond, Local Secretary; C. D. Smith, Secretary Descartes.

The meetings and programs were as follows:

December 12, 1933: Regular business and election of officers: "Outline of objectives" by C. D. Smith, Head of Mathematics Department.

March 8, 1934: Luncheon in Y.M.C.A. rooms; "Cultural and philosophical aspects of mathematics" by Dr. W. M. Patterson; Mathematical contest, directed by J. K. Upchurch.

May 1, 1934: Annual joint banquet with alpha chapter; "Some mathematical aspects of the coefficient of correlation" by Miss Annie Davis; "Perseverence in development of mathematics" by Miss Mabel Rupper; "Probabilities" by Miss Lucretia Davis; Vocal music, by Miss Elizabeth Grossnickle.

C. D. SMITH, *Corresponding Secretary*

PROBLEMS AND SOLUTIONS

EDITED BY OTTO DUNKEL, H. L. OLSON, AND W. F. CHENEY, JR.

ELEMENTARY PROBLEMS

Send all communications about Elementary Problems and Solutions to W. F. Cheney, Jr., Dept. Box 35, Storrs, Connecticut.

The Department of Elementary Problems welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. Problems may be submitted unaccompanied by their solutions.

PROBLEMS FOR SOLUTION

E 106. *Proposed by W. B. Campbell, Judson College, Rangoon, Burma.*

The minute and hour hands of a clock are indistinguishable in appearance. At what times does this make the time reading indeterminate?

E 107. *Proposed by J. B. Coleman, University of South Carolina.*

A straight line cuts two concentric circles in the points A, B, C and D in that order. AE and BF are parallel chords, one in each circle. CG is perpendicular to BF at G, and DH is perpendicular to AE at H. Prove that $GF=HE$.

E 108. *Proposed by Elmer Schuyler, Bay Ridge High School, Brooklyn, N.Y.*

Show how to construct a triangle when the orthocenter, the incenter and one vertex are given.

E 109. *Proposed by F. L. Manning, Ursinus College.*

Sum the infinite series,

$$\frac{1^2}{0!}+\frac{2^2}{1!}+\frac{3^2}{2!}+\cdots.$$

E 110. *Proposed by Robert McLaughlin, Ursinus College.*

A circular tower one hundred feet in diameter stands in a level field. A cow is tied by a rope a hundred feet long, the other end of which is fastened to a point at the base of the tower. Over what area may the cow graze?

E 111. *Proposed by W. R. Ransom, Tufts College.*

It is sometimes said that the differential equation $(D^2-1)y=0$ has the "general solution" $y=a \sinh (x+b)$. Show how this so-called general solution is related to the particular solution $y=ce^{-x}$.

E 112. *Proposed by Harry Langman, Cooper Union, New York.*

In a "round-robin" tournament, every entrant plays one round against each of the other entrants. If no one plays more than one round a day, how may the entries be paired so as to complete the contest in the fewest number of rounds?

$$(1)\qquad a/r_a + b/r_b + c/r_c = 2(\tan \tfrac{1}{2}A + \tan \tfrac{1}{2}B + \tan \tfrac{1}{2}C).$$

Since tangents from an external point to a circle are equal, $AB+BZ=AB+BX=AC+CY=AC+CX=\frac{1}{2}(a+b+c)=s$. Therefore $EZ=AZ \tan \frac{1}{2}A$ may be written: $r_a=s \tan \frac{1}{2}A$. Similarly $r_b=s \tan \frac{1}{2}B$ and $r_c=s \tan \frac{1}{2}C$. Therefore

$$(2)\qquad (a+b+c)/(r_a+r_b+r_c) = 2/(\tan \tfrac{1}{2}A + \tan \tfrac{1}{2}B + \tan \tfrac{1}{2}C).$$

Thus, by multiplying the equals of (1) and (2), the desired result is obtained, namely that

$$(a/r_a + b/r_b + c/r_c)(a+b+c)/(r_a+r_b+r_c) = 4.$$

Also solved by W. E. Buker, J. H. Butchart, W. B. Clarke, Bruce Johnson, L. M. Kelley, A. V. Richardson, J. Rosenbaum, C. D. Smith, C. W. Trigg, Simon Vatriquant and the proposer.

E 76 [1934, 103]. *Proposed by Raphael Robinson, University of California at Berkeley.*

Determine a so that the curve $y=a^x$ is tangent to the line $y=x$.

Solution by A. V. Richardson, Bishop's College, Lennoxville, Quebec.

Since $da^x/dx=a^x \ln a$, we have to determine a so that the equations $a^x=x$ and $a^x \ln a=1$ may have a common root.

Eliminating x, we have $a^{(1/\ln a)}=1/\ln a$. That is, $a^{1/m}=1/m$, where $m=\ln a$ and $a=e^m$. Consequently $e=1/\ln a$, $\ln a=1/e$, and $a=e^{1/e}$.

Also solved by Martha Allen, J. H. Butchart, Wm. Caughlin, David Colbert, Daniel Finkel, Elizabeth Giles, Margot Goodrich, W. W. Hurry, Herbert Levy, Roy MacKay, F. L. Manning, P. Morrison, F. R. Prescott, W. R. Ransom, J. Rosenbaum, Seymour Sherman, C. D. Smith, E. P. Starke, Woodrow Tichy, C. W. Trigg, M. J. Turner, Arthur Tyler, Simon Vatriquant and the proposer.

E 77 [1934, 104]. *Proposed by W. R. Ransom, Tufts College.*

Prove that in the factorial of each positive integer, the last non-zero digit is even.

Solution by C. W. Trigg, Cumnock College, Los Angeles.

With the exception of one factorial (for which the theorem is not true), the factorial of each positive integer less than five contains at least one even factor, so the last digit is even. The factorial of every other positive integer contains as factors at least two even integers for every integer ending in five. The product of one of these even integers by five introduces a terminating zero. Of the remaining factors at least one is even, hence the last non-zero digit is even.

Also solved by J. H. Butchart, Roy MacKay, F. L. Manning, P. Morrison, F. R. Prescott, B. D. Roberts, J. Rosenbaum, E. P. Starke, Simon Vatriquant, E. E. Whitford, Maud Willey and the proposer.

E 78 [1934, 104]. *Proposed by Ruth G. Mason, Berkeley, California.*

In this diagram of a long division problem, the x's denote unknown digits not all alike. One certain digit is represented by the letter A wherever that digit occurs. Determine all the digits and prove the solution unique.

$$
\begin{array}{rccccccccc}
x\,A) & x & x & x & x & x & x & x & x & x(x\,x\,x\,x\,x\,A\,x \\
 & x & A & x & & & & & & \\ \cline{2-4}
 & & & x & x & x & & & & \\
 & & & x & x & x & & & & \\ \cline{4-6}
 & & & & x & x & x & & & \\
 & & & & x & A & x & & & \\ \cline{5-7}
 & & & & & x & x & x & & \\
 & & & & & x & A & x & & \\ \cline{6-8}
 & & & & & & & x & x & \\
 & & & & & & & x & A & \\ \cline{8-9}
 & & & & & & & & x & x \\
 & & & & & & & & x & x \\ \cline{9-10}
\end{array}
$$

Solution by Frances P. Wallis, Connecticut College, New London.

Since the next to the last product shows xA times $A = xA$, it follows that A must be 1, 5 or 6. If A is 5 or 6, the divisor must be 15 or 16 in order that its product by A may be a two-digit number. But the first product is of the form xAx, and no multiple of 15 or 16 is of this form. Hence A is 1.

The only pairs of numbers that give a product ending in 1 are 3 times 7, and 9 times 9. Obviously 7 or 9 are impossible for the first digit of the divisor, since the last product shows a multiple of the divisor other than one times it, to be a two-digit number. Hence the divisor must be 31.

The first, fourth and fifth digits in the quotient are therefore each 7. The sixth digit of the quotient, being A, is 1. In the second step in the division, two digits from the dividend are brought down, so the second digit of the quotient is 0. The third and last digits in the quotient are respectively 9 and 2, since for these and only these values of the unknowns does the product of $70x771x$ and 31 yield a dividend free from the digit 1.

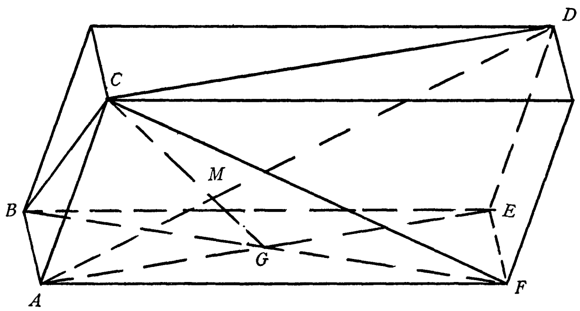
Consequently, the problem shows that if 220029072 be divided by 31, the quotient is 7097712.

Also solved by T. E. Benton, W. E. Buker, David Colbert, M. L. Constable, Elizabeth Giles, H. L. Krall, C. W. Munshower, W. R. Ransom, Caroline M. Reaves, C. B. Read, A. V. Richardson, C. A. Rupp, E. P. Starke, C. W. Trigg, Simon Vatriquant, Maud Willey, B. C. Zimmerman and the proposer.

E 79 [1934, 104]. *Proposed by W. C. Janes, Kansas State College.*

Prove that the centroid of the plane section of a parallelopiped thorugh the extremities of three concurrent edges, is a point of trisection of the diagonal concurrent with those edges.

Solution by L. M. Kelly, Northeastern University, Boston, Mass.



If the given parallelopiped is lettered as in the diagram, draw AE intersecting BF at G. Then draw CG intersecting AD at M, and connect C with D. Note that $AG=\frac{1}{2}AE=\frac{1}{2}CD$. Since the triangles AMG and DMC are similar (because corresponding angles are equal), $MG=\frac{1}{2}CM$ and $AM=\frac{1}{2}MD$. It consequently follows that $AM=AD/3$, and M is the centroid of the triangle CBF, as was to be proved.

Also solved by J. H. Butchart, E. W. Emery, Elizabeth Giles, Theodore Lindquist, Roy MacKay, W. R. Ransom, A. V. Richardson, J. Rosenbaum, C. D. Smith, E. P. Starke, Simon Vatriquant and the proposer.

E 80 [1934, 104]. *Proposed by H. E. Stelson, Kent State College, Kent, Ohio.*

A rectangular board has the length l (with the grain) and the width w (across the grain). From it a square table top is to be made with the grain running all one way. The board may be sawed both parallel and perpendicular to its length; but the table top may have joints only with the grain, and *not* across the grain. What is the largest table top that can be so made?

Solution by Elmer Latshaw, Philadelphia, Pa.

A complete solution involves three cases. The first is the special case in which there is no waste. In this case the square table would be of maximum area, and occurs when we can cut the length of the board into n equal pieces such that $l/n=wn$, or $l/w=n^2$. This is only possible when l/w is the square of an integer.

The remaining two cases involve waste in length or width cut off to produce a square table top. Cutting waste in both length and width need not be considered since both can only be done at the expense of area, thereby not giving a maximum.

The second case covers cutting off waste from the width of the board. If l/n is to be one side of the square table top, then in this case $l/n < wn$, and the waste is $(wn - l/n)l/n = wl - l^2/n^2 = wl(1 - l/wn^2)$. This waste will be a minimum when we select the smallest integer value of n which will make this last expression positive. Then l/n gives the length of one side, and l^2/n^2 gives the area of the table top, made up of l/nw boards. While the board is cut into n pieces of equal length, l/nw will not be an integer, indicating the necessity of cutting down the width of one board, and possibly discarding an entire board.

The third case covers cutting off waste from the length of the board. Here wn is one side of the square table top and $wn < l/n$. Then the waste is $(l/n - wn)wn = wl - w^2n^2 = wl(1 - wn^2/l)$, which will be a minimum when n is the largest integer which leaves it positive. Then wn gives the length of one side, the area is w^2n^2, and the number of equal pieces in the table top is n.

Hence the procedure is to see if l/w is the square of an integer, and if not, compute the waste in cases two and three to see which method of cutting to use to give the maximum table top.

Editor's Note. If the system of real positive numbers is considered as divided into intervals by the perfect squares and the products of pairs of consecutive integers, then l/w may be at a point of division, or in an even-numbered interval, or in an odd-numbered interval. If l/w is in an even-numbered interval, the lower bound of that interval is n^2 and we cut and use n pieces of width w and length wn. But if l/w is in an odd-numbered interval, the upper bound of that interval is n^2 and we cut the board into n equal lengths, split one piece lengthwise dividing its width in the same ratio that l/w divides its interval, and use the piece corresponding to the first segment of the interval, together with all the pieces which were not subdivided. If l/w is the product of two consecutive integers, $n-1$ and n, we cut the board into n equal lengths and use $n-1$ of them.

Also solved by W. R. Ransom.

ADVANCED PROBLEMS

Send all communications about Advanced Problems and Solutions to Otto Dunkel, Washington University, St. Louis, Mo. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide.

Problems containing results believed to be new or extensions of old results are especially sought. The editorial work would be greatly facilitated, if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known textbooks or results found in readily accessible sources, will not be proposed as problems for solution in this department. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

PROBLEMS FOR SOLUTION

3693. *Proposed by H. K. Fulmer, Georgia School of Technology.*

Solution of the differential equations, $dx/dt = k(a-x)^3$ and $dx/dt = k(a-x)(b-x)(c-x)$, $x=0$ when $t=0$, leads to

$$(1) \qquad k = \frac{2ax - x^2}{2a^2t(a - x)^2}$$

$$(2) \qquad k = \frac{(b - c)\log\dfrac{a - x}{a} + (c - a)\log\dfrac{b - x}{b} + (a - b)\log\dfrac{c - x}{c}}{t(b - c)(c - a)(a - b)}.$$

In certain applications it is physically necessary that the second expression for k should approach the first expression when b and c approach a simultaneously in any manner. Show mathematically that this is true.

3694. *Proposed by J. D. Leith, University of North Dakota.*

A given ellipse, $b^2x^2+a^2y^2=a^2b^2$, $a>b$, is the envelope of a family of circles of radii r and centers at points (o, c) on the y-axis, where r is a function of c. Determine this function, and discuss the variation of the circles as r and c vary.

3695. *Proposed by Robert C. Yates, University of Maryland.*

Given three simple closed curves A, B, C, on a sphere, find the shortest path from A to B to C and back to the starting point on A.

3696. *Proposed by J. B. Reynolds, Lehigh University.*

A dog directly opposite his master on the banks of a stream, flowing with uniform speed, swims at a still-water speed of two miles per hour heading directly towards his master at all times. The man notes that the dog does not stop drifting down stream until he is two-thirds across measured perpendicularly to the banks, and that it takes five minutes longer to make the trip than if the water had been still. How wide is the stream?

3697. *Proposed by J. Rosenbaum, Milford, Conn.*

In an orthocentric tetrahedron the centroid bisects the line segment joining the circumcenter with the orthocenter.

This is another analogy between an orthocentric tetrahedron and a triangle, for in a triangle the centroid trisects the segment joining the circumcenter with the orthocenter.

3698. *Proposed by Clifford W. Mendel and Gaylord M. Merriman, The University of Cincinnati.*

Prove the projective theorem: Given, in a projective plane, a complete quadrangle and a line l not passing through any vertex. On each side of the quadrangle consider the point which is the harmonic conjugate, with respect to the vertices on this side, of the point of intersection of the side with l. Among the six points thus determined call two points *opposite* if they lie on the opposite sides of the quadrangle. Prove that the three lines joining pairs of opposite

points are concurrent, and that on each of these lines this point of concurrency is the harmonic conjugate, with respect to the pair of opposite points determining the line, of the point of intersection of the line with l.

SOLUTIONS

3625 [1933, 427]. *Proposed by Robert E. Moritz, University of Washington.*

Show that

$$\sum_{k=1}^{n} (-1)^{k+1} {}_nC_k k^n = (-1)^{n+1} n!$$

for all positive integral values of n.

I. *Solution by M. S. Knebelman, Princeton University.*

We have in succession

$$\begin{aligned}\sum_{k=1}^{n} (-1)^{k+1} {}_nC_k k^n &= -\lim_{x\to 0} \frac{d^n}{dx^n} \sum_{k=0}^{n} (-1)^k {}_nC_k e^{kx} \\ &= -\lim_{x\to 0} \frac{d^n}{dx^n} (1-e^x)^n = -\lim_{x\to 0} \frac{d^n}{dx^n}\left(-x-\frac{x^2}{2!}-\frac{x^3}{3!}-\cdots\right)^n \\ &= (-1)^{n+1} \lim_{x\to 0} \frac{d^n}{dx^n}\left(x^n + n\frac{x^{n+1}}{2!}+\cdots\right) = (-1)^{n+1} n!.\end{aligned}$$

II. *Solution by W. P. Udinski, University of Texas.*

This problem is a special case of a class which may be solved by an elementary formula of the calculus of finite differences. Let Δ_h denote the difference operator with the unit h, which may be complex. Then for a positive integer m

$$\Delta_h{}^m f(x) = \sum_{k=0}^{m} (-1)^{m+k} {}_mC_k f(x+kh).$$

See, for instance, Boole's *Calculus of Finite Differences*, formula (3), page 19, where $h=1$. Setting $f(x)=x^n$, we obtain

$$\Delta_h{}^m x^n = \sum_{k=0}^{m} (-1)^{m+k} {}_mC_k (x+kh)^n. \tag{1}$$

Then

$$\begin{aligned}\sum_{k=0}^{m} (-1)^{m+k} {}_mC_k (x+kh)^n &= 0, \qquad \text{if } m > n, \\ &= n!h^n, \text{ if } m = n.\end{aligned} \tag{2}$$

For $x=0$ and $h=1$, the last equation above gives the result desired in the problem.

For proofs of identities in which $(-1)^k {}_mC_k$ occurs as a factor of a summand, it appears that in many cases the method of differences gives a very simple and brief proof, as may be seen from II after reducing it to the special case of the problem. The use of the unit $h\neq 1$ does not alter the simplicity of the proof of the fundamental formula of II. For we have merely to define the operator U_h so that $U_h^i f(y)=f(y+ih)$. Then.

$$\sum_{k=0}^{m}(-1)^{m+k}{}_mC_k f(x+kh) = \left[\sum_{k=0}^{m}(-1)^{m+k}{}_mC_k U_h{}^k\right]f(x)$$
$$= (U_h - 1)^m f(x) = \Delta_h{}^m f(x).$$

The method of differences may also be used to find the form of the expansion in the last line of I. For this purpose we may set $e^x-1=\Delta e^{0\cdot x}$. Then

$$(e^x-1)^n = \Delta^n e^{0.x} = \Delta^n \sum_{i=0}^{\infty}\frac{0^i\cdot x^i}{i!} = \sum_{i=n}^{\infty}(\Delta^n 0^i)\frac{x^i}{i!}\cdot$$

3626 [1933, 427]. *Proposed by Ruth G. Mason, Berkeley, Cal.*

Let all letters represent integers, $k>1$, $x\geqq 0$, $P>1$; and let $C_k(x)$ denote the sum of k consecutive integers, the smallest of which is $x+1$. Find the necessary and sufficient conditions that there exists a value of x such that $C_k(x)=P$.

Solution by E. P. Starke, Rutgers University.

A necessary and sufficient condition is that $2P/k-k$ be a positive odd integer. In fact, x is then given by

(A) $$2x = 2P/k - k - 1.$$

The proof is obvious from the formula for the sum of an arithmetic progression. Thus we have

(B) $$2C_k(x) = k(2x+k+1) = 2P.$$

A little more interest attaches to the problem in the following form: Find the necessary and sufficient condition that there exist values of x and k such that $C_k(x)=P$. Such a condition is merely that P be not a power of 2. The condition is necessary since one of the factors in (B) is odd. To prove the sufficiency, let $P=d\cdot f$, where d is odd and $d\neq 1$. If $d<2f$, set $k=d$; if $d>2f$, set $k=2f$. Then x as given by (A) will be a positive integer or zero.

The problem is noted by Carmichael, *Theory of Numbers*, p. 17.

Solved also by Frank Ayres, Jr., R. MacKay, R. E. Moritz, F. Underwood, S. Vatriquant and the proposer.

Note by the Editors. From the above solution it is clear that, in the first case, k is odd and P is divisible by k; in the second case, k is even and P is divisible by $k/2$ but not by k. In both cases $2P>k^2$. It is easily shown directly that these conditions on k are necessary. These conditions appeared in other solutions.

3627 [1933, 427]. *Proposed by W. H. Rasche, Virginia Polytechnic Institute.*

Given a cone of revolution and two ellipses E^h and F^0; by the processes of descriptive geometry so fix the position of the cone relative to the H- and V-planes of projection that E^h will be the H-base of the cone and F^0 its V-base. Show that there is no solution unless the sum of the acute angles which the axis of the cone makes with H and V is equal to or less than 90°.

Solution by A. Pelletier, Montreal, Canada.

Let the given cone with vertex S have γ for its half vertical angle, and suppose that a plane cuts from it one of the given ellipses. A plane through the axis of the cone and the major axis AA' of the ellipse cuts from the figure the triangle $SA'A$; for convenience suppose that $SA'<SA$. On SA' take $SC=SA$, and let the axis of the cone cut AA' in O. Then it is known that the side CA' of triangle ACA' is $2c$, the distance between the foci; and also angle $ACA'=\pi/2-\gamma$. Since the ellipse and cone are given, CA', $A'A$, and angle C are known; and thus the triangle ACA' can be constructed, and O can be determined. In a similar manner the intersection O' of the plane of the second ellipse with the axis can be determined. Hence OO' can be determined, and also the inclinations α and β of the axis OO' with the planes of the two ellipses. In order that the two ellipses be the intersections of the cone with the vertical and horizontal planes, we must have $\alpha+\beta\leqq\pi/2$. If the inequality is true there are four possible positions for OO'; if the equality is true there are two possible positions. The descriptive geometry construction then follows easily.

Solved also by the proposer.

3629 [1933, 428]. *Proposed by J. Rosenbaum, Milford, Conn.*

Find a formula for the sum

$$\sum_{x=1}^{n} x(x+1)(x+2)\cdots(x+p).$$

Note by the Editors. Find also the sum where the general term is multiplied by a^x, $a\neq 1$.

I. *Solution by Frank L. Manning, Ursinus College.*

The summand of the problem may be denoted by $(x+p)^{(p+1)}$; and, if Δ denotes the difference operator with the difference unity, we have

$$\Delta^{-1}(ax+b)^{(n)}=\frac{(ax+b)^{(n+1)}}{a(n+1)},\tag{1}$$

where Δ^{-1} is the operator of finite integration or summation corresponding to the indefinite integral in the ordinary calculus. Hence the given sum is

$$\left.\frac{(x+p)^{(p+2)}}{p+2}\right]_1^{n+1}=\frac{(n+p+1)^{(p+2)}}{p+2},\tag{2}$$

since for the lower limit the corresponding term is zero.

For the second part we may use the formula

$$\Delta^{-1}u(x)v(x) = \sum_{i=0}^{m} (-1)^i \Delta^i u(x) \Delta^{-i-1} v(x+i), \tag{3}$$

where $\Delta^{m+1}u(x)=0$, which is easily obtained by successive summations by parts. Here we set $u(x)=(x+p)^{(p+1)}$, $v(x)=a^x$, $\Delta^{-1}a^x=a^x/(a-1)$. Then

$$\sum_{x=1}^{n} (x+p)^{(p+1)}a^x = \sum_{i=0}^{p+1} (-1)^i \frac{(p+1)!(x+p)^{(p-i+1)}}{(p-i+1)!} \frac{a^{x+i}}{(a-1)^{i+1}}\Bigg]_{x=1}^{n+1}.$$

II. *Solution by J. M. Feld, Brooklyn College.*

For $|t|<1$ and $x=1, 2, \cdots, n$, we have

$$(1-t)^{-x} = \sum_{i=0}^{\infty} \frac{x(x+1)\cdots(x+i-1)}{i!} t^i, \tag{1}$$

and therefore

$$\sum_{x=1}^{n} (1-t)^{-x} = \sum_{i=0}^{\infty} \frac{t^i}{i!} \sum_{x=1}^{n} x(x+1)\cdots(x+i-1). \tag{2}$$

However, the left side of (2) may be evaluated as a geometric progression and this sum may be expanded in the same way as in (1). Thus

$$\sum_{x=1}^{n} (1-t)^{-x} = \frac{(1-t)^{-n}-1}{t} = \sum_{j=0}^{\infty} \frac{n(n+1)\cdots(n+j)}{(j+1)!} t^j. \tag{3}$$

Equating coefficients of t^{p+1} in (2) and (3) we have

$$\sum_{x=1}^{n} x(x+1)\cdots(x+p) = \frac{n(n+1)\cdots(n+p+1)}{p+2}.$$

Proceeding in the same manner we have

$$\sum_{x=1}^{n} a^x(1-t)^{-x} = \sum_{i=0}^{\infty} \frac{t^i}{i!} \sum_{x=1}^{n} x(x+1)\cdots(x+i-1)a^x. \tag{4}$$

The left side of (4) may be summed, and we have for its value

$$\begin{aligned} &\frac{a}{1-a}\left[1-a^n(1-t)^{-n}\right]\left[1-\frac{t}{1-a}\right]^{-1} \\ &= \frac{a}{1-a}\left[1-a^n\sum_{i=0}^{\infty}\frac{n(n+1)\cdots(n+i-1)}{i!}t^i\right]\sum_{j=0}^{\infty}\frac{t^j}{(1-a)^j}. \end{aligned} \tag{5}$$

Equating coefficients of t^{p+1} in (4) and (5) we have

$$\sum_{x=1}^{n} x(x+1)\cdots(x+p)a^x$$

$$= (p+1)!\left[\frac{a}{(1-a)^{p+2}} - a^{n+1}\sum_{i=0}^{p+1}\frac{n(n+1)\cdots(n+i-1)}{i!(1-a)^{p-i+2}}\right].$$

III. *Solution by E. P. Starke, Rutgers University.*

The given sum may be evaluated in the form

$$(1)\qquad (p+1)!\sum_{x=1}^{n}\binom{x+p}{p+1} = (p+1)!\binom{n+p+1}{p+2},$$

as is easily seen from the identity

$$(2)\qquad \binom{x+p}{p+2}+\binom{x+p}{p+1}=\binom{x+p+1}{p+2}.$$

Another interesting proof follows from the nature of the combinations of $n+p+1$ elements taken $p+2$ at a time. Of these there are ${}_{n+p}C_{p+1}$ which contain a given element e_1; ${}_{n+p-1}C_{p+1}$ which contain e_2 but not e_1; ${}_{n+p-2}C_{p+1}$ which contain e_3 but neither e_1 nor e_2; and so on. We finally end the count when n elements have been considered, and we then have

$$(3)\qquad \binom{n+p+1}{p+2}=\binom{n+p}{p+1}+\binom{n+p-1}{p+1}+\cdots+\binom{p+1}{p+1}.$$

For the proposed generalization, we set

$$(4)\qquad f(a)=\sum_{x=1}^{n}a^{x+p}=\frac{a^{p+1}-a^{n+p+1}}{1-a},\qquad a\neq 1.$$

Then the given sum is

$$(5)\qquad (p+1)!\sum_{x=1}^{n}\binom{x+p}{p+1}a^x = a\frac{d^{p+1}}{da^{p+1}}f(a) = a\frac{d^{p+1}}{da^{p+1}}\left(\frac{a^{p+1}-a^{n+p+1}}{1-a}\right).$$

The denominator in the result of the $p+1$ differentiations will be $(1-a)^{p+2}$, and this suggests the evaluation of the product

$$(6)\qquad (1-a)^{p+2}\sum_{x=1}^{n}\binom{x+p}{p+1}a^x = \sum_{y=0}^{p+2}(-1)^y\binom{p+2}{y}a^y\sum_{x=1}^{n}\binom{x+p}{p+1}a^x.$$

The coefficient of a^k in the product on the right is

$$(7)\qquad \sum(-1)^z\binom{p+k-z}{p+1}\binom{p+2}{z},$$

where the lower limit in the sum is $z=0$ for $k\leqq n$, and $z=k-n$ for $k>n$; the upper limit is $z=k-1$ for $k\leqq p+3$, and $z=p+2$ for $k>p+3$.

Let us digress a moment to consider this sum. It will be shown that

$$(8)\qquad \sum_{z=0}^{s}(-1)^z\binom{p+k-z}{p+1}\binom{p+2}{z}=(-1)^s\binom{p+k-s}{k-1}\binom{k-2}{s},\ s\neq 0.$$

For $s=1$ it is easily verified that (8) is true. We then show that

$$(-1)^s\left[\binom{p+k-s}{k-1}\binom{k-2}{s}-\binom{p+k-s-1}{p+1}\binom{p+2}{s+1}\right]$$
$$=\frac{(-1)^{s+1}(p+k-s-1)!}{(k-1)(p-s)!(s+1)!(k-s-3)!}=(-1)^{s+1}\binom{p+k-s-1}{k-1}\binom{k-2}{s+1}.$$

With this result the formula follows by mathematical induction. The sum in (8) is zero if $s=k-1$, or if $s=p+2$, which are the upper limits noted above.

Returning now to (7), we find that the coefficient of a is unity; the coefficient of a^k is zero for $0<k\leqq n$, since it is given by (8) for $s=k-1$ or $s=p+2$. For $k>n$ the lower limit in (7) is $z=k-n$, as stated above; and, in order to apply (8), we write (7) in the form

$$\sum_{k-n}^{s}=\sum_{0}^{s}-\sum_{0}^{k-n-1}=-\sum_{0}^{k-n-1}.$$

The coefficient of a^k is then

$$(-1)^{k-n}\binom{p+n+1}{k-1}\binom{k-2}{k-n-1},\qquad k>n.$$

We have finally from these results

$$(9)\qquad \sum_{x=1}^{n}x(x+1)\cdots(x+p)a^x$$
$$=\frac{(p+1)!}{(1-a)^{p+2}}\left[a+\sum_{k=n+1}^{n+p+2}(-1)^{k-n}\binom{p+n+1}{k-1}\binom{k-2}{k-n-1}a^k\right].$$

The sum on the right has $p+2$ terms, i.e., the number of terms on the right is independent of n. The same result may be obtained directly from (5) by using Leibnitz's formula for the derivative of a product with suitable reductions.

We shall now put the result (9) in a different form

$$\sum_{k=n+1}^{n+p+2}(-1)^{k-n}\binom{p+n+1}{k-1}\binom{k-2}{k-n-1}a^k$$
$$=\frac{(n+p+1)!}{(n-1)!(p+1)!}a\sum_{k=n+1}^{n+p+2}(-1)^{k-n}\binom{p+1}{k-n-1}\frac{a^{k-1}}{k-1}$$
$$=-\frac{(n+p+1)!a}{(n-1)!(p+1)!}\int_0^a a^{n-1}\sum_{i=0}^{p+1}(-1)^i\binom{p+1}{i}a^i da$$

$$= -\frac{(n+p+1)!a}{(n-1)!(p+1)!}\int_0^a a^{n-1}(1-a)^{p+1}da.$$

Hence

$$\sum_{x=1}^{n} x(x+1)\cdots(x+p)a^x$$

$$= \frac{a}{(1-a)^{p+2}}\left[(p+1)! - \frac{(n+p+1)!}{(n-1)!}\int_0^a a^{n-1}(1-a)^{p+1}da\right].$$

Solved also by H. E. H. Greenleaf, J. D. Leith, R. E. Mortiz, F. Underwood, S. Vatriquant, and the proposer.

Note by the Editors. The solutions by Greenleaf and Leith were similar to solution I above. Leith used the formula for summation on page 73 in Boole's *Finite Differences*. This formula is also given in connection with the solution of problem 3571 [1933, 434]. With exception of II and III above, the methods in general were essentially those of finite differences although the symbols of that calculus were not used by all the solvers. It is worth while noticing the importance of the formula for summation by parts in its simplest form as given in this MONTHLY, l.c. It may be written in a definite form as follows:

$$\text{(A)} \quad \sum_{x=0}^{n} v(x)\Delta u(x) = u(n+1)v(n+1) - u(0)v(0) - \sum_{x=0}^{n} u(x+1)\Delta v(x),$$

$$\text{(B)} \quad = u(n+1)v(n) - u(0)v(0) - \sum_{x=0}^{n-1} u(x+1)\Delta v(x),$$

where (B) easily follows from (A). Setting in (B), $v(x)=a_x$, $\Delta u(x)=b_x$, we have

$$\sum_{i=0}^{n} a_i b_i = a_n \sum_{i=0}^{n} b_i + \sum_{i=0}^{n-1}\left[(a_i - a_{i+1})\sum_{j=0}^{i} b_j\right].$$

This last form is of great value in the study of the convergence of series, and it is used in many treatises on analysis. For illustrations of the use of this formula consult any advanced text on analysis and refer to the index references to Abel or Abel's lemma. See for instance Goursat-Hedrick, vol. 1, pages 153, 348, 349. A large number of applications may be found in Tonelli's *Serie Trigonometriche*, page 73, where the given transformation of series is called *The Transformation of Brunacci-Abel*, and references are given for the origin of the designation. This transformation is simply summation by parts as given by formula (B).

The names of the following solvers were overlooked in the printed lists of solvers:

E. M. Berry, 3531; N. A. Court, 3598; J. M. Feld, 3535; V. F. Ivanoff, 3529; A. A. Shaw, 3530. Solutions of 3572 and 3592 by G. S. Jones were received after selected solutions had been sent in for printing.

NEWS AND NOTICES

Readers are invited to contribute to the general interest of this department by sending items to Professor J. H. Weaver, Ohio State University, Columbus, Ohio.

OUTLINE OF THE HISTORY OF MATHEMATICS

The Mathematical Association of America takes pleasure in announcing that The Society for the Promotion of Engineering Education has made over to it all rights in the booklet on the *Outline of the History of Mathematics* by Raymond Clare Archibald, and that the Association has just published a second thoroughly revised and enlarged edition, which it is offering for fifty cents a copy postpaid. The first edition of 1030 copies published by the S.P.E.E. was sold out in a year and a half. It is hoped that the new edition, containing much up-to-date and authoritative information, may meet a real need in the Association's constituency. Orders with remittances may be placed with the Secretary of the Association at Oberlin, Ohio. Believing that this *Outline* will be found extremely useful in connection with mathematics classes in schools and colleges, the Association will fill orders for ten or more copies to the same address at forty cents per copy.

The Southern Intercollegiate Mathematics Association was organized in Shreveport, Louisiana, last fall at the instigation of Dr. I. Maizlish of Centenary College. Its purpose is to stimulate an interest in mathematics in colleges and universities by holding annual competitions in algebra, trigonometry, analytic geometry, and calculus. The first annual meeting was held at Louisiana State Normal College, Natchitoches, Louisiana, May 5, 1934. Winners of this year's preliminary contests are Millsaps College, Jackson, Mississippi; Simmons University, Abilene, Texas; University of Arkansas, Fayetteville, Arkansas; and Louisiana State University, Baton Rouge, Louisiana. The final contest, held at Natchitoches, was won by Louisiana State University. The award was a silver cup which will become the permanent possession of the school first winning it three times. The second annual meeting of the association will be held next year in the early part of May at Shreveport, Louisiana.

Dean Tristram W. Metcalfe of Long Island University has announced the organization at the University of "The Galois Mathematics Institute." The purpose of the Institute is to give to undergraduates who are unusually gifted in mathematics an opportunity to pursue a special course of reading and study, and to hear addresses on current research in mathematics given by the research men themselves. Arrangements are being made to have prominent mathematicians address this group. The students of the Institute need not be students at Long Island University, but are to be chosen from among the student body of the various colleges in New York City, recommended for membership by the heads of the mathematics departments in these colleges. Students at Long

Island University will be given college credit for work in mathematics done at the Institute. It is hoped that the other colleges having students attending the Institute will also accord them this privilege. Dr. L. R. Lieber is the Director of the Institute.

The organization meeting was attended by thirty students from eight different colleges and eighty students from twenty-three different high schools. Other meetings held this spring are as follows: May 5: Professor Edward Kasner of Columbia University, "Mathematics"; May 12: Count Alfred Korzybski, "The Linguistic and Behavioristic Aspects of Mathematics"; May 26: Professor H. P. Robertson of Princeton University, "Geometry and Physical Space-Time"; June 9: Dr. James Singer of Princeton University, "Elementary Analysis Situs"; This program of the presentation and discussion of current research work is to be continued in the fall, the meetings taking place regularly once a week. This is the only institution of its kind in this country.

Professor R. D. Carmichael of the department of mathematics at the University of Illinois has been made Dean of the Graduate School.

Professor Arnold Dresden, of Swarthmore College, has been awarded a Guggenheim fellowship, for the preparation of a book on the calculus of variations, an attempt to unify the three points of view now dominant in this field; he will work at Pisa and at the Institute for Advanced Study.

The honorary degree of Doctor of Engineering was conferred by the Michigan College of Mining and Technology at the commencement in June on James Fisher, head of the department of mathematics and physics.

Professor E. T. Whittaker, of the University of Edinburgh, who was appointed Hitchcock Professor at the University of California for the second semester of 1933–34, lectured in Berkeley from March 19 to April 3, and in Los Angeles on April 5 and 6, at the University of California. At Berkeley, he delivered five public lectures on the general topic *The geometrical description of nature*, and five technical lectures on *Recent general field theories, especially those of Schouten and van Dantzig*. At Los Angeles he repeated his public lectures in briefer form. He returned immediately to Edinburgh to resume his duties there. The honorary degree of Doctor of Laws was conferred upon him by the University of California on the occasion of its Charter Day celebration on March 23, 1934.

Dr. H. S. Vandiver, associate professor of mathematics at the University of Texas, was appointed a temporary lecturer at the Institute for Advanced Study at Princeton, while on leave of absence.

Dr. H. B. Phillips, professor of mathematics at the Massachusetts Institute of Technology, has been appointed acting head of the department of mathematics.

Associate Professor George Rutledge of the Massachusetts Institute of Technology, has been promoted to a professorship, effective 1934–35.

Dr. H. P. Thielman, of the Ohio State University, has been appointed professor of mathematics at the College of St. Thomas, St. Paul, Minnesota.

Dr. J. H. Van Vleck, professor of physics at the University of Wisconsin, has been elected associate professor of mathematics at Harvard University.

The announcement in the June-July issue regarding C. O. Oakley should have stated that he has been appointed assistant professor, instead of professor, at Haverford College.

Dr. I. J. Schwatt, emeritus professor of mathematics at the University of Pennsylvania, died April 18, 1934.

The following seventy-five doctorates with mathematics or mathematical physics as major subject were conferred during 1933 in universities in the United States and Canada; the major subject is mathematics unless otherwise specified. The university, month in which the degree was conferred, minor subject (other than mathematics), and title of dissertation are given in each case if available.

Helen Schlauch Adams (Mrs. L. P.), Cornell, September, minor in physics. *The normal rational n-ic.*

Beatrice Aitchison, Johns Hopkins, June, *On mapping with functions of finite sections.*

H. H. Alden, Ohio State, June, *Properties of differential equations and integrals of differential equations arising from properties of the solution curves.*

E. W. Anderson, Iowa State College, June, minors in physics and engineering, *Statics of special types of homogeneous elastic slabs with variable thickness.*

H. M. Bacon, Stanford, June, minor in history, *An extension of Kronecker's theorem.*

S. F. Barber, Illinois, June, minor in astronomy. *Planar Cremona transformations.*

J. H. Binney, Rice Institute, June, *An elliptic system of integral equations on summable functions.*

Julia W. Bower, Chicago, June, *The problem of Lagrange with finite side conditions.*

Dorothy S. Brady, California (Berkeley), May, *On the solution of homogeneous linear integral equations.*

S. H. Caldwell, Massachusetts Institute of Technology, June, major in

electrical engineering, minor in physics, *The extension and application of differential analyzer technique in the solution of ordinary differential equations.*

Josephine H. Chanler, Illinois, June, minor in physics, *Poristic double binary forms.*

S. P. Cheo, Michigan, June, *Singularities of analytic vector functions.*

W. S. Claytor, Pennsylvania, June, *On the immersion of a Peanian continuum in a spherical surface.*

H. E. Crull, Illinois, June, minor in astronomy, *Quartic surfaces invariant in the symmetric group* G_{24}.

W. M. Davis, Chicago, August, *Contributions to the theory of conjugate nets.*

J. G. Estes, Massachusetts Institute of Technology, June, minor in aeronautical engineering, *The lift and moment of an arbitrary aerofoil—Joukovsky potential.*

H. M. Feldman, Washington University (St. Louis), June, minor in physics, *Product moments of samples drawn from a set of infinite populations.*

J. S. Frame, Harvard, June, *The theory of tables of group characteristics.*

Gordon Fuller, Michigan, February, *On the invariant character of certain systems of partial differential equations.*

A. T. Goble, Wisconsin, December, major in physics, *The four vector problem in quantum mechanics with applications to platinum-like spectra.*

H. S. Grant, Pennsylvania, June, *Concerning powers of certain classes of ideals in a cyclotomic realm which give the principal class.*

N. M. Gray, New York, June, major in physics. *On the calculation of the magnetic moment of the* Li^7 *nucleus from hyperfine structure data.*

T. N. E. Greville, Michigan, June, *Invariance of the property of admissibility under certain general types of transformations.*

D. F. Gunder, Wisconsin, October, *The flexure problem for rectangular beams with slits.*

Abraham Hackman, Rensselaer Polytechnic Institute, June, minor in mathematical physics, *An investigation of the applicability of Fourier's integral and Fourier's series to the summation of series of integral powers of the reciprocals of the natural numbers.*

E. H. Hadlock, Cornell, June, *On the progressions associated with a ternary quadratic form.*

D. L. Hibbard, Massachusetts Institute of Technology, June, minor in business and engineering administration, *Mathematical risks.*

R. C. Hildner, Ohio State, December, *Inverse problems of the calculus of variations involving the Hamilton and the Hamilton-Caratheodry functions.*

P. G. Hoel, Minnesota, March, minor in physics, *Certain problems in the theory of closest approximation.*

Roberta F. Johnson, Cornell, June, minor in philosophy, *Involutions of order* 2 *associated with surfaces of genera* $p_a = p_g = 0$, $P_2 = 1$, $P_3 = 0$.

H. K. Justice, Cincinnati, June, *Group theory applied to vector differential equations.*

S. C. Kleene, Princeton, October, *A theory of positive integers in formal logic.*

V. S. Lawrence, Jr., Cornell, June, *Closed orbits in central distance force fields.*

Joseph Lev, Cornell, September, minor in physics, *Effects of linear transformations on the divergence of bounded sequences and functions.*

J. F. Locke, Illinois, February, minor in philosophy, *Repeated sums of certain functions.*

E. R. Lorch, Columbia, April, *Elementary transformations.*

Emma W. McDonald, California (Berkeley), May, minor in astronomy, *Magic cubes which are uniform step cubes.*

M. L. MacQueen, Chicago, December, *A projective generalization of euclidean parallelism of surfaces.*

G. R. Magee, Chicago, June, *Conjugate nets of ruled surfaces in a congruence.*

J. R. Mayor, Wisconsin, June, *A generalization of the Veronese and Steiner surfaces.*

W. I. Miller, Pittsburgh, February, *Fundamental regions for the simple group of order* 168 *in* S_4.

Deane Montgomery, Iowa, January, minor in philosophy, *Sections of point sets.*

L. T. Moston, Harvard, June, *Invariant methods in the infinitesimal geometry of surfaces.*

D. S. Nathan, Cincinnati, June, *Groups of transformations in a composite function space.*

Abba V. Newton, Chicago, August, *Consecutive covariant configurations at a point of a space curve.*

G. W. Nicholson, North Carolina, June, *The generalized form of the Lorentz transformation.*

E. R. Ott, Illinois, February, minor in physics, *A modular manifold determined by the planar quartic curve.*

L. F. Palmer, Cincinnati, June, *On the strong summability of the double Fourier series.*

Ruth M. Peters, Radcliffe, June, *Parallelism and equidistance in Riemannian geometry.*

Melba N. Phillips, California (Berkeley), May, major in physics, *Problems in the spectra of the alkalis*: I *Photo-ionization probabilities in atomic potassium.* II *Theoretical considerations on the inversion of doublets in alkali-like spectra.*

H. S. Pollard, Wisconsin, October, minor in statistics, *On the relative stability of the median and arithmetic mean, with particular reference to certain frequency distributions which can be dissected into normal distributions.*

T. E. Raiford, Michigan, February, *Some geometric aspects of linear transformations.*

H. W. Raudenbush, Jr., Columbia, September, *Differential fields and ideals of differential forms.*

P. K. Rees, Rice Institute, June, *Transforms of Fuchsian groups.*

Casper Shanok, Yale, June, *Convex polyhedra and criteria for irreducibility.*

P. M. Singer, California (Berkeley), May, *Periodic continued fraction expansions of quadratic irrationalities.*

Abraham Sinkov, George Washington, June, *Families of groups generated by two operators of the same order.*

H. E. Spencer, Cornell, June, minor in physics, *On convergence and oscillation of transforms of sequences of vectors.*

Anna A. Stafford, Chicago, August, *Knotted varieties.*

Morris Stone, Pittsburgh, June, minors in physics and mechanical engineering, *The general torsion problem; solution by electric flow analogue.*

Alexander Tartler, Pennsylvania, June, *On a certain new class of orthogonal polynomials.*

Arthur Tilley, New York, June, minor in physics, *On spaces* (E) *of Frêchet.*

Y. Y. Tseng, Chicago, March, *The characteristic value problem of Hermitian functional operators in a non-Hilbertian space.*

H. L. Turrittin, Wisconsin, June, *Asymptotic solutions of certain ordinary differential equations associated with multiple roots of the characteristic equation.*

T. L. Wade, Jr., Virginia, June, minor in physics, *Irreducible complete systems of euclidean concomitants for some cubic curves with their syzygies and their geometric interpretations.*

G. P. Wadsworth, Mass. Inst. of Tech., June, minor in physics, *The geometry of linear homogeneous partial differential equations of the first order.*

T. O. Walton, Michigan, February, *Geometrical studies suggested by a proof of the fundamental theorem of algebra.*

C. H. Wheeler, Johns Hopkins, June, *A type of homogeneity for continuous curves.*

N. G. Whitelaw, Wisconsin, major in physics, *Theory of the quantum defect and anomalous multiple structure of Al II.*

Marian A. Wilder, Minnesota, July, minor in biometry, *Some problems in closest approximation over a discrete set of points.*

L. A. Wills, New York, June, major in physics, *Relativity corrections in the theory of hyperfine structure.*

Clement Winston, Pennsylvania, February, *On mechanical quadrature formulae related to the classical orthogonal polynomials.*

C. R. Worth, California Institute of Technology, June, minor in mathematical physics, *The sub-varieties of a field.*

Y. C. Yeh, Massachusetts Institute of Technology, June, minor in naval architecture, *The distribution of stresses in welded structures.*

P. T. Yuan, Michigan, February, *On the logarithmic frequency distribution and the semi-logarithmic correlation surface.*

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DIRECTORY

EDITORIAL CORRESPONDENCE should be addressed to the EDITOR-IN-CHIEF, W. B. CARVER, White Hall, Cornell University, Ithaca, N.Y.

BOOKS FOR REVIEW should be addressed to R. A. JOHNSON, Brooklyn College, 66 Court Street, Brooklyn, N.Y.

BUSINESS CORRESPONDENCE should be addressed to the SECRETARY-TREASURER of the Association, W. D. CAIRNS, 33 Peters Hall, Oberlin, Ohio.

CHANGE OF ADDRESS: Members should send notice of any change of address to the SECRETARY-TREASURER, W. D. CAIRNS, Oberlin, Ohio, before the 10th of each month.

MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Eighteenth Summer Meeting of the Association, Williamstown, Mass., Sept. 3-4, 1934.

The following is a list of the sections of the Association, with dates of those section meetings which have been scheduled for 1934 and reported to the Secretary.

ALLEGHENY MOUNTAIN, Pittsburgh, Pa., Feb. 10; Washington, Pa., May 5.
ILLINOIS, Jacksonville, May 4-5.
INDIANA, La Fayette, May 11-12.
IOWA, Des Moines, April 20-21.
KANSAS, Topeka, Mar. 17.
KENTUCKY, May.
LOUISIANA-MISSISSIPPI, Jackson, Miss., Mar. 23-24.
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Baltimore, Md., Dec. 8.
MICHIGAN, Ann Arbor, Mar. 17.
MINNESOTA, Northfield, May 12.
MISSOURI.
NEBRASKA, Crete, Apr. 27.
OHIO, Columbus, Apr. 5.
OKLAHOMA, Oklahoma City, Feb. 9.
PHILADELPHIA, Philadelphia, Dec. 1.
ROCKY MOUNTAIN, Colorado Springs, Apr. 20-21.
SOUTHEASTERN, University, Ala., Mar. 30-31.
SOUTHERN CALIFORNIA, Riverside, Mar. 3.
TEXAS, College Station, May 5.
WISCONSIN, Oshkosh, May 5.

AFFILIATED ORGANIZATIONS: THE NEW ENGLAND ASSOCIATION OF TEACHERS OF MATHEMATICS. THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS.

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THE FIRST ANNUAL MEETING OF THE OKLAHOMA SECTION

The first annual meeting of the Oklahoma Section of the Mathematical Association of America was held at the University Club in Oklahoma City, on Friday, February 9, 1934, Professor N. A. Court, presiding.

The attendance was sixty-nine, including the following fifteen members of the Association: E. F. Allen, J. C. Brixey, N. A. Court, E. P. R. Duval, F. C. Gentry, H. L. Hall, J. O. Hassler, J. E. LaFon, Clarence McCormick, Dora McFarland, S. W. Reaves, W. T. Short, A. M. Wallace, Grace West, J. H. Zant.

At the business meeting the following officers were chosen for next year: Chairman, N. A. Court, University of Oklahoma; Vice-Chairman, W. T. Short, Oklahoma Baptist University; Secretary, E. F. Allen, Oklahoma A. and M. College.

The next meeting will be held in connection with the Oklahoma Educational Association, at Tulsa, Oklahoma, in February 1935.

The following papers were presented:

1. "Notes on the history of the teaching of collegiate mathematics in Oklahoma" by Dean S. W. Reaves, University of Oklahoma.

2. "The achievements of mathematics in astronomy" by Professor W. E. Howard, University of Tulsa, introduced by Professor N. A. Court.

3. "Mathematics in the integrated curriculum" by Professor J. H. Zant, Oklahoma A. and M. College.

Abstracts of the papers follow:

1. A small part of the present State of Oklahoma was opened for settlement in 1889. Three years later three state educational institutions began operation: The University of Oklahoma, at Norman, the Agricultural and Mechanical College, at Stillwater, and the Normal School, at Edmond. The instruction in these schools at first was largely secondary. The first class to graduate with B.A. degrees at the University of Oklahoma was that of 1898 and contained two members. The earliest collegiate mathematics (calculus, or above) taught in Oklahoma, of which the writer has information, was given at the University of Oklahoma by Frederick S. Elder, now of Kansas City, Missouri, who was Instructor (1896–98) and Professor of Mathematics (1898–1905). He gave for the first time a course in calculus in the second semester of 1901–02, one in Engineering Mechanics, based on Church's Mechanics, throughout the year 1902–03, one in Projective Geometry in the first semester of 1903–04. Differential Equations (Murray) and Advanced Analytic Geometry (Salmon) were first given in the second semester of 1905–06 by the writer who was successor to Professor Elder. In 1909 calculus was being taught in the Oklahoma A. & M. College, but nothing above that subject. More advanced courses were given shortly after this date. In 1920 the number of normal schools had increased to six, these were then changed to teachers' colleges and calculus was offered for the first time in each

of them. At present the number of junior and senior colleges offering collegiate mathematics is as follows: thirteen state schools and five private schools.

2. Professor Howard's paper was of an expository nature. The importance of mathematics to the study of modern astronomy was explained. He then outlined the solution of a few astronomical problems.

3. Professor Zant stated that the aim of the proposed integrated curriculum is to "organize experiences into a unified whole in order to enable the individual to make appropriate responses." Various methods have been proposed to accomplish this aim. Many of them ignore the value and importance of subject matter. Teachers should have a clear idea of the schemes in order to use them to the best advantage and to guard against numerous dangers.

E. F. ALLEN, *Secretary*

THE ELEVENTH ANNUAL MEETING OF THE MICHIGAN SECTION

The eleventh annual meeting of the Michigan Section was held at the University of Michigan, Ann Arbor, Michigan, on Saturday, March 17, 1934, in conjunction with the Michigan Academy of Science. There were sessions both morning and afternoon for the reading of papers, both sessions being presided over by the chairman of the Section, Professor T. O. Walton.

The attendance was slightly over a hundred including the following forty-two members of the Association: W. L. Ayres, J. W. Baldwin, W. D. Baten, Henry Blumberg, J. W. Bradshaw, J. B. Brandeberry, R. V. Churchill, C. C. Craig, S. E. Crowe, Albertus Darnell, W. W. Denton, J. D. Elder, J. P. Everett, Peter Field, W. B. Ford, J. W. Glover, V. G. Grove, C. L. Herron, T. H. Hildebrandt, L. A. Hopkins, E. E. Ingalls, E. H. Johnson, L. S. Johnston, H. S. Kaltenborn, L. C. Karpinski, M. M. Lemme, Theodore Lindquist, C. E. Love, E. D. McCarthy, A. L. Nelson, J. A. Nyswander, H. L. Olson, J. E. Powell, G. Y. Rainich, C. C. Richtmeyer, L. J. Rouse, T. R. Running, E. R. Sleight, G. G. Speeker, A. G. Swanson, T. O. Walton, J. B. Winslow.

The annual business meeting was held at noon at a luncheon at the Michigan Union and was attended by forty-five persons. It was voted to change the by-laws to make the retiring chairman the third member of the Executive Committee instead of electing this member as formerly. There was a spirited discussion concerning a fall meeting of the section and it was voted to hold such a meeting next year at some point outside Ann Arbor as a one-year trial to determine its desirability. The nominating committee, consisting of V. G. Grove, C. C. Craig and A. L. Nelson, proposed the following officers for the year 1934–35 and these were unanimously elected: Chairman, L. S. Johnston, University of Detroit; Secretary-Treasurer, W. L. Ayres, University of Michigan; Member of Executive Committee, T. O. Walton, Kalamazoo College.

The following ten papers were read at the sessions:

1. "Nomography—an exposition" by Professor L. S. Johnston, University of Detroit.

2. "Bilinear forms and correspondences" by Professor H. L. Olson, Michigan State College.

3. "On the Euler-Maclaurin summation formula" by Donat Kazarinoff, University of Michigan, by invitation.

4. "The mathematics of 'Alice in Wonderland'" by Mrs. Dorothy Van Deusen, Battle Creek High School, by invitation.

5. "On the normals to the ellipse" by Donald Owens, Albion College.

6. "Mathematics in the Near East" by Professor L. C. Karpinski, University of Michigan.

7. "Mathematicians, right or left" by Professor L. C. Emmons, Michigan State College.

8. "The inductive principle" by Professor Henry Blumberg, Ohio State University, by invitation of the Program Committee.

9. "Econometric treatment of academic problems" by Doctor W. S. Kimball, Michigan State College, by invitation.

10. "Some recent problems in mathematical economics" by Dr. H. H. Pixley, Wayne University.

In the absence of the author the paper by Doctor Pixley was read by title. Abstracts of some of the papers follow:

1. The collinearity of three points is equivalent to the vanishing of a certain third order determinant. This paper presents nomography as the science of using this equivalence to construct in a plane a chart of an equation $f(x, y, z)=0$ in three variables, such that every set of three collinear points on this chart represents a solution of the equation. There were exhibited several illustrative charts with explanations of methods of construction, as well as the method of extending the process to equations in more than three variables. Some applications in engineering and physics were shown.

2. The paper discusses the bilinear correspondence

$$\sum_{j,k=1}^{n} a_{jk}x_jy_k = 0$$

between points and hyperplanes in $n-1$ space. By a choice of coordinates this bilinear form may be reduced to a few type forms and these may be classified geometrically. The cases $n=3$ and $n=4$ were discussed in detail.

4. "Alice in Wonderland," popularly known as the children's outstanding nonsense classic, is, in the last analysis, a book which appeals more strongly to the cultured adult mind than to younger people. Beneath the clever and fantastic creatures of Wonderland lies a rich vein of subtle humor which is the brainchild of a mathematical genius. The author, the Reverend Charles Ludwidge Dodgson, was a mathematical lecturer at Oxford and a man of rare scientific ability. His unique character makes his biography more than ordinarily

entertaining reading. The mathematician who reads this book with the realization that the background is mathematical will find a veritable storehouse of witty foolery which escapes the average reader.

5. In considering the normals to the ellipse two methods of attack were used: the application of Descartes' rule to the equation of the normals, and the consideration of that equation through delta, the discriminant function. The second method introduced a boundary curve which is a hypocycloid and which is of great assistance in determining the number and kind of normals that can be drawn to the ellipse from various portions of the plane.

6. The speaker told of his recent trip to the Near East, of the universities and mathematicians he visited there, and of the interest in the subject throughout that region. He also exhibited some books and manuscripts collected for the library of the University of Michigan on this trip.

8. The paper deals with the Inductive Principle for the general linear order. The essential relations of this principle with the existence theorems of analysis—such as the Dedekind Cut theorem, the Borel covering theorem, the Lebesgue chain theorem and the method of nets—are indicated, and it is pointed out that the point of view represented in the current literature is inadequate and at times untenable. Stress is laid on the systematic genesis of the various methods, and comparison is made of these methods as to such genesis. The ideas are extended to n-tuple order.

9. A graphical method is applied to a study of the functions and capabilities of a group of academic workers such as a mathematics department. A job chart may set forth the main duties of the various teachers and a talent chart their different capabilities when applied to different types of work. Thus the most probable distribution of the attention and ability of a department's teachers under a *laissez faire* policy can be calculated. Also the most efficient distribution of their interests and energies may be readily calculated.

10. The determination of demand functions has been a very difficult problem and most attempts in this direction have resulted in unsatisfactory functions in the sense that they did not represent the data satisfactorily or else this difficulty had been overcome by introducing time as a variable. This is an admission that some of the important economic factors are not well enough known to be studied explicitly. Recently two studies in demand with neither of these two faults have been presented. The first is a paper by C. F. Roos and V. Von Szeliski in which the demand for housing construction was determined as a function of interest rates, rents, building costs, and population. The second is a study by Victor Perlo and C. F. Roos in which the demand for gasoline was determined as a function of price, automobile registration, highway mileage, consumer purchasing power, and gasoline tax. A study of a different nature has been made by H. H. Pixley and C. F. Roos on the possibility of maximizing the profit from the sale of two commodities by selling one of them at a loss.

W. L. AYRES, *Secretary*

THE MARCH MEETING OF THE LOUISIANA-MISSISSIPPI SECTION

The eleventh annual meeting of the Louisiana-Mississippi section of the Mathematical Association of America was held at Jackson, Mississippi, on Friday and Saturday, March 23–24, 1934, Professor P. K. Smith presiding.

The attendance was eighty including the following twenty-two members of the Association; T. A. Bickerstaff, Leora Blair, H. E. Buchanan, Arnold Dresden, W. L. Duren, Margaret Harris, May Hickey, Ruth Johnson, C. G. Killen, Dorothy McCoy, Janet McDonald, R. L. Menuet, B. E. Mitchell, I. C. Nichols, W. V. Parker, S. T. Sanders, C. D. Smith, H. L. Smith, P. K. Smith, V. B. Temple, J. F. Thomson, B. A. Tucker.

The following officers were chosen for next year; Chairman, T. A. Bickerstaff, University of Mississippi; Vice-Chairman for Mississippi, D. S. Dearman, Mississippi State Teachers College; Vice-Chairman for Louisiana, V. B.Temple, Louisiana College. Resolutions adopted included thanks to the Mathematical Association of America for sending President Arnold Dresden to speak to us. The next meeting will be held in March 1935 at Louisiana College.

"The Mathematical Association of America and American mathematics" was Professor Dresden's subject at the annual dinner on March 23.

The following papers were presented:

1. "Trends of higher education" by Professor C. D. Smith, Mississippi State College.
2. "Combinations of abstract spaces" by Professor Dorothy McCoy, Belhaven College.
3. "On the degree of the highest common factor" by Professor W. V. Parker, Mississippi Woman's College.
4. "Some aspects of the calculus of variations" by Professor Arnold Dresden, Swarthmore College.
5. "Certain problems on characteristic exponents in the problems of mechanics" by Professor H. E. Buchanan, Tulane University.
6. "Problems of the calculus of variations with higher derivatives in the integrand" Professor W. L. Duren, Tulane University.
7. "A generalized helium atom problem" by Doctor J. F. Thomson, Tulane University.
8. "On a special type of problem of pursuit" by G. F. Cramer, Tulane University, by invitation.
9. "On a unifying theorem in Modern Geometry" by Professor C. D. Smith, Mississippi State College.

The papers by Professors Buchanan and Duren and by Dr. Thomson were read by title only. Abstracts of the papers follow, the numbers corresponding to the numbers in the list of titles:

1. The committee surveyed the educational system of England, France, Germany, and other large countries. A survey of the New Deal in America was

reported, indicating the trend toward national cooperation. A suggested program for a national education commission was set up.

2. A set of elements with a relation between the subsets is an abstract or topological space. The properties of a space obtained by adding two abstract spaces was discussed. For example the new space is not connected, and is monotonic provided both original spaces are. Interesting results were noted by combining spaces by other methods.

3. It is known (Frobenius, Journal für Mathematik, vol. 84, p. 11) that if $f(x)=0$ is the characteristic equation of a square matrix A, and $g(x)$ is any polynomial, the resultant of $f(x)$ and $g(x)$ is $R(f, g)=|g(A)|$. Wedderburn has shown (Annals of Mathematics, vol. 27 (1926), p. 247) that

$$f(x) = x^n + a_1x^{n-1} + a_2x^{n-2} + a_3x^{n-3} + \cdots + a_{n-1}x + a_n = 0$$

is the characteristic equation of the square matrix

$$A = \begin{Vmatrix} -a_1 & -a_2 & -a_3 \cdots & -a_{n-1} & a_n \\ 1 & 0 & 0 \cdots & 0 & 0 \\ 0 & 1 & 0 \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 \cdots & 1 & 0 \end{Vmatrix}.$$

By taking A to be a matrix of this type, the following theorem is proved in this paper.

Theorem: *If $f(x)$ is the characteristic function of the square matrix, A, of order n, and $g(x)$ is any polynomial, and r is the rank of the matrix $g(A)$, the highest common factor of $f(x)$ and $g(x)$ is of degree $n-r$.*

4. In this paper various problems of the calculus of variations were presented in a formulation which tended to bring out unifying elements among problems in spaces of two and more dimensions. Different methods of solution were presented in outline, the classical method as applied to various problems and the direct method of Tonelli, in connection with which the significance of lower semi-continuity was brought out.

8. The author considers a plane circular enclosure, E, containing no obstructions, and an object, A, moving with constant speed, k, along a continuous plane curve, C, contained within E and subject to the following conditions; (1) The curve C has at most a finite number of points of inflexion in any arc of finite length, (2) The curve has a continuously turning tangent line except possibly at a finite number of points in any arc of finite length. Object A is pursued by an object B whose speed is also equal to k. We choose a system of polar coordinates with pole at center of E and then show that B can always stay on the same radius vector with A after having proceeded first to the center of E. We also show that, if B uses this method of pursuit, there exists a finite time, t, in which the distance from A to B will become smaller than any preassigned positive constant. Hence, capture will always occur if A and B are objects whose dimen-

sions are all >0. It is conjectured that the ordinary method of pursuit in which B moves directly toward A at all times would result in even quicker capture.

9. A theorem of Miquel states that if a point be selected at random on each side of a triangle, and if a circle be drawn through two such points and the vertex of the angle on whose sides they fall the three circles thus determined have a point in common. Certain fundamental observations were given and the problem of the three circles was used to show how the theorems of Ceva, Menelaus, Wallace, Lemoine and Brocard are developed in a very simple way. Hence the theorem yields a unifying principle for important theories in modern geometry. Developments by this method are being continued with interesting consequences.

DOROTHY McCOY, *Secretary*

THE NINETEENTH ANNUAL MEETING OF THE OHIO SECTION

The nineteenth annual meeting of the Ohio Section of the Mathematical Association of America was held at the Ohio State University, Columbus, Ohio, April 5, 1934, with an afternoon session, a dinner, and an evening session. Professor I. A. Barnett, Chairman of the Section, presided at these sessions. Professor K. P. Williams of the University of Indiana was a guest of the Section.

Eighty persons registered attendance at the meeting, including the following forty-nine members of the Association: R. B. Allen, W. E. Anderson, A. H. Bailey, I. A. Barnett, P. E. Baur, H. M. Beatty, Henry Blumberg, M. G. Boyce, J. B. Brandeberry, R. S. Burington, E. H. Clarke, T. F. Cope, F. F. Crandell, Rufus Crane, Wayne Dancer, O. L. Dustheimer, P. S. Dwyer, F. J. Feinler, T. M. Focke, B. C. Glover, Harris Hancock, E. J. Hirschler, F. C. Jonah, Margaret Jones, H. W. Kuhn, Florentina Mathias, R. H. MacCullough, C. C. Morris, Max Morris, J. R. Musselman, R. L. Newlin, J. R. Overman, Jesse Pierce, H. S. Pollard, S. E. Rasor, C. E. Rhodes, Hortense Rickard, H. P. Rogers, S. A. Rowland, Mary E. Sinclair, S. A. Singer, Ruth B. Smyth, H. E. Stelson, C. F. Thomas, J. H. Weaver, R. B. Wildermuth, K. P. Williams, C. O. Williamson, B. F. Yanney.

The following officers were elected for the coming year: Chairman, Henry Blumberg, Ohio State University; Secretary-Treasurer, Rufus Crane, Ohio Wesleyan University; Member of Executive Committee, H. W. Kuhn, Ohio State University; Member of Program Committee, J. B. Brandeberry, Toledo University.

It is expected that the next meeting of the Section will be held at the Ohio State University on Thursday, April 4, 1935.

The following papers were read:

1. "Some suggestions for the improvement of the teaching of mathematics" by the Chairman of the Section, Professor I. A. Barnett, University of Cincinnati.

2. "The Hiram College concentration plan, as it affects mathematics" by Professor E. H. Clarke, Hiram College.

3. "A simple mathematical method in the study of a problem in crystal analysis" by Dr. C. E. Rhodes, University of Cincinnati.

4. Brief notes by several members.

5. "The problem of professional training" by Professor K. P. Williams, Indiana University, by invitation of the Program Committee.

6. "What is advanced mathematics?" by Professor Henry Blumberg, Ohio State University.

Abstracts of these papers follow:

1. If there is to be any improvement in the teaching of mathematics in this country, it must come through the efforts of the mathematicians themselves. Some of the larger universities are employing specialists in mathematics, who are giving adequate training to the future teachers of mathematics. In most colleges, this is not the case. For these it is proposed that the mathematics departments of the Liberal Arts Colleges shall offer two courses for students who plan to teach.

In the first course, which is primarily for those who have no special interest in mathematics, but who plan to teach in the elementary schools, a student should have the opportunity to learn some of the fundamentals of arithmetic and algebra, which he may be called upon to teach in the first eight grades. The emphasis here should be on the presentation by the students themselves of the more important topics of arithmetic and algebra. Such a presentation will of necessity bring out in sharp relief the meanings of the concepts and operations. Along with this, enough of the historical background should be given to show the student how the mathematical science has developed. This course should be given in the freshman year for two semesters, three meetings a week.

The second course should be required of every mathematics major, and should be a review of the high school and college mathematics through the calculus, which he will eventually be called upon to teach. This review should not in any sense be a repetition of any course, but rather a presentation of actual topics by students of the class. Such presentations will bring out the precise meanings of the concepts, and the student will have a chance to obtain a grasp of the fundamentals of algebra, geometry, and the calculus, which will be invaluable to him in his later work. This course should come in the senior year, and should take two semesters, meeting twice a week.

2. After three years of experimentation in the summer school, Hiram College has begun a plan of concentrated study, by which a student will be able to devote four-fifths of his time to the study of a single subject. Students with more than average ability are encouraged to do work of greater than ordinary difficulty, and have more opportunity to confer with their instructors. It will also be possible for the small college to offer a somewhat more flexible and possibly richer undergraduate sequence of studies. In mathematics, Professor Clarke has found that students of various types have shown a steadier interest and have carried over at least as great a facility and understanding of the subjects

studied under the concentrated study plan as under the usual plan of three-hour courses carried through two semesters.

3. An upper limit for the number of lines in an X-ray diffraction spectrum produced by a powdered crystal (Debye method) is given approximately by the volume of a certain ellipsoid whose equation is $Q(x, y, z)=1$. The coefficients of Q are functions of the edges and angles of the unit cells which make up the crystal. The volume of the ellipsoid is shown to be $32\pi V/3\lambda^3$, where V is the volume of the unit cell, and λ is the wave length of the X-ray. The problem furnishes an interesting physical application of solid analytic geometry.

4. Professor J. B. Brandeberry of Toledo University presented a summary of the results of a survey of the opinion of the students of Toledo University concerning the instruction that they are receiving, and showed what Toledo students think about mathematics.

Professor E. J. Hirschler of Bluffton College discussed a graphical approach to imaginary loci for beginners, beginning with the consideration of ordinates corresponding to real values $|x|<a$ in $b^2x^2-a^2y^2=a^2b^2$.

Professor F. C. Jonah of Western Reserve University discussed an operational formula in differential equations and presented an extension of one of the methods used by Hutchinson and Dow in notes in the October, 1933, and February, 1934, issues of the MONTHLY.

Professor C. F. Thomas of the Case School of Applied Science presented some specific suggestions as to those ideas which should be emphasized in presenting maxima and minima, related rates, curvature, and kindred subjects, to undergraduate students in calculus.

5. The extensive and rigid requirements in so-called "professional" subjects, against which there is so much complaint from teachers of academic subjects, are due, in part at least, to failure of university and college teachers to appreciate fully the problems of high school instruction. The bearing of advanced mathematics on more elementary instruction is not always so very clear, and mathematics departments have not dealt with it adequately. If a state, for instance, requires the study of calculus for a high school license to teach mathematics, instructors in calculus should leave their students with the conviction of the soundness of the requirement. Mathematics departments should provide adequately for the problem of high school teachers at the same time that they attack the trivialities in many Education courses and the foolishness of certain professional requirements.

A large part of the paper was devoted to an explanation of a systematic effort being made in Indiana to have "professional" requirements lowered.

6. Professor Blumberg discussed a variety of important attributes of "advanced mathematics," considered the hindrances to advance, and gave a list of brief characterizations of "advanced mathematics" in the form of paradoxes.

RUFUS CRANE, *Secretary*

THE APRIL MEETING OF THE IOWA SECTION

The twenty-third meeting of the Iowa Section was held at Drake University, Des Moines, Iowa, on Friday and Saturday, April 20–21, 1934 in conjunction with the meetings of the Iowa Academy of Science.

The attendance was fifty-three, including the following thirty members of the Association: F. A. Brandner, E. W. Chittenden, L. M. Coffin, N. B. Conkwright, A. T. Craig, Marian E. Daniells, C. W. Emmons, Annie W. Fleming, Cornelius Gouwens, M. E. Graber, Gertrude A. Herr, Dora E. Kearney, O. C. Kreider, F. M. McGaw, J. V. McKelvey, Martha M. McKelvey (Mrs. J. V.), E. E. Moots, I. F. Neff, Arthur Ollivier, J. F. Reilly, H. L. Rietz, Fred Robertson, W. J. Rusk, E. R. Smith, G. W. Snedecor, J. S. Turner, L. E. Ward, J. J. Westemeier, C. W. Wester, Roscoe Woods.

The Section Chairman, Professor J. F. Reilly, presided at both the Friday afternoon and the Saturday morning sessions, relieved for a time by Professor J.V. McKelvey. Dinner was enjoyed together with the members of the Physics Section of the Iowa Academy of Science on Friday Evening in the University dormitory. The officers elected for 1934–35 are as follows: Chairman, M. E. Graber, Morningside College; Vice-Chairman, E. E. Moots, Cornell College; Secretary-Treasurer, Cornelius Gouwens, Iowa State College. A resolution was adopted expressing the appreciation of the members of the section for the hospitality and courtesy extended to them by their host, Drake University, and particularly by Professor I. F. Neff, who did so much to make this meeting a happy occasion and a success.

The following fifteen papers were read:

1. "The use of expected numbers in analysis of variance with disproportionate sub-class numbers" by Professor G. W. Snedecor, Iowa State College.
2. "Certain sets of two-squares" by Professor W. J. Rusk, Grinnell College.
3. "The evolution of the function concept" by Professor M. E. Graber, Morningside College.
4. "History of the notation of mathematics" by Catherine Walliker, Wartburg College, by invitation.
5. "Iterated functions for the characteristic vibrations of a clamped beam" by Professor D. L. Holl, Iowa State College, by invitation.
6. "Computation of annuities with nominal interest rates," by Professor J. F. Reilly, University of Iowa.
7. "The International Mathematical Congress at Zurich" by Professor J. S. Turner, Iowa State College.
8. "Applications of elliptic functions to certain problems in plane cubics" by Professor Marian E. Daniells, Iowa State College.
9. "Mechanical grade averagers" by Karl E. Gaylord, Colo High School, by invitation.
10. "Convergence of series defined by recurrence relations" by C. C. Hurd, Iowa State College, by invitation.

11. "The circular solutions of the two body problem in electrodynamics" by Professor J. S. Turner, Iowa State College.

12. "Conic sections in the elliptic plane" by Dwight Goodner, Penn College, by invitation.

13. "The dilution method for bacterial census" by Professor E. S. Allen, Iowa State College, by invitation.

14. "An elementary method for the computation of logarithms" by Professor E. W. Chittenden, University of Iowa.

15. "The effect of increasing the size of samples in certain distributions" by Professor F. A. Brandner, Iowa State College.

Abstracts of the papers follow, the numbers corresponding to the numbers in the list of titles:

1. A method is proposed for estimating the main effects and interactions in n-way tables with disproportionate sub-class numbers. Such numbers are replaced by proportional expected numbers derived from the border frequencies. The sub-class means remain unchanged, but the means for the main effects are altered in accordance with the weighting of the expected sub-class numbers. The analysis of variance then proceeds as usual, yielding all estimates and tests of significance. The proposed method is appropriate if the assumption is made that the disproportionate sub-class numbers constitute a random sample from a population with proportional sub-class numbers. The assumption may be tested by means of chi-square. In tables with large numbers of sub-classes, the computation is much simpler than that required by the method of fitting constants.

2. By a simple re-arrangement of $u_1^2=x_1^2+y_1^2+z_1^2$, and $u_2^2=x_2^2+y_2^2+z_2^2$ it is proved that certain lineo-linear sets of their elements are two-squares. One such set would be $u_1u_2\pm x_1y_2\pm y_1z_2\pm z_1x_2$. From these any lineo-linear combination containing u_1u_2 can be expressed as a four-square, and any lineo-linear combination not containing u_1u_2 as a two-square minus a two-square. If $u_1=u_2$ then the above set when multiplied by two yields the theorem that $(x\pm y)^2+(y\pm z)^2+(z\pm x)^2=$ a two-square where $u^2=x^2+y^2+z^2$.

3. The unifying concept of mathematics relating to value, form, and formulae is that of function. By its use and operation, the most heterogeneous parts of mathematics are brought into connectivity and unity. The germ of the integral calculus and the function concept is to be found in the works of Archimedes. It is also implied in Descartes's Géometrie, but it seems to be agreed that we owe the term function to Leibniz. From this point on, the concept has undergone an evolution at the hands of Bernoulli, Euler, Lagrange, Fourier, Weierstrass, and Dirichlet. From the logical frame of reference, Russell and Peano have developed a comprehensive conception of propositional function based on some 20 undefined elements. For practical purposes of instruction in collegiate mathematics, a variable may be considered as an aggregate of mathematical quantities enumerable or of the order of the continuum. Two variables, x and y, are in functional relation when as x varies over its domain there is a definite correspondence between its values and the values assumed by y in its

domain; the order of arrangement of x and y being preserved. Professor P. W. Bridgman of Harvard questions the methods of analysis which attribute functional relationships to infinite classes in number set theory, and points out inconsistencies in the fundamental notion of mathematics. Professor E. T. Bell also questions the uniqueness of the results of functional analysis where only statistical inferences may be drawn. A re-evaluation of function theory as applied to mathematical physics seems to be in order at this time.

4. A survey is made of the origin and development of the three types of mathematical symbols: (1) picture symbols; (2) abbreviation of words; (3) ideographic or arbitrary symbols. It is observed that notation always grows too slowly; we should assist in hastening improvements.

5. The characteristic frequency numbers, corresponding to the normal modes of the free vibrations of a uniform beam which is clamped at the ends and supports a fixed mass within the span, are the roots of a complicated transcendental equation. Rayleigh's energy method is employed to approximate these roots by constructing iterated functions satisfying the boundary conditions and generated by the relation

$$f_{n+1}(x) = \int_0^a \sigma(s) f_n(s) G(x, s) ds,$$

where σ is given, $f_0(x) = 1$, and $G(x, s)$ is the Green's function for a clamped beam.

6. In computing the present value or the amount of an annuity, certain tables are available which enable us to obtain the result easily when either m/p or p/m is an integer, where the annuity is payable in p instalments annually, and where the interest rate is nominal convertible m times a year. In his paper Professor Reilly indicates how the existing tables can be extended to apply to cases in which neither m/p nor p/m is integral.

7. The subject of the Friday evening after-dinner address by Professor J. S. Turner was "The International Mathematical Congress at Zurich." The speaker gave an interesting account of his trip and impressions of the Congress.

8. The chief facts developed in this paper are; (1) the possibility of expressing the coordinates of a point on a cubic as a function of a single parameter by means of the Weierstrass "p" function; (2) the condition for three points of a cubic being on a straight line; and (3) the condition for $3n$ points of a cubic being on a curve of order n. These principles are applied to the solution of several well known problems.

9. The author discusses two difficulties of a mathematical nature which confront every teacher: (1) How may the large amount of time and energy required for accurately averaging grades be reduced? (2) In schools where grades are indicated by letters A, B, C, etc., how may such grades be averaged quickly, easily, and accurately? This may be simply done by means of a mechanical contrivance easily made at home. Two types of mechanical averagers are exhibited. A number of possibilities for additional developments and improvements are suggested.

10. The object of this paper is to obtain a method for testing the convergence of a series whose coefficients g_i are determined by the relation

$$g_{i-1} = A_i g_i + B_{i+1} g_{i+1}.$$

11. Let E be a sphere of mass m_1, radius ϵ, superficial charge $-e$, rotating with angular velocity Ω_1. Let P be a sphere of mass m_2, radius ϵ_2, superficial charge e, rotating with angular velocity Ω_2. Let E, P describe circles of radii a_1, a_2 and fixed center O, with angular velocity ω, the center of gravity of E and P being at O. Assume that E, P act on each other according to the laws of classical electrodynamics. There are solutions if and only if, to a close approximation

$$\frac{\epsilon_1}{m_1} = \frac{\epsilon_2}{m_2} = \frac{2e^2\sqrt{2}}{c^2}, \quad 2(a_1 + a_2)a_1\omega\Omega_1 = 2(a_1 + a_2)a_2\omega\Omega_2 = c^2.$$

12. Standard forms of equations of the conic sections in the elliptic plane are formed by elementary methods. These are found to be similar to the usual forms in the parabolic plane when the center of the conic is at the origin. The Klein transformation carrying points from the elliptic to the parabolic plane is applied and some interesting figures are obtained.

13. The dilution method in bacteriology aims to determine the density of micro-organisms in a substance, from the observation of their mere presence or absence in successively smaller samples of the substance. The mathematical treatment uses inverse probability. The present paper considers the conditions underwhich this is justifiable, and also investigates the probable error when these conditions are fulfilled. For instance 5% accuracy requires the use of some thousand samples.

14. The rule employs only the operations of squaring and division by the base and yields a succession of binary approximations to the desired logarithm. It can be presented to classes in college or even third semester algebra, and is chiefly of pedagogical interest, because it is not practical for use in the computation of logarithms beyond the third decimal place.

15. The significance of the differences of frequencies in an A by B table is known, expressed as a probability. An expression is evolved which shows the relation between this probability and one for which each cell of the table is multiplied by an integral value K.

CORNELIUS GOUWENS, *Secretary*

THE ELEVENTH ANNUAL MEETING OF THE NEBRASKA SECTION

The eleventh annual meeting of the Nebraska Section of the Mathematical Association of America was held in conjunction with the annual meeting of the Nebraska Academy of Sciences at Doane College, Crete, Nebraska, Friday afternoon, April 27, 1934, with Professor J. M. Earl of the University of Omaha as chairman.

There were ten visitors present as guests and the following ten members of the Association: A. K. Bettinger, Jessie W. Boyce, C. C. Camp, A. L. Candy, J. M. Earl, J. D. Fitzpatrick, A. L. Hill, E. Marie Hove, J. M. Howie, T. A. Pierce. Officers for the ensuing year were elected as follows: Chairman, A. L. Candy, University of Nebraska; Secretary-Treasurer, J. M. Howie, Nebraska Wesleyan University; Member of Executive Committee, J. M. Earl, University of Omaha.

The following program was presented:

1. "A modern modification of Napier's Rabdologia" by W. N. Halsey, Omaha, by invitation.

2. "Mathematics in Nebraska colleges" by Professor A. L. Hill, State Teachers College, Peru.

3. "On a number of magic squares by the method of current groups" (A continuation) by Professor A. L. Candy, University of Nebraska.

4. "Approximate factors" (Read by title) by Professor H. C. Feemster, York College.

5. "A solution of the heat equation" by Professor A. K. Bettinger, Creighton University.

6. "On the order of summability of a series of Legendre polynomials" by A. P. Cowgill, University of Nebraska, by invitation.

Abstracts of these papers follow according to the numbering above:

1. Baron John Napier's "Stick-calculator" was a great contribution and aid to accountants, tradesmen and calculators of the sixteenth and seventeenth centuries. The device was an adaptation of an invention of the Hindus passed on through the Persians and Arabs to European arithmeticians. Napier's age called for fuller and more rapid skill in reckoning, because of the developments in astronomy, geographical discovery, navigation, trade-routes, markets and standards of living. Multiplication and division were not taught to children in the schools and few adults knew the multiplication tables. The Rabdologia was a practical contrivance for calculating without knowledge of the tables of multiplication. The book of instructions and explanation of the device was sold in several editions throughout Europe in English, Italian, Dutch, German and French, and was in general use for more than 100 years. The present methods of multiplication and division, dependent on memorizing tables, came into vogue because the device of Napier was too limited, in spite of its usefulness.

The Kalc is a simple modification made during the past decade which makes possible the building of any product without referring to memorized tables. The five-column Kalc,[1] weighing less than half an ounce, contains the equivalent of 900,000 products, which if printed would fill 1,000 pages of 900 products to a page, a volume as large as a school dictionary. Long division and "root" problems as well as multiplication are facilitated by its use. It is the speaker's belief, after employing this method of calculation for more than ten years, that there is still a large field of usefulness for the improved device.

[1] For information address Mr. W. N. Halsey, 5320 N. 28th Ave , Omaha, Neb.

2. Professor Hill presented the results of a questionnaire which summarized the various kinds of courses being taught in Nebraska colleges, the number of colleges teaching the courses and the textbooks in general use. Objectives as determined by the texts used were discussed. The general status of college mathematics for the entire country as described in current literature was reviewed and the following questions proposed for discussion:

(a) Are we, the members of the Nebraska section, ready to admit that what is being said generally in regard to mathematics teaching is true of our state?

(b) What constructive plans can we set into motion to remedy the situation?

3. After a more careful study of the method of "Current Groups," I find that there are 18 (instead of 3) possible "Order Diagrams" by means of which the columns and rows can be balanced in the 6×6 square. Using all of these I get 60 normal 6×6 squares (instead of 12). These in turn—by selecting the groups for the 12×12 squares in 36 different ways—give me 768 normal 12×12 squares, that is, about six times as many as I reported last year.

Furthermore, I find that any Order Diagram can be rotated through ninety degrees counter-clockwise, and the new square (location of groups not changed) will be balanced as to rows and columns precisely as the normal square is balanced as to columns and rows. Hence, by rotating the Order Diagrams of both the 6×6 square, and the 12×12 square we can get four 12×12 squares from any given 12×12 square.

As a result of these two discoveries I now have 24 times as many 12×12 squares as I reported one year ago. Thus making a grand total of more than 9600 trillions.

Also, by using all the above mentioned 18 Order Diagrams, and their rotated forms, and interchanging pairs of numbers at the ends of middle rows and middle columns, I have actually written down 288 different 6×6 squares. By reversing, and inverting, one or more pairs of symmetric groups in each of these, I get a grand total of 29,568 6×6 squares.

5. Professor Bettinger discussed the solution of the differential equation of heat conduction in the cases of linear and radial flow. Illustrations were given showing how the Dirichlet condition in a finite interval is satisfied in applied problems.

6. In this paper it is proved that $\sum_{n=1}^{\infty} n^p X_n(x)$, where $X_n(x)$ is a Legendre polynomial and p a positive integer, is summable (C, k), $k > p - \frac{1}{2}$, $-1 < x < 1$. It had been previously proved summable $k = p$. In the proof the sum of n terms of the given series is transformed by the recursion formula for Legendre polynomials into a new sum of n terms plus four additional terms. Convergence factors for summability $(C, p-1)$ are applied. This causes the highest ordered part of the sum of the two additional terms involving n to take the form of a series

$$(n + 1)\cdot(X_n - X_{n-1}) = (1 - x)\sum_{r=0}^{n}(2r + 1)X_r,$$

by a formula due to Christoffel. Chapman proved $\sum_{r=0}^{\infty}(2r+1)X_r$ was summable (C, j), $j>\frac{1}{2}$, to the value zero, so the second application of summability (C, j) necessary to evaluate the additional terms causes the total order of summability to be $k>p-\frac{1}{2}$.

J. M. HOWIE, *Secretary*

THE MAY MEETING OF THE MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA SECTION

The May meeting of the Maryland-District of Columbia-Virginia Section of the Mathematical Association of America was held at the College of William and Mary in Williamsburg, Virginia, on Saturday, May 12, 1934.

The annual Fall meeting will be held at the Johns Hopkins University on Saturday, December 8, 1934.

The following officers for the year 1934–1935 were elected: Chairman, F. M. Weida, George Washington University; Secretary, John Williamson, Johns Hopkins University; additional members of Executive Committee, J. M. Stetson, College of William and Mary, and C. H. Wheeler, University of Richmond.

Forty-nine persons attended the meeting, including the following thirty members of the Association: O. S. Adams, J. W. Blincoe, W. E. Bryne, Emily E. Calkins, G. R. Clements, Abraham Cohen, Alexander Dillingham, J. L. Dorroh, Mary Ewin, R. E. Gaines, Isabel Harris, F. E. Johnston, L. M. Kells, C. H. Keulegan, W. D. Lambert, B. Z. Linfield, J. J. Luck, F. D. Murnaghan, H. A. Perkins, W. T. Puckett, Beulah Russell, J. M. Stetson, J. H. Taylor, John Tyler, F. M. Weida, C. H. Wheeler, G. T. Whyburn, John Williamson, Oscar Zariski, R. T. Zoch.

Professor J. H. Taylor of George Washington University was the invited speaker. He addressed the afternoon session on "Some remarks concerning Finsler geometry."

The following papers were presented at the morning session:

1. "An algebraically complete system of Euclidean invariants for the ternary quartic" by J. W. Blincoe, University of Virginia.
2. "Elementary introduction to quaternions" by Professor F. D. Murnaghan, Johns Hopkins University.
3. "Some interesting features of frequency curves" by R. T. Zoch, George Washington University.
4. "A theorem on the resultant of two polynomials" by Oscar Zariski, Johns Hopkins University.
5. "An irreducible complete system of Euclidean concomitants for three points" by W. T. Puckett, University of Virginia.
6. "The torsion in bars of parallelogram cross section" by G. H. Keulegan, U. S. Bureau of Standards.

Abstracts of some of the papers follow, the numbers corresponding to the numbers in the list of titles:

1. An algebraically complete system of Euclidean invariants for the ternary quartic consists of twelve invariants. The twelve invariants chosen to form the system may be expressed as functions of the coefficients of the ground form, or in the symbolism of Aronhold and Clebsch. A geometrical significance may be attached to the vanishing of each invariant.

3. It is well known that in the normal error curve the points of inflection are equidistant from the mode. However, it has never been pointed out that this is also a characteristic of all of the bell-shaped Pearson frequency curves. This fact can be most easily shown by placing the mode at the abscissa $x=0$. Many rough checks have been developed for use in applying the Theory of Least Squares. The latter part of this paper develops a rough check on the computation for use when fitting a Pearson frequency curve to a set of observations. No rough checks on computation are given in textbooks on Pearson's frequency curves.

5. This irreducible complete system is found to consist of ten invariants, eighteen covariants, seven contravariants, and four mixed concomitants. The vanishing of each member is interpreted. A set of twenty-seven independent syzygies is found connecting the thirty-nine irreducible concomitants.

6. The torsion in bars of parallelogram cross-section is considered from the so-called Baussinesq analogy existing between the torsion of bars and the viscous flow of tubes of the same cross-section as the bars. In the case at hand the problem of the viscous flow is solved with the aid of a theorem stating that for a given quantity of flow the rate of dissipation is minimum. This theorem is nothing other than a physical interpretation of Ritz's method for dealing with the Dirichlet problem of the plane. The solution is given as a summation of polynomials.

F. M. WEIDA, *Secretary*

MATHEMATICS CONFERENCE OF THE S.P.E.E.

In connection with the forty-second annual meeting of the Society for the Promotion of Engineering Education at Cornell University, Ithaca, New York, there was held a mathematics conference on June 19–20, 1934. The general topic of the conference was "Advanced Courses in Mathematics for Engineering Students." The three sessions of the conference were well attended and the discussions were very animated.

Thirteen papers were read, the first being an introductory paper by the chairman of the section, Professor R. S. Burington, Case School of Applied Science. This paper pointed out the fact that a syllabus on the mathematics needed by the engineer was prepared by a joint committee of mathematicians and engineers appointed in 1907; that this syllabus, which included only topics through the introductory course in calculus, was published by the Society for the Promotion of Engineering Education in 1912; that since the publication

of this syllabus engineering science has expanded to the point where topics in mathematics beyond the elementary course in calculus seem to be a necessary part of the equipment of the engineer of today; and that the purpose of the conference was to consider the topics needed, where such topics should be included in engineering curricula, and how and by whom they should be presented.

The topics needed and used in industry were considered by representatives of a number of industries. Dr. Theodore Theodorsen, N.A.C.A., Langley Field, Va., showed how the properties of conformal transformation, Bessel's functions and integral equations enabled the aeronautical engineer to solve the problems of air flow around airplane wings, and emphasized the need for a more thorough training in theoretical analysis as opposed to teaching a mass of detached facts in a multitude of subjects. W. R. Griswold of the Packard Motor Company pointed out the fact that courses in advanced dynamics, theory of vibrations and others of like nature were becoming more and more necessary in the work of the automotive engineer. A. M. Dudley of the Westinghouse Electric and Manufacturing Company listed certain topics that the electrical engineer should know, such as differential equations, theory of functions of a complex variable, symmetric coordinates, etc., but laid greater emphasis on training the student of engineering to apply his mathematical knowledge to the solution of the numerous practical problems with which he is confronted. S. Dushman of the General Electric Company stressed the value of mathematics in setting in order the experimental facts of science, thus enabling the scientist to generalize and increase the efficiency of his experimental observations. To illustrate his point he referred to the kinetic theory of gases, which by mathematical generalization has led into such fields as theories of conduction in gases and metals, statistical mechanics and quantum theory. P. L. Alger of the General Electric Company emphasized the need for mathematical research which develops new symbolisms to express in simple form highly involved problems; which produces new methods for handling equations unsolvable by existing methods; and better ways of getting approximate numerical results. In his opinion the application of tensor analysis or non-Riemannian dynamics to engineering problems offers the most fruitful field of future progress.

Professor J. B. Scarborough of the U.S. Naval Academy outlined the developments of engineering during the past thirty years and listed the mathematical topics necessary to adequately explain these developments. In his opinion about 75% of the students now taking engineering get about all the mathematics they need, but the other 25% should have the opportunity of taking courses in partial differential equations, theory of functions of a complex variable, Fourier series, theory of vibrations, elliptic functions, theory of potential, hydrodynamics, theory of elasticity, etc., and educational institutions should make provision for such courses.

The place where these topics should be included in the curriculum was discussed by Professor I. S. Sokolnikoff of the University of Wisconsin and Professor J. B. Reynolds of Lehigh University. Professor Sokolnikoff is of the

opinion that the first course in mathematics following the elementary course in calculus should be of the nature of a survey course and include as many topics as possible. This course could then be followed by more substantial courses determined by the needs and desires of the students. Professor Reynolds seems to think that, in general, there is no place for advanced courses in mathematics in existing engineering curricula and that there is not much hope of their being included in such curricula. His solution is the establishment of a curriculum in general engineering with specialization in mathematics. This curriculum would stress careful analytical thinking in a few fundamental mathematics courses rather than in a number of specialized courses.

How and by whom these topics should be presented was discussed by Dean Langsdorf of Washington University, St. Louis, Professor J. H. Weaver of the Ohio State University, and Professor Philip Franklin of the Massachusetts Institute of Technology. Dean Langsdorf believes that the teaching of mathematics should be colored by the spirit of applied science, but that the "color" should be used as a stimulant outside the main trend of thought, in order to make the problem a living issue in the minds of the students. Professor Weaver believes that the teacher of mathematics to engineering students should possess the following qualities:

(a) An unbounded love and enthusiasm for mathematics because of its beauty and the universality of its methods; and its usefulness when attacking practical problems.

(b) The qualities of a salesman, involving a knowledge of the future possible needs of the student and the usefulness of mathematics in satisfying those needs.

To supplement these qualities he should have a thorough training in mathematics, a broad training in science and some training in the applications of mathematics to problems in industry. Professor Franklin made a plea for the policy of fostering pure mathematics in technical institutions. He pointed out the fact that this policy has rich historical precedent, that it has immediate and ultimate effects on the advancement of technology, and that it makes for better teaching.

Professor A. A. Bennett of Brown University discussed the possibilities of mathematicians acting in a consulting capacity with industry. He discovered that there are no mathematicians in the United States whose primary profession is that of consultant in mathematics unattached to a commercial organization, and that mathematical consulting work seems to reduce to the sporadic assistance afforded by professional teachers of mathematics to inquirers and cranks, or to the regular services of employees, particularly evident in the cases of telephone and electrical manufacturing industries.

The conference was closed with a brief talk by Dr. T. C. Fry of the Bell Telephone Laboratories, summarizing the previous discussions.

J. H. WEAVER, *Secretary*

IS THERE A CRISIS IN MATHEMATICS?

WITH REFERENCE TO THE NOTION OF EXISTENCE AND A DOUBTFUL APPLICATION OF THE LAW OF EXCLUDED MEAN

By ROLIN WAVRE, University of Geneva[1]

"Men do not understand each other because they do not speak the same language and because there are languages which are not learned."

In saying that, Poincaré sought to describe the irreconcilable nature of that clash of temperament manifested in the very heart of mathematics at the appearance of Cantorian ideas and logistic. The clash has taken form under various captions.[2] Du Bois-Reymond called the divergent tendencies empiricist and idealist; Poincaré, pragmatist and realist; Brouwer, intuitionist and formalist. We shall here use intuitionist for the one, but according to circumstances, idealist, formalist, or even realist for the other.

The intuitionist is at present identified by his extreme caution. Being anxious to attain the greatest intelligibility and clarity, he challenges such propositions as Zermelo's axiom of selection or such reasoning as does not seem rigorous. The idealist claims to depart in nothing from the rigor which the intuitonist rightly demands, and this without evidencing the least distrust with regard to those modes of reasoning—even on the contrary conceding their perfect legitimacy and applying them literally and without restriction. Russell, Hilbert, Zermelo, and Hadamard are idealists; Lebesgue, Borel, and Baire are intuitionists.[3]

The clash is very much more evident now than in the past, as a result of the several publications, on the one hand of Brouwer and Weyl, two new and undeniably revolutionary intuitionists, and on the other of Hilbert. We should like to summarize here the essence of these publications and show how clear-cut the clash becomes in connection with the notion of existence and a doubtful application of the law of excluded mean. We shall see in particular why, and in what system of definition, the intuitionist denies us the right to say:

Two mathematical points are either coincident or distinct.

Two functions defined for the same values of the variable coincide for all those values or else there exists one for which they do not coincide.

That the intuitionist formulates such paradoxes in the name of truth seems curious. But the paradox is bound up with the words and not with the ideas.

Weyl, however, believes he has discerned at the basis of the theory of real numbers a vicious circle which endangers its value, and that he should therefore declare that mathematics is undergoing a crisis. Brouwer and Weyl have at-

[1] Summarized translation by Alice Ambrose, authorized by R. Wavre for publication in this MONTHLY. By permission of La Revue de Métaphysique et de Morale in which it appears, vol. 31 (1924), pp. 435–470.

[2] Poincaré, *Dernières pensées*, ch.V; Brunschvicg, *Les Étapes de la Philosophie mathématique*, pp. 527–537; Brouwer, "*Intuitionism and Formalism*," Bull. of the American Math. Soc., vol. 20, (1913), trans. by A. Dresden.

[3] Borel, *Introduction à la théorie des fonctions*, 1914, p. 150.

tempted to reconstruct the theory of sets and that of functions on new foundations, carefully avoiding the suspected law and vicious circle. In this they have carried distrust as to certain modes of reasoning, whose legitimacy has created no doubt for three centuries, further perhaps than the French school of the theory of sets.

Hilbert is not of the opinion that it is the turn of mathematics, after physics, to go through a revolution. Observing that the consequences of the idealistic attitude would never lead to the slightest contradiction, however far they be pushed, he considers that the principles suspected by the intuitionists, although not having perfect evidence, are nevertheless legitimate. Unlike Brouwer and Weyl, Hilbert does not believe in the necessity of reconstructing the foundations of mathematics, for he prescribes a radical remedy in order to prevent the crisis and to legitimate a frankly idealistic attitude. This means is the axiomatic method. In substance he does this: to the axioms accepted by the intuitionists, he adjoins a new proposition called the axiom of the transfinite which contains in itself all the doubtful principles united. And from this would be deduced the doubtful applications of the law of excluded mean. But to justify such a procedure, it must be demonstrated that this total system of axioms does not imply contradiction. By this artificial union of its idealistic foundations present-day mathematics would not forfeit the renown of being the discipline whose truth is above suspicion.

The Intuitionist Point of View

Mathematics and Logic

I do not conceal the difficulty encountered in trying to express the exact meaning of the relation Brouwer establishes between mathematics and logic. However, I shall try to set off the essentials in the thought of the great Dutch mathematician.[1] In 1907 Brouwer made bold to reverse the roles which Russell and Couturat would have had mathematics and logistic play. Instead of its being the second which accounts for the first, it is on the contrary the part of mathematics to comprise logistic and even traditional logic. The Aristotelian logic, born of natural classification, would be adequate to the theory of finite collections and would not go beyond it; it would be concerned exclusively with the relations of whole and part. *Its complete self-evidence is the cause of the a priori character conferred on it; but, taken in by this a priori character, we would have called traditional logic to a function it is incapable of exercising.*

Traditional logic can at most claim to conform discourse to rules, but language itself becomes more and more inadequate to the genuine understanding of the facts of present-day mathematics; thus the word "all," for example, despite the subtle distinctions of Russell,[2] has an unprecise meaning as soon as one is concerned with "all" the objects of an infinite collection. The mathe-

[1] See A. Dresden, "*Brouwer's Contributions to the Foundations of Mathematics,*" Bull. of the American Math. Soc., vol. 31, 1924.

[2] *The Principles of Mathematics*, Ch. V.

matical intuition is on the contrary, guarantee of the autonomy of this science, and is not bound to respect always and everywhere in its translation into discourse the rules of syntax which traditional logic prescribes. Logistics, on the other hand, would be only the algebra of the *language* by which the reasoning is translated. But a logical construction of mathematics independent of the mathematical intuition is impossible, according to Brouwer, because one would obtain thereby only a verbal construction irrevocably divorced from the science. Further, this would be a vicious circle, *for logic itself requires the fundamental intuition of mathematics.*

According to Brouwer the field of application of traditional logic would be limited to finite collections. The crux of the issue then is the right to reason on finite and infinite sets in the same manner. For Brouwer does not admit the law of excluded mean, expressed *in abstracto* and *universally* in the form: "A thing either is or is not" or "A proposition is either true or false."

The Question of Logic

The laws of contradiction and of excluded mean express two fundamental relations between an attribute A and its contradictory, *non-A*, both well defined relatively to the same object, such as even and odd for an integer. The law of contradiction forbids attribution of A and *non-A* at the same time to the same object. The law of excluded mean compels assertion of one or the other. They can be regarded as defining the words "well defined attribute" or "attribute *non-A*" or simply negation; bound up as they are with the definitions, there can be no question of doubting them. *The sole question at issue is to know when they are applicable and to what attributes.* In more precise terms, it is a matter of recognizing whether two distinct attributes A and B are such that one is entitled to make one the logical equivalent of the negation of the other. It is a matter of knowing, for example, whether for an integer the attribute of being factorable is well defined and whether the attribute of being prime is by definition equivalent to non-factorable. If it is by definition, these laws would apply with full right; if not, a special examination of the two attributes becomes necessary, and it is only in virtue of an indirect evidence that the laws do apply. In the numerous examples which mathematics furnishes of well defined attributes, or which one considers as such, *only an intuitive, extra-logical evidence permits us to consider two attributes A and B as the negation one of the other.*

The following example is suggestive and of great importance in the present issue: Let us call a *fundamental aggregate* a *sequence of integers*: 1, 3, 4, 5, 7, for example. Are the following propositions, which state an attribute of the sequence, related as A and *non-A*?

a. All the numbers of the sequence are odd.

b. There exists in the sequence an even number.

No one doubts that they are so related, but that is in virtue of indirect evidence. I run through the sequence; this done, at some determinate position I either have or have not encountered an even number, and I cannot at the same time

have found and not found one. I then have the right, from the point of view of their logical function, to assimilate a to *non-b* and b to *non-a*. The logistician, the formalist, makes "there exists" equivalent by definition to "not all." That is his right. But in doing so, he introduces a new axiom or a new definition.

To the question: Does there exist an even number in the sequence?, whether I refer to the intuitive sense of the word "exist" or to its logistic sense, I cannot refuse to answer with yes or with no; yes, if I affirm b, no, if I affirm a. Generalized, this becomes: 1. *To the question: Does there exist in a fundamental aggregate a number having a well defined attribute A?, I can only reply with yes or no.* This is the application of the law of excluded mean which we had in mind.

Likewise the following undoubted proposition, "The numbers of the sequence 1, 3, 4, 5, 7 which are odd form a new set 1, 3, 5, 7," becomes when generalized: 2. In a fundamental aggregate, a well defined attribute A suffices to characterize a sub-class of the elements which possess it. With Brouwer, we call this principle 2 Zermelo's axiom of inclusion. It states that in a sequence of integers a definition by intension is equivalent to a definition by extension. *The formalists believe that they have the right to apply principles* 1 *and* 2 *without any restriction to the case where the fundamental aggregate is composed of an infinity of elements, such as the natural series of positive integers. The intuitionists refuse this right.*

In exposition of the arguments for the respective positions, consider the following imaginary dialogue:

The idealist: Either there exists a factorable number in the sequence $m_n = 2^{2^{n+38}}+1$ $(n=1, 2, 3, \cdots)$ or else one such does not exist.

The Intuitionist: I can only say one exists by exhibiting such a number, say m_{1000}, which is factorable; and I can only deny it by deducing from the definition of the numbers m_n that they are all prime.[1] But I do not see that the rejection of one of the parts of the alternative compels me to affirm the other; as I cannot exclude *a priori* every *tertium*, I refuse to be reduced to your alternative.

The Idealist: Suppose I take successively the numbers 1, 2, 3, $\cdots$; then either I shall or shall not come across a number n giving rise to a factorable number m_n. My encounter either will or will not occur.

The Intuitionist: To deny that all the numbers m are prime does not imply that there exists a determinate one, e.g., m_{263}, which is factorable. And in order to be certain of not having met one, you must have exhausted the series of integers, and that you will never do. You will take several steps in the series, and can perhaps say one exists, but you will never take enough to make a denial. *I should not deny that the number does or does not exist if the existence of an object in an infinite set were a well defined attribute of the set; but existence is not a well defined attribute. There are perhaps several modes of existing.*

Let us introduce the two following propositions which state two attributes of the fundamental aggregate:

a. All the numbers of the aggregate possess the attribute A.

[1] Prime is here equivalent to non-factorable.

b. There exists a number of the aggregate possessing the attribute non-A. As soon as an infinity of objects is concerned, the word "all" is suspect to the intuitionist. The proposition a can only have the precise meaning which its demonstration confers on it. This meaning will vary according to the demonstration. The most complete meaning would be such that it would follow just from the definition of the numbers of the set that they all have the attribute A. The "there exists" can only have meaning if the object said to exist is actually found. "There exists a number having such an attribute" signifies for the intuitionist: "Here is a number possessing this attribute." "All" from the formalist and logistic point of view means: and this, and this, . . . and this. Likewise, "there exists means: or this, or this, . . . or this. They are logical product and sum. But if the fundamental aggregate is infinite, the handling of such product and sum requires precautions as to convergence analogous to those with the infinite products and series of analysis. Intuitionists (such as Weyl) seem to confer meaning only on the implication b implies *non-a*, which suffices to establish that propositions a and b cannot both be affirmed. But as the inferences *non-b* implies a and *non-a* implies b, maintained by formalists, are doubtful or no longer meaningful, they see no need of their being related by the law of excluded mean. Consider the intuitionist meaning of general and existential demonstrations and the fact that the expressions "not all" and "there does not exist" have a doubtful meaning, and one will perhaps no longer be surprised at the paradoxes of the most extreme intuitionists. Even Hilbert recognizes that these phrases are devoid of a clear and immediate meaning. *The intuitionists refuse to make an alternative of the affirmation of a universal affirmative or of a singular negative,* when an infinite class is in question; of saying, for example, *either all the integers of a class are factorable or else there exists a determinate one which is prime.*

Here are two examples illustrative of Brouwer's attitude:

1. Let $m=\phi(n)$ and $m'=\phi'(n)$ be two laws of correspondence between a positive integer n and two integers m and m'. Let us say that the two functions ϕ and ϕ' are identical if the numbers m and m' are equal for all values of n, and that the two functions are different if there exists a value of n giving two unequal numbers m and m'. To demonstrate the identity of the two functions, we should have to be able to reduce the one to the other algebraically or analytically; to demonstrate their difference we should have to discover a number n giving rise to two distinct numbers m and m'. Considering what such discovery entails, it perhaps will not seem surprising that the functions are not *a priori* identical or different.

2. Fermat's theorem that the sum of two nth powers of two positive integers is never equal to the nth power of another integer as soon as n is greater than 2 has no known demonstration. One is tempted to say: if the proposition is false, I can assure myself of it by a finite number of trials on the integers. For then, there exist three integers and a power larger than 2 giving rise to the equality, and I can order my experiences in one series in such a way as to be certain of

finding the numbers in question within a finite range in the series.[1] Brouwer absolutely refuses to argue this. For him, even the demonstration that Fermat's theorem led to a contradiction would not imply the existence of four numbers invalidating it.

The crux of the intuitionist thesis is the meaning to attribute to the existential judgment and the general judgment. For Weyl, a true judgment is the attribution of a predicate to a singular subject. Where the formalist is content to affirm some sort of ideal existence of an object possessing such an attribute (e.g., there exists an even number), the intuitionist requires the discovery of such an object as will enable him to replace the existential judgment by a true judgment (the number 4 is even).

As Lebesgue has already said, we can only prove the existence of a mathematical entity by constructing it. The "there exists" would be only a check, without value in itself, so long as we cannot find the bank. And perhaps we shall never find it to convert its nominal value into its effective one. The "there exists" of the idealist is only an incitation to formulate a true judgment; ideal existence is worthless so long as it is not converted into actual existence. It is also to be noted that general judgments have meaning only through the fact that they imply an indefinite number of singular judgments; they are only true because they are constantly verifiable.[2]

The Mathematical Continuum Conceived as a Becoming

According to the classical conception, the mathematical continuum is only the aggregate of real numbers, all given as its ultimate elements. With Brouwer appears a new conception. Without, we hope, distorting his meaning, we shall here follow a path a little different from his in the exposition of the intuitionist view. We shall identify the real number and the decimal expansion.[3] The irrational number and its expansion are infinitely more common than the rational number. The irrational number has no repetend, and we shall never conceive it except approximately inasmuch as we cannot think simultaneously of an infinity of numbers (the numbers of the decimal expansion).[4] We shall never have anything but a finite number of figures of the expansion before us. To define a number with precision, this expansion must be given with as many figures as we wish.

We have now two distinct questions to consider: 1. By what process can a real number be defined? 2. What idea can be formed of the continuum?

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[1] See Richard, *Philosophie des Mathématiques*, p. 107.

[2] Within an infinite class, the rejection of the universal affirmative does not imply the singular negative. The idealist does not take precautions when dealing with infinite classes, conferring a meaning on the particular even in the case where he is not assured of being able to convert it into a singular. To affirm the particular amounts to affirming an ideal existence. The singular alone would correspond to "intuitionistic" existence.

[3] Brouwer has defined the real number independently of the decimal expansion as a series of intervals, each enclosing the next, the series approaching zero. So that he is led to put the question, Cannot one define a real number which does not have a decimal expansion?

[4] We could have taken the binary system; it is evident the system adopted is immaterial.

1. Since an infinity of numbers cannot be given one by one, we must have recourse to a law of generation of the decimal expansion, $m=f(n)$, which prescribes that the nth decimal be equal to $m(0 \leqq m \leqq 9)$. Such a law of generation will be given, for example, by the arithmetic process of extracting the square root. Only such a law can define an irrational number, and in a general way, we can say that a real number is a law f of generation of a decimal expansion.

2. One would be tempted to reply to the second question as to the first, giving oneself a law g of generation, no longer of the decimal expansion, but of the real numbers themselves, of the laws f. Proceeding thus, one would construct the continuum. It would be denumerable and considerably mutilated. But this will not do. The laws f must remain entirely general, so that they shatter the frame in which a law g which presided over their genesis would confine them. Nothing ought to restrict the freedom of choice of the laws f. Hence if we wish to preserve for the continuum the richness of which it allows, we must avail ourselves of this freedom. We are here led to the following definition: The mathematical continuum is the decimal expansion in the freedom of its genesis.[1] The idea of the continuum then comes to that of a series of arbitrary choices, or better, to that of a *free sequence*, in the sense that nothing prejudges, after n choices, what will be the next.

A new question arises. How conceive the relation between continuum and real number? If the construction of the continuum which we just sketched had succeeded, we should be given the real numbers as ultimate elements of the continuum; and the latter would have been only the aggregate of real numbers, it would have been the law g. But since this attempt has failed, we have given a distinct definition of the continuum, even opposed in certain respects to that of the real number, opposed as liberty to law. In no case now can the continuum be envisaged as the simple union of real numbers. The notion of sequence oscillates constantly between two extremes, on the one hand, the given sequence, the law, being; on the other, becoming, the free sequence or continuum.

Strictly speaking, the real number is in the continuum in this sense, that being free to choose the decimals as I please I can choose them conformably to a law f. But it would be absolutely meaningless to make of the continuum a simple aggregate of real numbers and to see in it a relation analogous to that of a whole and its parts. One cannot rise from the number to the continuum as from the elements to the class, and see nothing in the class other than the aggregate of elements. And then the numbers are not all given; one must be satisfied with a representation of particular numbers by a law f and not imagine a given anterior to this representation and containing them all.

Should we wish to parcel out the continuum at any cost, the ultimate elements with which we should end would not be points, but intervals; we should not pass beyond these, and the interval is a new continuum. From the intuitionist point of view, the stopping place must be the interval, for we can have before us only a finite number of figures of the decimal expansion. The real number is a

[1] Brouwer makes of the continuum the "Medium freien Werdens."

sequence of intervals (i, n) each enclosing the next, their length approaching zero as one progresses in the sequence.[1] It is in a way a passage to the limit of a series of intervals, an approximation whose error we are sure of making as small as we wish. But we shall never make this passage to the limit; we shall never do away with this error. The interval still remains infinitely divisible. Weyl thinks this conception can claim to do justice in a lasting way to the notion of becoming.

Another objection to the atomistic conception which sees in the continuum only an aggregate of points appears in the following considerations: Whether the elements be points or intervals, can one say of a given couple whether they are identical or different? Now this one can say of intervals, but not of points, whereby we are brought back to the question of logic. Two points are given by two laws $m=f(n)$, and $m'=f'(n)$, that is, by two functions of integers, and we cannot affirm them to be either identical or different. The main point of the matter is in the rejection of the application of the law of excluded mean, which brings us to this: it is false to say, "either a given point is interior to an infinity of intervals or it is exterior to one of them."[2] The realist-atomist says: Two points are identical or distinct. With that, he places himself in the presumed given, the aggregate of points, and not in the act generating such points, which is alone intelligible. He is heedless of the fact that a point is only defined by an indefinite sequence of integers, by a process implying an infinity of operations. If this series of operations could be thought in its entirety and not in its law of generation, the decision would perhaps not fail to be forthcoming. But this exhaustion is inconceivable.

The Intuitionist Reconstruction

If, as Leibniz thought, systems are false by what they deny and true by what they affirm, I cannot escape the duty of sketching here the most important points of the intuitionist reconstruction.

The intuitionist, who rejects Zermelo's axiom of inclusion, cannot define the set by means of an attribute characteristic of all its elements; it must be constructed. To construct, in the narrow sense of the word, would consist in exhibiting the elements one by one; but here again one would find oneself within too restricted limits. One would risk confinement in the domain of the law g of which I spoke just now. A construction of sets is possible, starting from the free sequence. Now the free sequence or sequence in becoming consists in choosing arbitrarily a first number, then a second arbitrarily, and so on as long as desired. Let us call such a sequence s. The fundamental mathematical operation

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[1] Calling i the number formed by the n figures reached in the decimal expansion, (e.g., for 0.265, $n=3$, $i=265$) the number in question is comprised between two rational numbers $i/10^n$ and $i+1/10^n$, i.e., within the interval (i, n) of length $1/10^n$ limited by these two abscissae. The approximation is made closer with each additional figure n.

[2] In his memoir of 1912, "*Le calcul des intégrales définites,*" Journal des Mathématiques, fasc. II, Borel put forward a similar idea on the equality of two numbers.

which is substituted for the notion of set and of function is a law of construction $c(s, k)$ making a determinate number correspond to each sequence s and to each order number k in the sequence. This well defined number could indeed depend very well not only on the kth number of the sequence, but on the k first ones. The expression $m_k = n_1 + n_2 + \cdots + n_k$ furnishes an example of such a law. One can make k vary in such a series as well as change the series. This law of correspondence $c(s, k)$ Brouwer calls a set, Weyl a function. A very special case of the function would be the correspondence between two integers of the series S, that is, a law $N_k = c(S, k)$. That would be a *functio discreta*, to use Weyl's phrase. The law $N = 2k$ is an example; it defines the set of even numbers. In the general case the law c makes a new series correspond to each given infinite series. Weyl calls this dependence a *functio continua*. But it is not equivalent to the continuous function of analysis. That, from the intuitionist point of view, is as Weyl shows, a *functio discreta*.

THE IDEALIST POINT OF VIEW OF HILBERT

The Absence of Contradiction among the Axioms of Arithmetic

One willingly supposes that reasoning in which one starts from such evidence as certain mathematical axioms and continues according to the rules of the strictest logic could not lead to a contradiction. To start from evidence, to reason logically, is that a guarantee of never being inconsistent? As contradiction is the worst thing which a mathematician can encounter, it is important to demonstrate that however far one pushes mathematico-logical deductions, a contradiction will never arise. And this task is all the more urgent since the axioms are not all equally evident, especially those which the intuitionists reject.

Hilbert was fully aware of this, and this task has been thrust upon him all the more forcibly since the time when he claimed to reinstate the inferences doubtful to the intuitionists, by means of a new axiom containing all the doctrine of Zermelo and Cantor, plus four axiom-definitions of the notions all, not all, there exists, there does not exist—propositions whose evidence is not at all immediate.

The problem of non-contradiction among the axioms is stated as follows: Given a system of axioms, to show that from this two propositions reducible to the form A and non-A will never be deduced. In his axiomatic of geometry, Hilbert has fully succeeded in solving this problem so far as the axioms of geometry are concerned, but it was through postulating that the axioms of arithmetic implied no contradiction. At the international congress of mathematics in 1904 he tried to establish that the latter in turn were exempt from contradiction. Poincaré made essentially two objections[1] to this attempt of Hilbert: 1. Your sole means of showing that you will never meet a contradiction would be to establish that, if at the nth deductive operation no contradiction appears, it will not appear in the following deduction either; and then to reason by complete induction so as to be sure of never meeting one. It must at least be postu-

[1] *Science et Méthode; Les Logiques nouvelles.*

lated that the axiom of induction taken in itself does not imply contradiction. 2. You must at least admit that the series of integers does not imply contradiction.

After these criticisms of Poincaré, Hilbert's attempt might seem vain. Hilbert avoids the first objection provided one understands clearly what he wishes to establish. He does not seek to demonstrate that should deductions be infinite in number they will all be exempt from contradiction, but simply that one will not actually meet such a contradiction. Man is capable of only a finite number of actual deductions; so if he meets a contradiction, it is after a finite number of operations. And if one succeeds in establishing that this circumstance cannot occur, the problem will be humanly solved.

We shall give an account of how Hilbert recognizes the objections and qualifies their importance, by expounding the essence of his procedure. To conduct a Hilbertian demonstration properly, the following precautions seem indispensable:

1. Each word must be rid of the dangerous richness of ordinary language—of what any notion, as that of number or of implication, for example, conjures up before imagination. To this end it is best to reduce each notion to an easily recognizable sign.

2. Mathematical axioms and logical principles must be represented by complexes of signs.

3. A symbolic representation must be given of the elementary deduction itself. The latter is the hypothetical syllogism of the propositional calculus: if A (is true) and if A implies B, then B (is true).[1] All mathematical demonstrations would be reduced to this unique procedure repeated a convenient number of times. The premises A and A implies B would spring directly from axioms, or would be propositions previously established.

4. A criterion of contradiction must be given, also representable by a sign complex, for example, $a \neq a$.

In other words, it is necessary to represent by signs first notions, first propositions, demonstration, and contradiction in such a way that the sign in the deduction outlined plays the same logical role as in the demonstration thought. This outline, or formalized demonstration, is a skeleton of the reasoning establishing a certain proposition by proceeding from axioms. And the intuitive sense of each term no longer intervenes, but only its logical function. However, it is impossible, as Hilbert explicitly recognizes, to abstract from all intuition; we must indeed be able to recognize the same sign in two patterns which are not completely identical. More than that, it must be required that men agree on the meaning of the following words: the first sign of such and such a sort which is met in running through a sequence of signs in a determinate order, from the axioms to the final proposition. This being the case, we must require Hilbert to demonstrate that the sign of contradiction does not occur in his pattern; so

[1] A implies B means: B is true or A is false.

that his formalized demonstration, once made, becomes the object of a concrete reflection. Hilbert requires that this concrete reflection be itself free from contradiction. It is little enough to ask in one sense, for his formalized demonstration contains only a finite number of signs which he can exhibit.

The Axiom of the Transfinite

In his memoir of 1922 Hilbert shows how, by his axiomatic method, he is in a position to reinstate the principles doubtful in the eyes of Brouwer and Weyl.

Let E be a set assumed to be well defined, e any element of the set and A any property well defined relatively to each element of the set. It is postulated that there exists an element t of the set such that if it has the property A, that implies that all the elements likewise possess it. For example: If E is the set of men and A the property of being corruptible, the object t would be a man of such inviolable incorruptibility that if he is corruptible; all men are. Hilbert's idea is this: it will not always be possible to discover the element whose existence is postulated, but one can without risk of error act as if this discovery had already taken place. In other words, ideal existence would have the same logical value as actual existence.

Conclusion

In his axiom of the transfinite set-up to reinstate suspected principles, Hilbert tries to give logical value to ideal existence, which the intuitionist refuses to recognize. The divergences of viewpoint which were manifest in connection with the axiom of selection cannot fail to spring up again as soon as the admission of that axiom is in question.

Ideal existence is for the intuitionist only a false window for the logical symmetry of propositions, bearing on the finite set on the one hand and on the infinite set on the other, a fiction of the logicians imagined not to save logic—it is not in danger—but to extend its domain. Now it is only an intuitive analysis of each particular case which determines whether one happens to be under its jurisdiction. The intuitionist proceeds from the representation to the object represented, while the idealist does not hesitate to deal with a system of things, such as the set of real numbers, of which he is incapable of giving a representation. Intuitionistic existence is just that representation, while idealistic existence seems in the final count to vanish in the idea of the non-contradictory. The intuitionist is more prudent; he seems to enjoin us not to assert existence of an indefinable object, or even of a non-constructible one, just as the physicists of the relativistic school enjoin us not to invent hypotheses which do not correspond to a physical set-up which is at least imaginable if not practically realizable. Mathematics ought perhaps to give up the classic conception of the continuum. But the free sequence and the operations on it are a new canton open to it.

The intuitionists through their great concern for rigor and evidence occupy the stronger position and run the least risk. But so long as they have not put their finger on a sort of *tertium datur*, they cannot convince the idealists of the necessity of slackening their pace. And it indeed seems improbable that idealists will arrive at any contradiction. It seems to me that we shall not give up the language and even the reasoning of the idealist, but shall require intuitionist verifications. Often in the past, intuitionists (notably Lebesgue) have succeeded in replacing idealist demonstrations, in which the axiom of selection was invoked, by others where one dispensed with it. It will perhaps be the same with reference to the doubtful application of the law of excluded mean. The divergence of temperament, as Brunschvicg has so well shown, has its roots deep in history. It has not constituted, properly speaking, a great danger. And even today, when the intuitionist is more uncompromising, the word "crisis" in the foundations of mathematics is inappropriate.

NOTES ON THE ORTHOCENTRIC TETRAHEDRON

By N. A. COURT, University of Oklahoma

1. The "rectangular" or "orthogonal" tetrahedron was first considered by Simon Lhuilier (1750–1840) in his book De relatione mutua etc., Varsaviae, 1782, and by L. A. S. Ferriot in the Annales de Gergonne, vol. 2 (1811–1812), p. 123. The term "orthocentric tetrahedron," now in general use, was proposed by G. de Longchamps (Mathesis, 1890, p. 50).

An extensive bibliography on the orthocentric tetrahedron may be gathered from

I. Encyclopaedie der Mathematischen Wissenschaften, vol. III, part I, pp. 1061–1062.

II. M. Simon: *Ueber die Entwicklung der Elementar-Geometrie im XIX Jahrhundert*, pp. 207–208. Leipzig, Teubner, 1906.

III. Intermédiaire des mathématiciens, 1917, p. 117.

Some more recent contributions are: Philip Franklin; This MONTHLY, 1919, pp. 146–151. V. Thébault; The Tôhoku Mathematical Journal, vol. 19 (1921), p. 25, and vol. 21 (1922), p. 232; Mathesis, 1932, p. 150. C. Servais; Mathesis, 1923, p. 452, and 1925, p. 145. Goormaghtigh; Mathesis, 1926, p. 126. N. A. Court; Mathesis, 1928, p. 337, and Annals of Mathematics, vol. 30 (1929), p. 613.

Some shorter contributions are: Mathesis, 1922, p. 34, p. 334; 1923, p. 45, p. 454; 1924, p. 139, p. 316; 1928, p. 173, p. 321; 1930, p. 269, p. 318. This MONTHLY 1927, p. 540; 1929, p. 449.

2. It is impossible to examine the literature on the orthocentric tetrahedron without being struck by the fact that the same properties recur time and again, being rediscovered by various authors quite independently. In spite of this frequent overlapping practically no applications have been made, in the study of the orthocentric tetrahedron, of the relation, pointed out by Gaspar Monge,

between the tetrahedron and its circumscribed parallelepiped.[1] This omission is regrettable, as may be seen from the two examples in sections 3 and 4.

3. A tetrahedron $(T) \equiv ABCD$ inscribed in a rhomboid (R) (i.e., a parallelepiped whose faces are rhombuses), is orthocentric, for the diagonals of the faces of (R) are orthogonal, and, conversely, the parallelepiped circumscribed about an orthocentric tetrahedron is a rhomboid. Now if in a parallelepiped two pairs of opposite faces are rhombuses, the same is true of the third pair of faces, and the parallelepiped is a rhomboid, hence:

If in a tetrahedron two pairs of opposite edges are orthogonal, the same is true of the third pair of edges, and the tetrahedron is orthocentric.

4. The vertices A', B', C', D', of the rhomboid (R) diagonally opposite to A, B, C, D, determine the tetrahedron (T') twin to (T). The two planes BCD, $B'C'D'$, being parallel, the perpendicular h from the vertex A of (T) upon the opposite face BCD is also perpendicular to $B'C'D'$. Now (R) being a rhomboid, we have $AB' = AC' = AD'$, hence h passes through the circumcenter of the triangle $B'C'D'$, and therefore through the circumcenter H of (T'). Thus:

If the pairs of opposite edges of a tetrahedron are orthogonal, its four altitudes are concurrent.

Furthermore, from the relation between the two twin tetrahedrons (T), (T'), follows immediately that

The orthocenter H of (T) is the symmetric of its circumcenter with respect to its centroid.

5. I shall now proceed to give a property of the orthocentric tetrahedron which I believe to be new. In order to simplify its proof I shall first establish the following proposition:

The two lines joining a point of intersection of two orthogonal circles to the ends of a diameter of one of these circles determine in the second circle the ends of the diameter which is perpendicular to the diameter first considered; and conversely, given two orthogonal diameters of two orthogonal circles, the lines joining the ends of one of these diameters to the ends of the other pass through the points of intersection of the two circles.

Let the lines EA, EB, joining the common point E of the two orthogonal circles (P), (Q), to the ends A, B, of the diameter AB of (P) meet (Q) in the points C, D (see Fig. 1). Since CED is a right angle, CD is a diameter of (Q). In the two isosceles triangles AEP, CEQ, where P, Q are the centers of the given circles, we have

$$\angle EAP = \angle AEP, \qquad \angle ECQ = \angle CEQ,$$

and hence

$$\angle EAP + \angle ECQ = \angle AEP + \angle CEQ = 90°,$$

for the angle PEQ is a right angle, by assumption. Thus the angle ARC of the triangle ARC, where $R \equiv (AB, CD)$, is a right angle, which proves the proposition.

[1] This MONTHLY, 1932, p. 196; more extensively in Mathematics Teacher, 1933, pp. 46–52.

It follows immediately that the lines FA, FB, where F is the second point common to the two circles, pass through the points C, D, for there is only one diameter of (Q) which is perpendicular to AB.

The converse proposition follows from the fact that each of the lines AEC BFC, ADF, EDB, is determined by two of its points.

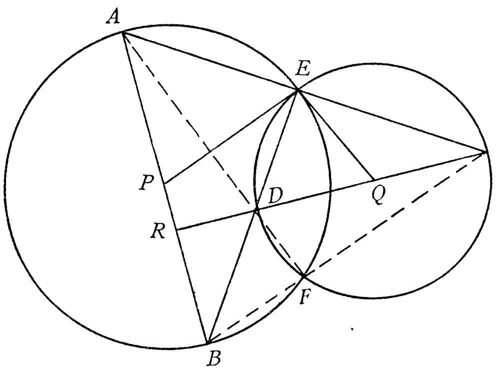

Fig. 1

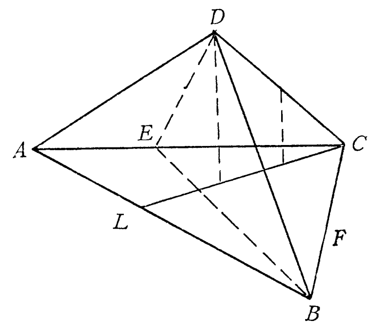

Fig. 2

6. THEOREM. *The ends of two skew orthogonal diameters of two orthogonal spheres are the vertices of an orthocentric tetrahedron.*

Let AB, CD, be two orthogonal diameters of two orthogonal spheres (AB) (CD) (See Fig. 2). The plane through CD perpendicular to AB cuts the plane ABC along the orthogonal projection CL of CD upon ABC, and CL is perpendicular to AB. The plane ABC passes through the center of (AB), hence ABC cuts (AB) along a great circle, and (CD) along a small circle, and these two circles are orthogonal, for the spheres are orthogonal.

The center of the small circle is the projection of the mid-point of CD upon CL. Hence C is one end of the diameter of the small circle which is perpendicular to the diameter AB of the great circle. Consequently, by the theorem of section (5), the lines CA, CB, pass through the points E, F, common to the two circles.

The point E thus belongs to the sphere (CD), hence AB and CD subtend each a right angle at E, i.e., AEC is perpendicular to both ED and EB, and therefore also to BD.

Considering the point F it may be shown in a similar way that AD, EC are rectangular. The proposition is thus proved, superabundantly in view of the theorem of section (3).

7. If the two spheres (AB), (CD), remain fixed, while the diameters AB, CD, vary, we obtain a variable orthocentric tetrahedron $ABCD$, and the geometric elements associated with this tetrahedron will describe various loci. For lack of space I shall point out only the following.

(a) The sphere (G) having for diameter the line of centers of the given spheres is the first twelve point sphere of $ABCD$.[1] As $ABCD$ varies the sphere (G) re-

[1] Rouché et Comberousse, *Traité de Géométrie* (Paris, 1900), vol. 2, pp. 661–662.

mains fixed. The center G of (G) is the centroid of $ABCD$. The sphere (G) is the locus of the mid-points of the edges of $ABCD$ as well as the locus of the feet of the bialtitudes (i.e., common perpendiculars) of the pairs of opposite edges of $ABCD$.

(b) The vertices A, B, of $ABCD$ are a pair of conjugate points with respect to the polar sphere (H) of $ABCD$, hence the sphere (AB) is orthogonal to (H), and the same is true for the sphere (CD), for the same reason. Hence the center H of (H), which point is the orthocenter of $ABCD$, has for its locus the radical plane of the spheres (AB), (CD); the sphere (H) describes the coaxal net of spheres conjugate to the coaxal pencil of spheres determined by the given spheres (AB) and (CD).

(c) The bialtitude of $ABCD$ relative to the pair of opposite edges AB, CD, describes a tetrahedral complex.[1]

A PROOF OF THE WEIERSTRASS CONDITION IN THE CALCULUS OF VARIATIONS

By L. M. GRAVES, University of Chicago

The simplest type of problem of the calculus of variations is that of minimizing an integral

$$I = \int_{x_1}^{x_2} f(x, y, y')dx$$

in a class of admissible curves

$$y = y(x) \qquad (x_1 \leqq x \leqq x_2) \tag{1}$$

joining two fixed points 1 and 2 in the xy-plane. We shall suppose that the integrand function $f(x, y, y')$ is defined and continuous together with $f_{y'}$, its first partial derivative with respect to y', for (x, y) in a region R of the plane (which may be closed or not) and for all y'. A curve (1) will be called *admissible* provided it lies in the region R, is continuous, and has a continuously turning tangent except possibly at a finite number of corners.

In deriving necessary conditions which a minimizing curve for the integral I must satisfy, it has been customary first to obtain the Euler differential equation

$$\frac{d}{dx} f_{y'} = f_y \tag{2}$$

which must be satisfied by the arcs of the minimizing curve lying interior to the region R. The Weierstrass condition, which states that

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[1] Nathan Altshiller-Court, *Some tetrahedral complexes*, this MONTHLY, vol. 35 (1928), p. 471, art. 15.

$$E(x, y, y', Y') = f(x, y, Y') - f(x, y, y') - (Y' - y')f_{y'}(x, y, y') \geqq 0 \tag{3}$$

for all elements (x, y, y') of the minimizing curve and all Y', is then proved by a method depending on the validity of the Euler equation (2) on an arc of the minimizing curve neighboring the point (x, y).[1] Such a proof is not applicable to arcs of the minimizing curve lying along the boundary of the region R, for which the Euler equation (2) need not hold. The proof of the Weierstrass condition (3) to be given below makes no use of the Euler equation, and hence is valid for arcs of the type just mentioned. It is also believed to be simpler than the usual proofs. The method may be applied in any number of dimensions and to parametric problems.

Let the minimizing arc be denoted by

$$E_{12}: \quad y = y(x) \qquad\qquad (x_1 \leqq x \leqq x_2),$$

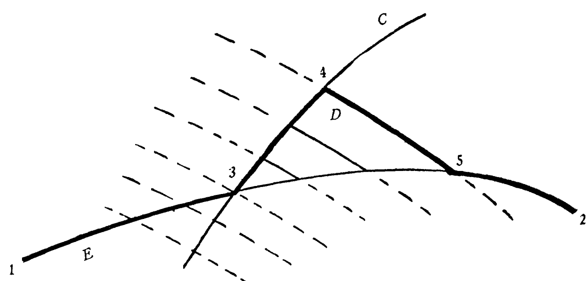
and let

$$C: \quad y = Y(x)$$

be an arbitrary curve passing through a point 3 on E_{12}. Let

$$y = \phi(x, t)$$

be the equation of a one parameter family of curves D, whose intersections with the curves C and E have the abscissas $x_4(t)$ and $x_5(t)$ respectively. In the case illustrated in the figure, we may suppose $x_4(t_3) = x_5(t_3) = x_3$, $x_4'(t_3) > 0$, and



$x_4(t) < x_5(t)$ for $t > t_3$. The incidence relations of the curves are expressed by the equations

$$Y[x_4(t)] = \phi[x_4(t), t], \qquad y[x_5(t)] = \phi[x_5(t), t].$$

By differentiating these and setting $t = t_3$ we obtain[2]

$$Y'x_4' = \phi'x_4' + \phi_t, \qquad y'x_5' = \phi'x_5' + \phi_t,$$

from which by subtraction there results

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[1] See Bliss, *Calculus of Variations* (Carus Monograph No. 1), Ch. V.

[2] We shall denote all derivatives by primes, except that the partial derivative of ϕ with respect to t is denoted by ϕ_t and the partial derivative of f with respect to y' by $f_{y'}$. In particular, the partial derivative of ϕ with respect to x is denoted by ϕ'.

$$(4)\qquad (Y' - \phi')x_4' = (y' - \phi')x_5'.$$

Now consider the admissible comparison curve $\Gamma_t = E_{13} + C_{34} + D_{45} + E_{52}$, of the type indicated by the heavy line in the figure. Let

$$I(t) \equiv I(\Gamma_t) = \int_1^3 f(x, y, y')dx + \int_3^4 f(x, Y, Y')dx$$
$$+ \int_4^5 f(x, \phi, \phi')dx + \int_5^2 f(x, y, y')dx.$$

Then since $I(E_{12}) = I(t_3)$ is supposed to be a minimum value of I,

$$I'(t_3) = [f(x, y, Y') - f(x, y, \phi')]x_4' + [f(x, y, \phi') - f(x, y, y')]x_5' \Big|^3 \geqq 0.$$

By use of equation (4) and the inequality $x_4'(t_3) > 0$, this inequality becomes

$$(5)\qquad \frac{I'(t_3)}{x_4'(t_3)} = f(x, y, Y') - f(x, y, \phi') + [f(x, y, \phi') - f(x, y, y')]\frac{Y' - \phi'}{y' - \phi'}\Big|^3 \geqq 0,$$

which holds for all Y' and all ϕ' such that $(Y' - \phi')/(y' - \phi') > 0$. The familiar form (3) of the Weierstrass condition is obtained from (5) by letting ϕ' approach y'.

While the condition contained in the inequality (5) appears to be stronger than the familiar form of the Weierstrass condition, it is easy to see that it is equivalent. For, if we multiply the inequality

$$f(x, y, \phi') - f(x, y, y') - (\phi' - y')f_{y'}(x, y, y') \geqq 0$$

by $(Y' - y')/(y' - \phi')$ and add to (3), we obtain by a simple algebraic reduction exactly the inequality (5).

QUESTIONS, DISCUSSIONS AND NOTES

EDITED BY R. E. GILMAN, Brown University, Providence, Rhode Island

The department of Questions and Discussions in the Monthly is open to all forms of activity in collegiate mathematics, including the teaching of mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.

ON A METHOD FOR CALCULATING SQUARE ROOTS

By D. C KALBFELL, University of California at Los Angeles

It has been known for a long time that, if a is an approximation to the square root of r, then $(1/2)(a + r/a)$ is a better approximation. This procedure was used by Heron of Alexandria about 200 A.D.; it may be justified in various ways, for example, by applying Newton's method to the equation $x^2 - r = 0$. The most complete investigation of this process is due to Bouton.[1] He derives expressions

[1] Annals of Math., ser. 2, vol. 10 (1909), pp. 147–72. See also this MONTHLY, vol. 39 (1932), pp. 533–535.

for the upper and lower limits of the error made by stopping with any given approximation after the first; his expressions use two successive approximations. The present paper develops equivalent expressions which are simpler in form, and which employ a single approximation. It may be added that Bouton uses decimal approximations, while this paper keeps them in fractional form.

Let the first approximation be p_1/q_1. Then the second approximation will be

$$p_2/q_2 = (\tfrac{1}{2})(p_1/q_1 + rq_1/p_1) = (p_1^2 + rq_1^2)/(2p_1q_1).$$

Let the difference $p_1^2 - rq_1^2$ be denoted by α; whereupon[1] $p_2 = 2rq_1^2 + \alpha$, $q_2 = 2p_1q_1$.

By elementary algebra we now find that $p_2^2 = rq_2^2 + \alpha^2$, and, by induction, assuming that no p_n/q_n is reduced to lowest terms, we obtain

$$p_n^2 = rq_n^2 + \alpha^{2^{n-1}}. \tag{1}$$

If p_2/q_2, for example, is reducible, as when $r=3$, $p_1/q_1 = 9/5$, $\alpha = 6$, we may reduce it, take it as a new first approximation with a new α, and continue the approximating process until a further reduction is possible. With this interpretation, the result (1) is always true, though n may not represent the actual number of approximations used, and α may not be the original α.

We shall now consider the error made by stopping with a certain approximation. Let E_n be the error in p_n/q_n; that is,

$$E_n \equiv p_n/q_n - r^{1/2} = (p_n^2 - rq_n^2)/q_n(p_n + q_nr^{1/2}) = \alpha^{2^{n-1}}/q_n(p_n + q_nr^{1/2}).$$

This is an exact expression, but it is not suitable for our purposes as it contains the square root of r. However, we may derive upper and lower limits for E_n as follows. Notice that the above result for E_n shows that $p_n/q_n > r^{1/2}$, when $n > 1$. To find a lower limit for E_n, increase its denominator by replacing $q_nr^{1/2}$ by p_n. This gives

$$E_n > \alpha^{2^{n-1}}/(2p_nq_n). \tag{2}$$

To find an upper limit for E_n, decrease its denominator by first replacing $r^{1/2}$ by rq_n/p_n. We find[2]

$$E_n > \frac{\alpha^{2^{n-1}}p_n}{2q_n^3 r}. \tag{3}$$

Bouton's upper and lower limits for E_n are given by

$$\frac{\left(\dfrac{p_{n-1}}{q_{n-1}} - \dfrac{p_n}{q_n}\right)^2}{2p_n/q_n} < E_n < \frac{\dfrac{p_n}{q_n}\left(\dfrac{p_{n-1}}{q_{n-1}} - \dfrac{p_n}{q_n}\right)^2}{2r}.$$

[1] That is, p_2/q_2 is not reduced to lowest terms. If r is an integer, we can choose p_1/q_1, by the use of continued fractions, for example, so that $\alpha=1$. It is then not difficult to show that all later approximations are in lowest terms. This is, as a matter of fact, true whenever p_2/q_2 is taken as any convergent in the continued fraction expansion of $\sqrt{r}$.

[2] For a different treatment of the case $\alpha=1$ see this MONTHLY, vol. 39 (1932), p. 535.

It is simply a matter of algebra to show that Bouton's limits are equivalent to those of the present paper.

By subtracting the lower limit in (2) from the upper limit in (3) we obtain, after reduction

$$\alpha^{2^n}/(2p_nq_n^3r). \tag{4}$$

This allows us to measure the accuracy obtainable at any stage of our process in a convenient way. For p_n/q_n diminished by the lower limit as obtained in (2) cannot differ from $r^{1/2}$ by more than the expression (4). The fact that this expression takes such a simple form is worthy of note. If the reader carries through a numerical case, for example the approximation of $3^{1/2}$ with $p_1/q_1=7/4$, $\alpha=1$, he will see how quickly the expression (4) which in this case is equal to $1/(6p_nq_n^3)$ can be estimated at each step, and how easy it then is to secure any required accuracy.

Note Concerning Two Problems in Geometrical Probability

By O. K. Bower, University of Illinois

Most of the six solutions given by E. Czuber[1] for Bertrand's Paradox, "What is the probability that a chord chosen at random on a circle is greater than the side of the inscribed equilateral triangle?", are special cases of infinite systems of solutions. It is noticed in Czuber's solutions that the totality of all chords on the circle is defined in various ways, and, after the definition, a "typical" subset of the totality is chosen. A subset is typical, the conception being almost essentially intuitive, if the totality of all chords can be broken up into a finite number or an infinitude of similar subsets for each of which the probability in question is the same. The probability for the typical subset is determined as the probability sought. This procedure may easily be followed to obtain infinite systems of solutions by bringing a parameter into the definition of the totality of all chords on the circle.

Take the unit equal to the radius, $r=1$, and the equation of the circle $(x-b)^2+y^2=1$, $b>1$. The totality of all chords, extended, consists of chords which cut a concentric circle of radius b. Take those cutting the concentric circle at a given point, which for convenience will be chosen as the origin, as a typical subset. Then the probability, p, that a chord on the circle is greater than the side of the inscribed equilateral triangle depends on the equilikely element. If this is $d\theta$,

$$p=\frac{\int_0^{\arcsin(1/2b)}d\theta}{\int_0^{\arcsin(1/b)}d\theta}=\frac{\arcsin\dfrac{1}{2b}}{\arcsin\dfrac{1}{b}},$$

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[1] Czuber, *Wahrscheinlichkeitsrechnung*, (1924) pp. 116–118.

$$P_1 = \frac{1}{\arcsin(1/b)}\left[2\arcsin\frac{\sqrt{4-a^2}}{b} - \arcsin\frac{\sqrt{1-a^2}}{b} + \frac{2a}{b}E(\arcsin\sqrt{1-a^2}, b^{-1}) - \frac{a}{b}E\left(\arcsin\frac{\sqrt{4-a^2}}{2}, b^{-1}\right)\right],$$

$$P_2 = 1 - P_0 - P_1.$$

As in the case of Bertrand's Paradox other infinite systems of solutions may be obtained.

Ruled Surfaces Tangent along a Curve

By J. H. Butchart, Indianapolis, Indiana

All the ruled surfaces which are tangent to a skew surface S along an arbitrary directrix C may be generated from S by rotating the generators of S about the points of C in the planes tangent to S at these points. From the fact that the tangent planes for all these surfaces are coincident, it follows at once that if C is a line of curvature, a geodesic, or an asymptotic line on S, it is a line of the same sort for every one of the surfaces. We ask whether, among this array of tangent skew surfaces, there are any for which C is the line of striction. The condition which the unit vector in the direction of the generators, $\boldsymbol{d}$, must satisfy is that $\boldsymbol{t}\cdot\boldsymbol{d}'=0$, where $\boldsymbol{t}$ is the unit tangent vector to C, and the differentiation is with respect to the arc of C. Since the determination of $\boldsymbol{d}$ involves a quadrature, there is a singly infinite system of skew surfaces, among those tangent to S along C, for which C is the line of striction.

The case where C is a geodesic and that in which it is the line of striction of S offer interesting illustrations. If C is a geodesic, by Bonnet's theorem for skew surfaces, all the surfaces whose generators cut C under a constant angle have C for their line of striction. If, on the other hand, C is the line of striction of S, then any surface whose generators make a constant angle with the generators of S has C for its line of striction. For, using θ to denote the angle between the generators of S and the tangent to C, and $\bar\theta$ as the corresponding angle for the new surface, we have $\bar\theta=\theta+c$. Moreover, since the geodesic curvature of C is the same for both surfaces, we have from Bonnet's formula $1/\rho_g=d\theta/ds+b\csc\theta$ that $b=\boldsymbol{t}\cdot\boldsymbol{d}'$ and $\bar b$ vanish simultaneously for both surfaces.

If C is an asymptotic line, from the theorem of Beltrami and Enneper that the Gauss curvature of a surface along an asymptotic line is equal to the negative square of the torsion of this line, all the tangent surfaces have the same curvature at points of C. Incidentally this is the same as the curvatures of the surfaces of principal normals and binormals evaluated at points of C. (Weatherburn, *Differential Geometry*, p. 142.)

Combining two previous considerations, we remark that if C is both the line of striction and an asymptotic line, rotation of the generators through a constant angle leads to another ruled surface for which C is both the line of striction

and an asymptotic line. This transformation has a clear connection with the following simple geometrical construction for a skew surface whose line of striction is an asymptotic line: upon a system of parallel lines in the plane draw an arbitrary curve C. Then, with the lines fixed to the trihedral of C, give the curve arbitrary torsion. The lines are the generators of the required surface. It is evident that C is an asymptotic line of the surface, for its principal normal lies in the tangent plane. Its curvature $1/\rho$, by the construction, may be taken as $d\theta/ds$, where θ is the angle between the parallels and the tangent to the curve. Hence, from Bonnet's formula, $1/\rho_g = 1/\rho + b \csc\theta$, but since $1/\rho_g = 1/\rho$, we have $b = 0$, and C is the line of striction.

Note on Vector Identities

By H. V. Craig, University of Texas

It is the purpose of this note to point out that many of the identities of the older vector analysis may be established concisely by means of certain of the notations[1] used in tensor analysis.

The symbolism to be employed is as follows. The ith component of a vector V will be represented, interchangeably, by V^i and V_i; partial derivatives will be denoted by means of subscripts or superscripts preceded by a comma; while summations will be indicated by repeated lower-case indices only, the index of summation appearing once as a superscript and once as a subscript. For example, $V_2{}^{,3} = V_{2,3} = V^{2,3} = V^2{}_{,3} = \partial V_2/\partial z$ (V_2 is the second component of the vector V, z the third letter of the set x, y, z); $U^s W_s = U^1W_1 + U^2W_2 + U^3W_3 = U\cdot W$; $V^i{}_{,i} = V_i{}^{,i} = \partial V_1/\partial x + \partial V_2/\partial y + \partial V_3/\partial z = \text{div } V$. Finally, $e^{ijk} = e_{ijk}$; $\delta^{jk}_{rs} = e^{jkt}e_{rst}$; e_{ijk} being defined by the statements: e_{123} is unity; e_{ijk} is skew-symmetric in each pair of indices. Thus, $e_{123} = -e_{132} = e_{231}$ etc., while e_{ijk} is zero if any two integers of the set i, j, k are the same. From these definitions it follows that $e^{1jk}\,V_jW_k = V_2W_3 - V_3W_2$ and in general that $e^{tjk}V_jW_k$ is the tth component of $V\times W$. Also, $e_{rst}\;e^{tjk} = \delta^{jk}_{rs} = e^{jkt}\;e_{rst} = e^{jk1}\;e_{rs1} + e^{jk2}\;e_{rs2} + e^{jk3}\;e_{rs3}$ and consequently δ^{jk}_{rs} is one if $j=r$, $k=s$, $r\neq s$; is minus one if $j=s$, $k=r$, $r\neq s$; and otherwise is zero. This symbol may be regarded as a kind of substitution operator; to illustrate, $\delta^{jk}_{rs}\;V_jW_k = V_rW_s - V_sW_r$.

To indicate the applicability of the notation it will perhaps suffice to list a few examples.

To prove $$U\times(V\times W) = U\cdot WV - U\cdot VW.$$

Proof: $$e_{rst}U^s(e^{tjk}V_jW_k) = U^s\delta^{jk}_{rs}\,V_jW_k = U^sV_rW_s - U^sV_sW_r.$$

To prove $$\text{curl }(SV) = \text{grad } S\times V + S \text{ curl } V.$$

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[1] The notation and essence of the method are in McConnell's *Applications of the Absolute Differential Calculus*, see pp. 8, 47, 71, 151.

Proof:

$$e^{ijk}(SV_k)_{,j} = e^{ijk}S_{,j}V_k + Se^{ijk}V_{k,j}.$$

To prove $\qquad$ curl curl V = grad div V − grad ${}^2V$.

Proof:

$$e_{rst}(e^{tjk}V_{k,j})^{,s} = \overset{jk}{\delta_{rs}}V_{k,j}{}^{,s} = V_{s.r}{}^{,s} - V_{r,s}{}^{;s}.$$

RECENT PUBLICATIONS

EDITED BY R. A. JOHNSON, Brooklyn College of the City of New York

All books for review should be sent directly to the editor of this department, at Brooklyn College, 66 Court Street, Brooklyn, N. Y., and not to any of the other editors or officers of the Association.

Tables of Integrals and other Mathematical Data. By H. B. Dwight. New York, The Macmillan Company, 1934. viii+222 pages. $1.50.

This book is essentially a reordering of material common to many collections of formulas. The grouping is primarily according to the type of function concerned and secondarily according to the operation. Thus, differentiation of trigonometric functions appears with other trigonometric manipulation and not with other differentiation. The group headings range through functions from Algebraic Functions (65 pages) and Trigonometric Functions (34 pages) to Bessel Functions (11 pages) and Surface Zonal Harmonics (2 pages). There follow: Definite Integrals (8 pages), Differential Equations (5 pages), Tables of Numerical Values (35 pages), References and Index.

The numbering is in a decimal notation which facilitates grouping of the items and location of the cross-references. References to other texts, tables, and original sources and also cross-references are very numerous and make the book much more useful. No count of the formulas was made due to the non-sequential numbering. The index is too short to be of great service but is hardly needed in view of the careful classification throughout.

The printing is clear and attractive except in some of the numerical tables taken from old plates and in a few places where characters show through the page.

H. C. HICKS

Mathematical Facts and Formulae. By A. S. Percival. London, Blackie and Son, 1933. v+125 pages. 4sh. 6d.

The contents of this book indicate that it is aimed at the field between text-books and formula tables, being more discursive than the latter and more condensed than the former. In scope the book ranges from some very elementary material, such as interest rules, to some very advanced, such as a table for log $\Gamma(p)$. Much of the space is taken up by illustrative examples from special fields and by description of methods. In the reviewer's opinion the student or researcher trained in American schools will prefer a book in which this descriptive material is either amplified or omitted; those who have not studied a sub-

ject will find insufficient details to follow it here, while previous study should make all but the formulas unnecessary. There is neither index nor detailed table of contents and very little cross reference or reference to other sources.

While this selection of "mathematical facts" is presumably the list found most useful by its author, the reviewer doubts whether many others will prefer this assortment to already available collections.

H. C. HICKS

Differential Equations. By N. B. Conkwright. New York, The Macmillan Company, 1934. xiv+234 pages. $1.90.

This book presents few novelties in the treatment of a subject which has become standardized in this country. Its table of contents shows that the customary topics are included and are given about the same relative space as is usual. However, applications, higher degree equations, solution in series receive less treatment than might be expected. A chapter on numerical approximation treats the elements of the methods of Runge, Picard, and Milne, and is a welcome addition to the more classical material. In it there are many references to the literature. The omission of even the simplest of boundary value problems is regrettable although it conforms to most elementary texts. In view of the author's statement in the preface "students are prone to regard the subject of Differential Equations as a collection of special devices without any perceptible unifying principle," an outline or résumé might well have been included in the book. The problem lists are ample and well selected.

The presentation is simple and direct. The large number of references throughout both to journals and to other texts is noteworthy. By careful abbreviations of derivations, proofs and geometric interpretations and by relegation to appendices of topics from analysis, the author has succeeded admirably in his stated object, "The topics discussed, as well as the method of presentation, have been so selected that they should be intelligible to students who have a working knowledge of a first course in the calculus."

H. C. HICKS

The Mathematical Atom, Its Involution and Evolution Exemplified in the Trisection of the Angle. By J. Gliebe, O.F.M., San Francisco, St. Boniface Franciscan Friary, 1933. 87 pages, $1.50.

This modern attempt to solve an ancient problem is remarkable for the admirable spirit in which it is written, for the beauty of its language and for the excellent typography and arrangement of its setting. The author is broadminded enough to state on page 55 the objection which has been made to his proof. He admits that he is unable to prove deductively that points A, Y and B' of his diagram, which look to be collinear, really are so. Perhaps some ingenious reader will prove by elementary methods the fact that they are not. It is easier to detect the fallacy in the result. In the author's notation, $\angle$KOJ is

3/8 of the given $\angle$KOL, and $\angle$KOA is alleged to be 1/3 of $\angle$KOL. If this is correct, then $\angle$AOJ is 1/24 of the given angle. Now if $\angle$AOJ is represented by $\theta/24+\epsilon$, it is not difficult to find the value of tan ϵ; and to prove that ϵ cannot be zero for any other value of θ other than a multiple of 48 right angles. However, ϵ is extremely small for values of θ less than a straight angle; and the construction is thus a very close approximation to the true trisection of such angles. On the other hand, if θ is approximately 1645°, ϵ is a right angle—a monstrous deviation.

It is too bad that the impression can remain among intelligent people that the impossibility of trisecting an angle with ruler and compasses rests upon an assertion by Felix Klein, or that there is something dubious about the proof. Until these impressions can be corrected we shall have angle-trisectors. Let us hope that all the members of that clan will be as genial and orderly in stating their claims as is the author of this little book.

J. W. CLAWSON

MATHEMATICS CLUBS

EDITED BY F. M. WEIDA, The George Washington University, Washington, D.C.

All reports of club activities, suggestions and topics for club programs should be sent to F. M. Weida, The George Washington University, Washington, D. C. All manuscripts should be typewritten with double spacing, and with margins at least one inch wide.

CLUB ACTIVITIES

1933—1934

THE PI MU EPSILON MATHEMATICAL FRATERNITY

Pi Mu Epsilon is an academic fraternity in institutions of university grade. Its primary aim is the advancement of mathematics and scholarship. It is a living, active, working fraternity of scholars in which the members are actively engaged in study and research and the exchange of ideas in the field of mathematical science.

Pi Mu Epsilon of New York University

On October 28, 1933, Director General F. W. Owens installed the Epsilon Chapter of New York at Washington Square College of New York University. Fourteen alumni, six seniors and ten members of the mathematics department were initiated. The following officers were elected for the year 1933–1934: Professor C. K. Payne, Director: Morris Levenson, Vice Director; Selma Blazer, Secretary; Solomon Hoberman, Treasurer; Murray Ritterman, Librarian.

The mathematics department invited the members of Pi Mu Epsilon to attend the departmental colloquia, and the undergraduate members had a very active part in the leadership of the two mathematics clubs in the college.

The following lectures were delivered at the meetings of the fraternity:
November 1, 1933: "Hypernumbers" by Morris Kline.
December 6, 1933: "Some topics in vector analysis" by Al. Miller.
December 15, 1933: "The geometrical solution of $dy/dx=f(x, y)$" by Beatrice Edison.
January 3, 1934: "Abridged notation" by Sidney Roth.
February 14, 1934: "Symbolic logic" by J. C. C. McKinsey.

March 7, 1934: "Polygenic functions" by Louis Baron.
April 23, 1934: "Summation of non-convergent series" by David Gans.
May 21, 1934: "The problem of correlation" by R. C. Stearnes.

During the year, nine alumni, one senior, and fourteen members of the faculty were initiated.

At the meeting on January 3rd, Director Payne proposed that the chapter sponsor an interscholastic mathematics contest. This idea met with enthusiastic support, and a committee was appointed to take charge of the contest. Four hundred ninety-two contestants from eighty-seven high schools in New York City, New York State, and New Jersey took the three-hour comprehensive examination in Elementary Algebra, Plane Geometry, and Intermediate Algebra on Saturday, April 28, 1934. Following the contest, the contestants and their faculty advisers were entertained at luncheon by the chapter. The results of the contest were announced and the prizes were awarded by Dean Loomis of Washington Square College at the annual initiation dinner on May 12, 1934. A school was permitted to enter seven contestants, the highest ranking five being considered the team from the school. The team of the DeWitt Clinton High School of New York City obtained first place and was awarded a silver loving cup which they may retain for one year. When a school wins the cup three times it becomes the permanent possession of the school. The Boys' High School of New York City placed second, and Evander Childs High School of New York City placed third. The highest individual score of 409 out of a possible 490 was attained by Helene E. Parnass of Evander Childs High School, who was awarded a gold medal; the second highest score, 401, was attained by Geoffrey Keller of DeWitt Clinton High School, who was awarded a silver medal; and the third highest score, 394, was attained by David Gilbarg of Boys' High School, who was awarded a bronze metal. A certificate was awarded to the contestant attaining the highest score in each school which sent three or more contestants. The second annual interscholastic mathematics contest will be held in April, 1935. Further information about the contest may be had from the Director of the chapter.

The new members initiated on May 12th were: William Jaffee, Ernest Israels, Julian Rachele, Johanna Pruslin, Sol Ducorsky, Jack Wolfsie, Jack Northam, Leon Goroff, Albert Schnitzer, Simon Arnold, Stephen Moore, Andrew Weikert, and Edith Wechsler.

SELMA BLAZER, *Secretary*

Pi Mu Epsilon of the University of Illinois

The Illinois chapter of Pi Mu Epsilon at the University of Illinois reports a very successfu year.

The officers for 1933–1934 were: R. A. Baumgartner, Director; Grace E. Ford, Vice Director; Mildred A. Norval, Secretary; Georgia P. Searle, Corresponding Secretary; R. B. Watson, Treasrer. The Executive Committee was composed of J. M. Glass, Marie I. Leonard, Carmen Parr, and Byron T. Darling. All officers were elected at the regular Spring election, May 23, 1933. They were elected by vote of the members present at the meeting.

The chapter had thirty active members, five of which were initiated on November 21, 1933.

The meetings and programs were as follows:

October 10, 1933: "Remarks about Bernoulli numbers and the Euler-Maclaurin sum formula" by M. T. Bird.
October 24, 1933: "Partition of n into r parts" by Gerald B. Huff.
October 31, 1933: Masquerade rushing party.
November 8, 1933: Pledge luncheon held in the Southern Tea Room, Champaign, Illinois.
November 21, 1933: Initiation banquet held in the Southern Tea Room, Champaign, Illinois. Professor A. B. Coble gave the address on "Mathematical societies."
November 28, 1933: "Famous problems in probability" by Professor A. R. Crathorne.
February 7, 1934: "Some observations in curve fitting" by J. W. Cell.
February 27, 1934: "Fermat's last theorem" by R. K. Cook.
March 13, 1934: "Mathematics in nature" by Professor Jacob Kunz.
March 27, 1934: "Aims in research" by Professor J. H. Bartlett.

April 10, 1934: "Magic squares" by Edith Lytle.

April 24, 1934: "Some elementary applications of Legendre polynomials to physical problems" by J. M. Glass.

Mrs. GEORGIA P. SEARLE, *Secretary*

LOCAL MATHEMATICS CLUBS

The Van Vleck Mathematics Club of Wesleyan University

The Van Vleck Mathematics Club was named in honor of Professors James M. Van Vleck and E. B. Van Vleck, who taught mathematics and astronomy respectively at Wesleyan University. The purpose of the club is to provide a means for social gathering and for stimulation of interest in mathematics and mathematical applications.

The officers for 1933–1934 were: Sterling Tooker, President: George Williams, Vice President; A. Alfred Fisher, Secretary-Treasurer.

We had thirty active members in the organization, membership being open to all those taking advanced courses in mathematics.

The meetings and programs for the first semester were as follows:

October 5, 1933: Fall gathering and dinner at the Historical C. A. Cabin. Professor Malcolm C. Foster was the toastmaster.

November 6, 1933: "Mathematics and its applications to cards and match tricks" by Albert Hickman Taylor.

November 20, 1933: "Inversion" by Professor Malcolm C. Foster.

December 11, 1933: "Dependencies" by Professor Frederick Slocum.

A. A. FISHER, *Secretary*

The Mathematics Club of the Oshkosh State Teachers College

The purpose of the club is to promote interest in the subject of mathematics, and to afford an opportunity to study interesting topics connected with mathematics that do not find a place in the usual class discussion. Membership is open to all students who have completed at least one year of college mathematics.

The officers for 1933–1934 were: Carl Rohde, President; Josephine Katzka, Vice President; Alma Gensch, Secretary; Clarence DeGroot, Treasurer. Dr. Beenken and Dr. Price acted as faculty advisers.

The meetings and programs were as follows:

October 3, 1933: "The history of mathematics to 1000 B.C." by Josephine Katzka.

November 6, 1933: "The history of mathematics from 1000 B.C. to 100 B.C." by Helen Ewert; "Elementary principles of the slide rule" by Clarence Discher.

December 12, 1933: "Methods of extracting square roots" by Margaret Farin; "Mathematical wrinkles" by Clarence DeGroot.

January 16, 1934: "Simpson's rule for finding areas" and "Methods of determining Pi" by Lester Lunsted; "Mathematics in a liberal education" by Irene Timm.

February 13, 1934: "Mathematics in the dairy industry" by Hubert Wetak; "How maps are made" by Frank Simpson.

March 6, 1934: "Mathematics and the honey bee" by Vialor Dumdie; "Mathematics and Biology" by Anita Leitzke.

April 3, 1934: "The history of the calendar" by Alma Gensch; "Descriptive geometry" by Carl Rohde.

May 1, 1934: "The empty column" by Gordon Kester; "Astrology" by Joseph Blank.

May 29, 1934: The annual picnic was held at Ruth Van Keuren's cottage on Lake Winneconne.

ALMA GENSCH, *Secretary*

PROBLEMS AND SOLUTIONS

EDITED BY OTTO DUNKEL, H. L. OLSON, and W. F. CHENEY, JR.

ELEMENTARY PROBLEMS

Send all communications about Elementary Problems and Solutions to W. F. Cheney, Jr., Dept. Box 35, Storrs, Connecticut.

The Department of Elementary Problems welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. Problems may be submitted unaccompanied by their solutions.

PROBLEMS FOR SOLUTION

E 113. *Proposed by E. T. Krach, Hasbrouck Heights, N. J.*

Prove that if three circles are so arranged that their six external tangents are real (each tangent touching two circles), then the three points of intersection of the three pairs of corresponding tangents are collinear.

E 114. *Proposed by Maud Willey, Long Beach, Mississippi.*

What is the locus of the parametric equations,

$$x = k + h \sin A,$$
$$y = k + h \sin (A + 2\pi/3),$$
$$z = k + h \sin (A + 4\pi/3),$$

where h and k are constants and A is the parameter?

E 115. *Proposed by J. E. Trevor, Cornell University.*

Four rectangular buildings have lengths x_i, widths y_i, and heights z_i, $i=1, 2, 3, 4$. These dimensions are positive integers in a unit equivalent to ten feet, and $z_i<y_i<x_i$. The owner proposes to add to each building a top story of height y_i-z_i. The values of x_i, y_i, and z_i are such that, for each new story, the numbers expressing its volume, the combined areas of its east and south faces, and its height, add up to 165. What are the dimensions in feet of each building before this construction?

E 116. *Proposed by V. Thébault, Le Mans, France.*

Find the number whose square and cube together require the ten symbols 0, 1, 2, 3, 4, 5, 6, 7, 8 and 9, once each, to write them, and show that the solution is unique.

E 117. *Proposed by Otto Dunkel, Washington University.*

The shorter parallel side of a trapezoid is produced each way a distance equal to the length of the longer parallel side, and the longer parallel side is produced each way a distance equal to the shorter parallel side. Prove that the diagonals of the new trapezoid built on these new parallel sides, intersect at the centroid of the original trapezoid.

SOLUTIONS

E 81 [1934, 188]. *Proposed by Rev. Clete Adams, Quincy College, Quincy, Ill.*

A "Poosh-up" game with eleven balls contains pockets marked "100," "200," "300," "400," "500," "600," "700," "800," and "Double." Every ball in the "Double" pocket doubles the entire score in all the other pockets. What is the greatest score that can be rolled if the numbered pockets can each hold only one ball and the "Double" pocket any number? What is the greatest score possible if each pocket is large enough to hold all the balls?

Solution by Theodore Lindquist, State Normal College, Ypsilanti, Mich.

The meaning is interpreted to be that any value of a ball in any pocket whatever is double once for each ball in the "Double" pocket. For instance, the value of a ball in the 800 pocket becomes, when three balls are in the "Double" pocket, 800×2^3, or 6,400.

Let x represent the number of numbered pockets with a ball in each, beginning with the 800 pocket. Their sum is $50x(17-x)$, and there remain $11-x$ balls in the "Double" pocket. This makes the total score

$$S = 50x(17 - x)2^{11-x} \tag{1}$$

and equating the first derivative to zero gives the equation

$$(\ln 2)x^2 - (2 + 17 \ln 2)x + 17 = 0. \tag{2}$$

This equation has two irrational roots, one greater than eleven, which disqualifies it, and the other between 1 and 2. Substituting 2 for x in (1) gives $S=768{,}000$, and substituting 1 for x in (1) gives $S=819{,}200$, which is the maximum possible score if no numbered pocket can hold more than one ball.

In case a numbered pocket may hold an unlimited number of balls, equation (1) becomes $S=(800x)2^{11-x}$ and equating the derivative to zero gives $x=1/\ln 2 = 1+$. If either 1 or 2 is substituted for x in the revised equation for S, the value is again 819,200, while higher values of x give smaller values of S, as was to be expected.

Editor's note. No other correct solutions to this problem were received, but several readers submitted solutions based on the doubling factor, "two times the number of balls in the Double pocket," instead of "two exponent the number of balls in the Double pocket." Perhaps they have played the game under those rules instead of the rules cited in this problem.

E 82 [1934, 188]. *Proposed by L. S. Johnston, University of Detroit.*

If the points, $P, A, B, C, D, \cdots, K$, lie on a straight line, and the point O is off that line, show that if the segments, $PA, PB, PC, PD, \cdots, PK$ form a harmonic sequence, then the tangents of the angles, $AOP, BOP, COP, DOP, \cdots KOP$, also form a harmonic sequence.

Solution by A. V. Richardson, Bishop's College, Lennoxville, Quebec.

Let S be the foot of the perpendicular from O to the given line, and let $OS=h$ and $SP=k$. Since the given segments are in harmonic progression, we can let $PA=1/a$, $PB=1/(a+d)$, $PC=1/(a+2d)$, $\cdots$. Denote the angles, $SOP, SOA, SOB, SOC, \cdots$, by $A_0, A_1, A_2, A_3, \cdots$, respectively.

Now $\tan POA=\tan(A_1-A_0)=(\tan A_1-\tan A_0)/(1+\tan A_1 \tan A_0)$, which reduces to $h/[k+a(h^2+k^2)]$. Similarly, $\tan POB$ reduces to $h/[k+(a+d)(h^2+k^2)]$, $\tan POC$ reduces to $h/[k+(a+2d)(h^2+k^2)]$, $\tan POD$ reduces to $h/[k+(a+3d)(h^2+k^2)]$, etc. Since these fractions are obviously in harmonic progression, the proposition is proved.

Also solved by L. M. Bauer, H. E. H. Greenleaf, Roy MacKay, E. P. Starke, Simon Vatriquant and the proposer.

E 83 [1934, 188]. *Proposed by Morgan Ward, California Institute of Technology.*

Show that in any arithmetic progression of integers with common difference less than two thousand, at most ten consecutive integers can be primes.

Solution by E. P. Starke, Rutgers University.

There exists a single exception to the proposition as given. The progression with first term -11 and common difference 210 has its first eleven consecutive terms all primes.

Let $a+id$, $i=0, 1, 2, \cdots$, be the terms of the progression. Then a and d have no common factor, for if they did, at most one term would be prime. If p is any prime which does not divide d, then integers r and s can be found such that $dr-ps=1$. If we then take $i=kp-ar$, we will have $a+id$ divisible by p for every integer k. There are then at most $p-1$ consecutive terms of our arithmetic progression not divisible by p, if p does not divide d. Hence a necessary condition that eleven or more consecutive terms should not be divisible by one of the integers, 2, 3, 5, 7 and 11, is that d should have them all as factors, and hence be a multiple of their product, 2310, which is greater than two thousand.

However, a number may be divisible by a prime and still be prime, by equaling that prime or its negative. From the above, there might be as many as $2p-1$ consecutive terms which are prime, even though d is prime to p, provided p or $-p$ is one of the terms. The discussion is then completed by examining the terms of the progressions for which p is 7 or 11 and a is p or $-p$. If p is 7, d is any multiple of 330 under 2000, and if p is 11, d is any multiple of 210 under 2000. However, of these few progressions, only the one noted above has more than ten consecutive primes.

Also solved by Hansraj Gupta, J. Rosenbaum, Simon Vatriquant, Maud Willey and the proposer.

E 84 [1934, 188]. *Proposed by J. D. Leith, University of North Dakota.*

A given parabola, $x^2=4py$, envelopes an infinite family of circles centered on

the Y-axis. Determine the ordinate of the center of such a circle as a function of its radius and p.

Solution by Hansraj Gupta, Government College, Hoshiarpur, India.

Let the ordinate of the center of such a circle be z and its radius r. Then the circle is

$$x^2 + (y - z)^2 = r^2. \tag{1}$$

Since $x^2 = 4py$ envelopes (1), the equation

$$(y - z)^2 + 4py - r^2 = 0 \tag{2}$$

must have two equal roots in y. This gives $4p^2 = 4pz - r^2$, whence $z = (4p^2 + r^2)/4p = p + r^2/4p$.

Also solved by H. E. H. Greenleaf, Roy MacKay, Nicholas Petroff, A. V. Richardson, E. P. Starke, C. W. Trigg, M. J. Turner, and Simon Vatriquant.

E 85 [1931, 188]. *Proposed by R. E. Moritz, University of Washington, Seattle, Wash.*

Show analytically that the two polar coordinate equations, $r = k/(1 + m \cos\theta)$ and $r = -k/(1 - m \cos\theta)$, represent the same curve.

Solution by M. J. Turner, Ball State Teachers College, Muncie, Ind.

If θ is replaced by $180° + \theta$, the curve will be rotated through $180°$ and the equation becomes $r = k/(1 - m \cos\theta)$. Replacing r by $-r$ serves to rotate the curve through an additional $180°$ ane thus return it to its original position. This is equivalent to the equation, $r = -k/(1 - m \cos\theta)$.

Also solved by L. M. Bauer, Hansraj Gupta, Theodore Lindquist, Roy MacKay, E. P. Starke, C. W. Trigg, Simon Vatriquant and Maud Willey.

E 86 [1934, 189]. *Proposed by Roy MacKay, Albuquerque, N. M.*

If the faces of a tetrahedron are congruent triangles, prove that the circumcenter and the centroid are coincident.

Solution by Simon Vatriquant, Athênée Royale d'Ixelles, Brussels.

A simple inspection shows that the opposite edges of the tetrahedron are equal. If we draw through them pairs of parallel planes, we form the circumscribed parallelopiped, in the faces of which the diagonals are equal. This parallelopiped is thus rectangular and has the same circumsphere as the tetrahedron. But the centroid of the parallelopiped coincides with the centroid of the tetrahedron as well as with the center of the circumsphere. Therefore the circumcenter and centroid of the tetrahedron coincide.

In his solution, J. Rosenbaum calls attention to the relationships between this problem and numbers 3512, 3624, and 3671 in the advanced problem section.

Also solved by J. E. LaFon, E. P. Starke and C. W. Trigg.

ADVANCED PROBLEMS

Send all communications about Advanced Problems and Solutions to Otto Dunkel, Washington University, St. Louis, Mo. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide.

Problems containing results believed to be new or extensions of old results are especially sought. The editorial work would be greatly facilitated, if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known textbooks or results found in readily accessible sources, will not be proposed as problems for solution in this department. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

PROBLEMS FOR SOLUTION

3699. *Proposed by V. Thébault, Le Mans, France.*

In the tetrahedron $ABCD$ let G be its centroid and set $a=BC$, $b=CA$, $c=AB$, $a'=DA$, $b'=DB$, $c'=DC$. If the tetrahedron $GABC$ is trirectangular at G, show that

(a) $$2(b^2+c^2) = a^2+a'^2,\quad 2(c^2+a^2) = b^2+b'^2,$$
$$2(a^2+b^2) = c^2+c'^2;$$

(b) the edges DA, DB, DC are equal, respectively, to the medians of triangle $A_2B_2C_2$, the anticomplement of ABC; and the tetrahedron $DA_2B_2C_2$ has equal opposite edges.

Note by the Editors. Anticomplement means here that the side A_2B_2 passes through C and is parallel to AB; and, similarly, for the other two sides.

3700. *Proposed by J. H. M. Wedderburn, Princeton University.*

Find a basis h_{pq} for matrices of order n such that each element of the basis is idempotent and

$$h_{pq}h_{rs} = k_{pqrs}h_{ps}$$

where k_{pqrs} is a rational number.

3701. *Proposed by Harry Langman, Brooklyn, N. Y.*

If the C's denote binomial coefficients, show that

(a) $$\sum_{i=0}^{n-1}(-1)^i\frac{{}_{n-1}C_i}{i+1} = \frac{1}{n},$$

(b) $$n\sum_{i=0}^{n-1}(-1)^i\frac{{}_{n-1}C_i}{(i+1)^2} = \sum_{i=1}^{n}\frac{1}{i},$$

3702. *Proposed by W. P. Udinski, University of Texas.*

In the domain of real numbers, let $f(x)$ be a function such that $f(\alpha+mh)$ (h is a constant) is finite and of the same sign or zero for every $m=0, 1, 2, \cdots, n-1$. Further, let $P_s(x)$ be a polynomial of degree s such that

$$\sum_{m=0}^{n-1} f(\alpha + mh)P_s(\alpha + mh)(\alpha + mh)^k = 0 \tag{1}$$

for $k = 0, 1, 2, \cdots, s-1$.

Finally, we assume, that at some point of the set involved in the summation in (1) neither $f(x)$ nor $P_s(x)$ is zero.

Under the above hypothesis, prove that the roots of $P_s(x)$ are all real and distinct and all lie on the interval $\alpha < x < \alpha + (n-1)h$.

The above lemma is applicable to each member of the family of orthogonal (with respect to a finite summation) polynomials introduced by Tschebyscheff. See: C. Jordan, *Lond. Math. Soc. Proc.* series 2, vol. 20, page 297.

3703. *Proposed by V. Thébault, Le Mans, France.*

Prove that the integral part of the fourth root of the product of eight consecutive integers is equal to x^2+7x+6, where x is the smallest of the eight integers. This result may be used to show that the product of eight consecutive integers is never the fourth power of an integer.

3704. *Proposed by V. Thébault, Le Mans, France.*

Show that in any tetrahedron the sum of the squares of the distances of the center of the hyperboloid of altitudes (Monge's point) to the vertices is equal to the square of the diameter of the circumscribing sphere.

SOLUTIONS

3628 [1933, 427]. *Proposed by Mannis Charosh, New Utrecht High School, Brooklyn, N. Y.*

M_1, M_2, M_3 are the midpoints of sides A_2A_3, A_3A_1, A_1A_2 of a triangle $A_1A_2A_3$. B_1, B_2, B_3 are the symmetrics of any point P with respect to M_1, M_2, M_3. Prove the following theorems:

(*a*) A_1B_1, A_2B_2, A_3B_3 are concurrent in some point Q which bisects each of these lines.

(*b*) Triangles $A_1A_2A_3$ and $B_1B_2B_3$ are congruent and their corresponding sides are parallel.

(*c*) If P be taken as the orthocenter of $A_1A_2A_3$, Q will be its circumcenter.

(*d*) If P be taken as the circumcenter of $A_1A_2A_3$, Q will be the center of its nine point circle.

(*e*) If N_1, N_2, N_3 are the midpoints of sides B_2B_3, B_3B_1, B_1B_2 respectively, then A_1N_1, A_2N_2, A_3N_3 are concurrent in some point R.

(*f*) M_1N_1, M_2N_2, M_3N_3 are concurrent in Q, which bisects each of them.

(*g*) P, Q, R are collinear, and Q bisects PR.

(*h*) If M is the centroid of $A_1A_2A_3$ and N is the centroid of $B_1B_2B_3$, then M and N lie on the line PQR, and Q bisects MN.

(*i*) If P describes any locus, Q and R describe similar loci.

Solution by A. S. Householder, Washburn College, Topeka, Kansas.

These theorems are all simple consequences of the well known theorems which may be stated as follows:

If the lines A_iA_i' $(i=1, 2, 3, \cdots)$ are all concurrent in a point S which divides each of them in the ratio $SA_i'/SA_i=k$, then the figure $A_1A_2A_3 \cdots$ is similar to $A_1'A_2'A_3' \cdots$, the corresponding lines are parallel, and the ratio of corresponding segments is $|k|$. Conversely, if the figures $A_1A_2A_3 \cdots$ and $A_1'A_2'A_3' \cdots$ are similar with corresponding lines parallel, then there exists a point S of concurrence of A_iA_i', dividing these segments in a fixed ratio k.

If we denote by the expansion $S(k)$, or, where there is no ambiguity, simply S, the transformation which brings the first figure, A, into coincidence with the second, A', then it follows that the resultant of two expansions $S_1(k_1)$ and $S_2(k_2)$ is again an expansion $S(k_1k_2)$. The center S can be determined as follows: The point S_1 under the expansion S_1 is invariant, under S_2 it becomes S_1' where $S_2S_1':S_2S_1=k_2$; the resultant S carries S_1 into S_1', and $SS_1':SS_1=k_1k_2$. Since S_1, S_2, and S_1' as well as S_1, S_1', and S are collinear, it follows that S_1, S_2, and S are collinear. Moreover, the above proportions give, *dividendo*, $S_1S_1':S_2S_1=k_2-1$, and $S_1S_1':SS_1=k_1k_2-1$, whence $S_1S:S_1S_2=k_2-1:k_1k_2-1$.

Applying these results we observe that $M_1M_2M_3$ is obtained from $A_1A_2A_3$ by the expansion $M(-1/2)$; $B_1B_2B_3$ from $M_1M_2M_3$ by $P(2)$; whence $B_1B_2B_3$ is obtained from $A_1A_2A_3$ by $Q(-1)$, where Q is on PM and $QP=3QM$. Since the ratio $k=-1$, Q is the midpoint of the segments A_iB_i. Thus we obtain (a) and (b). $N_1N_2N_3$ is obtained from $B_1B_2B_3$ by $N(-1/2)$, and hence is obtained from $A_1A_2A_3$ by $R(1/2)$, with R on QN and $NR=2QN$. Theorem (f) follows from the fact that M_i and N_i are homologous points of figures A and B. The expansion Q is its own reciprocal and hence may be regarded as sending the M_i and B_i into the N_i and A_i, hence the point of concurrence of M_iB_i into the point of concurrence of N_iA_i, i.e. P into R. Thus Q is the midpoint of PR. Again the same expansion sends M into N, whence the last part of (h). Moreover M is known to be on PQ, so that N is likewise. Since M is fixed, and the ratios $MQ:MP=-1/2$ and $MR:MP=-2$ are fixed, the expansions $M(-1/2)$ and $M(-2)$ send the locus of P into the loci of Q and R respectively. There remain only (c) and (d), and these follow from the above relations between P, Q and M, and the fact that the orthocenter H, the circumcenter C, the centroid M, and the center to the nine-point circle O are collinear, and $MC:MO:MH=-2:1:4$.

Solved also by Frank Ayres, Jr., J. W. Clawson, T. C. Esty, F. C. Gentry, R. MacKay, A. Pelletier, S. Vatriquant, and the proposer.

3630. [1933, 428]. *Proposed by Rufus Crane, Ohio Wesleyan University.*

It is known that there are three spheres escribed to the tetrahedron each of which corresponds to a pair of opposite edges. Let the tetrahedron be $A_1A_2A_3A_4$, let the area of the face opposite to A_i be α_i, and let the dihedral angle whose edge is A_iA_j be δ_{ij}. Prove that the sphere corresponding to the pair of edges A_iA_j and A_kA_l lies external to the edge A_iA_j if $\alpha_k+\alpha_l<\alpha_i+\alpha_j$, a condition which may be reduced to $\alpha_k\alpha_l\cos^2(\delta_{ij}/2)<\alpha_i\alpha_j\cos^2(\delta_{kl}/2)$.

Solution by the Proposer.

Let the required center be J, supposed to be external to the edge A_iA_j, let the incenter be I, and let the line IJ cut A_iA_j at Q and A_kA_l at P. In any tetrahedron, the plane bisector of any dihedral angle divides the opposite side into segments proportional to the areas of the adjacent faces. Hence:

$$\frac{A_jQ}{QA_i}=\frac{\alpha_i}{\alpha_j}\quad\frac{A_kP}{PA_l}=\frac{\alpha_l}{\alpha_k}\,.$$

Let the perpendicular from P upon α_k be g, the altitude from A_k upon α_k be h, and the radius of the sphere (J) be r_j. Then:

$$\frac{JQ}{QP}=\frac{r_j}{g}$$

and

$$\frac{g}{h}=\frac{\alpha_k}{\alpha_k+\alpha_l};$$

but

$$\frac{r_j}{h}=\frac{\text{vol. } JA_iA_jA_l}{\text{vol. } A_iA_jA_kA_l}=\frac{\alpha_k}{\alpha_i+\alpha_j-\alpha_k-\alpha_l};$$

hence

$$\frac{r_j}{g}=\frac{\alpha_k+\alpha_l}{\alpha_i+\alpha_j-\alpha_k-\alpha_l}$$

and

$$\frac{JQ}{JP}=\frac{r_j}{r_j+g}=\frac{\alpha_k+\alpha_l}{\alpha_i+\alpha_j}\,.$$

Thus, $JQ<JP$ if $\alpha_k+\alpha_l<\alpha_i+\alpha_j$, and the first relation is proved.

To prove the second relation we use a relation deduced by Painvin (*Nouvelles Annales*, ser. 2, vol. 1, 1862, p. 267 ff.) namely:

$$\alpha_k^2+\alpha_l^2-2\alpha_k\alpha_l\cos\delta_{ij}=\alpha_i^2+\alpha_j^2-2\alpha_i\alpha_j\cos\delta_{kl}$$

Combining this with the square of the inequality above deduced, we find that

$$\alpha_k\alpha_l(1+\cos\delta_{ij})<\alpha_i\alpha_j(1+\cos\delta_{kl})$$

or

$$\alpha_k\alpha_l\cos^2(\delta_{ij}/2)<\alpha_i\alpha_j\cos^2(\delta_{kl}/2)$$

which is the second relation proposed.

This condition is obviously both necessary and sufficient.

I have been unable to find that a valid criterion for the position of the sphere here discussed has ever been published. In the editorial note appended to the solution of problem 421 (1914, 301), it is stated that it the dihedral angles at opposite pairs of edges are equal the corresponding ex-center is at infinity, while if they are unequal it is in the compartment of the greater angle. That this is incorrect is evident from a specific case of this problem which was discussed

by G. Fontené in paragraph 13 of a paper in the *Nouvelles Annales*, ser. 4, vol. 9, 1909, p. 57. He shows that for a trirectangular tetrahedron it is possible for the escribed sphere to lie either in the compartment outside a right dihedral or in the compartment outside the opposite acute dihedral. Escribed spheres of this type, as well as those that are escribed to a face, are discussed in various places, but the question of a criterion for their position seems to be nowhere mentioned.

Note by the Editors. The first part of the problem results from the harmonic properties of the incenter I and an excenter such as J. For the four planes having A_jA_l as axis and passing through J, Q, I, P form an harmonic set. Hence

$$\frac{JQ}{QI} = \frac{JP}{IP} = \frac{r_j}{r}, \tag{1}$$

where the first equality results from the harmonic division of JI, and the last equality easily follows from the figure. Also from the definitions of J and I, and their positions, we have

$$r(\alpha_i + \alpha_j + \alpha_k + \alpha_l) = r_j(\alpha_i + \alpha_j - \alpha_k - \alpha_l) = 3V, \tag{2}$$

where V is the volume of the tetrahedron. Setting the ratios in (1) equal to ρ, we have in succession

$$JQ + JP = \rho QP,$$

$$\frac{JQ}{JP} = \frac{\rho - 1}{\rho + 1} = \frac{r_j - r}{r_j + r} = \frac{\alpha_k + \alpha_l}{\alpha_i + \alpha_j}.$$

The Painvin equality is easily obtained as follows: Let $\boldsymbol{a}_i$ denote the vector area of a face of a given polyhedron, $\boldsymbol{a}_i = \alpha_i \boldsymbol{n}_i$, where α_i is the area of the face and $\boldsymbol{n}_i$ is the unit vector normal to that face and directed towards the interior of the polyhedron. It is well known and easily proved that the sum of these vector areas is zero for a closed figure. In this case we have

$$(\boldsymbol{a}_i + \boldsymbol{a}_j)^2 = (-\boldsymbol{a}_k - \boldsymbol{a}_l)^2,$$

$$\boldsymbol{a}_i^2 + \boldsymbol{a}_j^2 + 2\boldsymbol{a}_i\cdot\boldsymbol{a}_j = \boldsymbol{a}_k^2 + \boldsymbol{a}_l^2 + 2\boldsymbol{a}_k\cdot\boldsymbol{a}_l,$$

where the dot and exponent mean scalar products of the two vectors. For example, $\boldsymbol{a}_i\cdot\boldsymbol{a}_j = \alpha_i\alpha_j\ \boldsymbol{n}_i\cdot\boldsymbol{n}_j = -\alpha_i\alpha_j\cos\delta_{lk}$. Hence

$$\alpha_i^2 + \alpha_j^2 - 2\alpha_i\alpha_j\cos\delta_{lk} = \alpha_k^2 + \alpha_l^2 - 2\alpha_k\alpha_l\cos\delta_{ij}.$$

3631 [1933, 495]. *Proposed by Norman Anning, University of Michigan.*

On the Argand diagram a regular pentagram is constructed with its center at the origin and one vertex on the axis of positive real numbers. Find the equation whose roots are the numbers associated with the ten points where the lines intersect. Can the scale be so chosen that the equation will have integral coefficients?

Solution by B. D. Roberts, New Mexico Normal University.

Let the radius of the circle circumscribing the pentagon be a. The vertices of the pentagon then represent the five distinct fifth roots of a^5, and (x^5-a^5) is a factor of the required function. The remaining five points represent the five distinct fifth roots of negative b^5, where by the law of sines, $b=a\sin 54°/\sin 18°$. $(x^5+b^5)(x^5-a^5)=0$ is one form of the required equation.

Inserting the values $\sin 18°=(\sqrt{5}-1)/4$ and $\sin 54°=(\sqrt{5}+1)/4$, we reduce the equation to the form:

$$x^{10}+x^5a^5(121+55\sqrt{5})/2-a^{10}(123+55\sqrt{5})/2=0.$$

The question of integral coefficients reduces to that of rational coefficients in the above form. Letting the second and third coefficients above equal, numerically, b/c and d/e respectively, and eliminating the a we get $b^2/121c^2=d/e$ which is to have integral solutions in b, c, d, and e. Obviously there are many such solutions, the simplest of which gives $b/c=11$ and $d/e=1$, and the corresponding $a=(-1+\sqrt{5})/2$.

The equation then is

$$x^{10}+11x^5-1=0.$$

More generally $x^{10}+11Kx^5-K^2=0$, in which K is any rational number, satisfies the requirements.

Solved also by W. B. Campbell, D. C. Duncan, R. MacKay, A. A. Rood, E. P. Starke, and the proposer.

Note by the Editors

In MacKay's solution any regular polygon with an odd number, n, of vertices was considered; and it was found that

$$\frac{b}{a}=\frac{\cos\frac{\pi}{n}}{\cos\frac{2\pi}{n}}$$

where a and b have the meaning in the above solution. And, if

$$\left[\left(\frac{\cos\frac{\pi}{n}}{\cos\frac{2\pi}{n}}\right)^n-1\right]^2\left(\frac{\cos\frac{\pi}{n}}{\cos\frac{2\pi}{n}}\right)^{-n}$$

is rational, then a and b may be given real positive values so that the equation for the $2n$ vertices has rational coefficients.

The result obtained by Starke may be put in a slightly modified form. If we set $a=kt_1$, $b=-kt_2$, the required equation is

$$x^{10} - k^5(t_1^5 + t_2^5)x^5 + k^{10}(t_1t_2)^5 = 0, \tag{1}$$

where t_1 is the positive, and t_2 is the negative root of

$$4x^2 + 2x - 1 = 0. \tag{2}$$

Since the required symmetric functions of t_1 and t_2 are rational, $t_1^5+t_2^5 = -11/2^5$, $(t_1t_2)^5 = -1/2^{10}$, the rest easily follows.

The proposer in his solution stated that, if the problem had read "regular star 7-gon" instead of pentagram, the corresponding equations would be

$$x^{21} + 57x^{14} - 289x^7 - 1 = 0,$$

$$x^3 + x^2 - 2x - 1 = 0$$

where the second equation gives the points on the axis of reals.

The regular pentagon and its related equation (2) are of interest in connection with the famous Fibonacci sequence, 1, 1, 2, 3, $\cdots$, u_i, $\cdots$, and the rabbit problem. These numbers are given by

$$u_i = \frac{1}{\sqrt{5}}[(-2t_2)^i - (-2t_1)^i], \quad u_{i+2} = u_{i+1} + u_i.$$

Consult the solution of problems 2809 [1921, 329] and 3360 [1930, 199]; and also a note on *A Fibonacci Series* in this MONTHLY, vol. 25 (1918), p. 235.

3633 [1933, 495]. *Proposed by N. A. Court, University of Oklahoma.*

If the three face angles of a trihedral angle of a tetrahedron are right angles, the line joining this vertex of the tetrahedron to the centroid of the opposite face is equal to two thirds of the circumradius of the tetrahedron.

Solution by A. D. Bradley, Hunter College.

Let the tetrahedron be $ABCD$ with right angles at vertex A. Let M be the midpoint of BC. Let MO be perpendicular to ABC, on the same side as AD, and equal to one half AD. Then O is the circumcenter of $ABCD$, AD and MO are coplanar, and AO intersects DM, a median of BCD, in N. Triangle ADN is similar to triangle OMN, with ratio of sides 2:1. Hence $DN=2NM$ and N is the centroid of BCD. Hence $AN=2NO$, or $AN=2/3\,AO$.

Solved also by Frank Ayres, Jr., L. M. Bauer, Ruth S. Berkow, J. W. Clawson, T. C. Esty, W. C. Janes, L. M. Kelly, R. MacKay, A. Pelletier, P. W. Allen Raine, O. J. Ramler, A. A. Rood, J. Rosenbaum, W. P. Udinski, F. Underwood, S. Vatriquant, and Maud Willey.

3634 [1933, 495]. *Proposed by J. Rosenbaum, Milford, Connecticut.*

In the triangle ABC, the bisectors of the angles A and B meet the opposite sides at D and E. Prove that if DE divides the angles CDA and CEB into parts having equal ratios then the sides CA and CB are equal.

Solution by L. M. Kelly, Lawrence, Mass.

Denote by e and d the angles DEB and EDA, and by $2x$ and $2y$ the angles at A and B of triangle ABC: Then we have

$$\text{(1)} \qquad \frac{e}{\angle CEB} = \frac{d}{\angle CDA}, \quad \frac{e}{2x+y} = \frac{d}{2y+x}.$$

Let the lines BE and AD meet in O, and the internal bisectors of the angles at E and D of triangle CED meet in M. Then C, M, O lie on the internal bisector of angle C. From two triangles with the common vertex O we have $e+d=x+y$; and this equation with the second of (1) gives

$$d = \frac{2y+x}{3} = \frac{1}{3} \angle CDA, \quad e = \frac{1}{3} \angle CEB,$$

Thus the triangles MED and OED are symmetric; and hence MO is perpendicular to ED, or the bisector of angle C is perpendicular to ED. The triangle CED is therefore isosceles and $e=d$. This with the second equation of (1) gives $x=y$; and from this it follows that $CA=CB$.

Solved also by Ruth S. Berkow, A. G. Clark, J. W. Clawson, S. Vatriquant, and the proposer.

NEWS AND NOTICES

Readers are invited to contribute to the general interest of this department by sending items to Professor J. H. Weaver, Ohio State University, Columbus, Ohio.

During the sessions of the American Association for the Advancement of Science at Pittsburgh, beginning December 27, 1934, there will be held the annual science exhibition in the Mellon Institute of Industrial Research Building. This exhibition is expected to be the largest ever held. It will include apparatus used in the study of cosmic rays, deuterium, neutrons, induced radioactivity, stratosphere, and recent advances in physics.

On June 19 and 20, the Mathematics Section of the Society for the Promotion of Engineering Education held a conference at Cornell University, Ithaca, New York, in connection with the annual meeting of the Society. The general topic of the conference was Advanced Mathematics Courses for Engineering Students. A number of interesting papers were read by both teachers of mathematics to engineering students and men from the industries. A digest of these papers appears elsewhere in this issue.

The Philosophy of Science Association has begun publication of a quarterly journal, Philosophy of Science, under the editorship of W. M. Malisoff. Among the articles in the first issue are the following: Quantum mechanics as a basis for philosophy, by J. B. S. Haldane; The foundations of the theory of probabili-

ties, by D. J. Struik; The character of philosophical problems, by Rudolf Carnap. Subscriptions may be sent to Williams and Wilkins, the Waverly Press, Baltimore.

At a recent meeting of the American Mathematical Society held at Chicago, the dinner program was in honor of Thomas F. Holgate, emeritus professor of mathematics at Northwestern University, and H. E. Slaught, emeritus professor of mathematics at the University of Chicago. Each had been secretary of the Chicago section for a period of ten years and each had served his university continuously for a period of more than forty years. Addresses were made by Professors F. R. Moulton, L. E. Dickson, D. R. Curtiss, G. A. Bliss, and W. C. Graustein. Professors Holgate and Slaught responded.

Columbia University has conferred the degree of Doctor of Science on E. W. Brown, emeritus professor of mathematics at Yale University.

The degree of Doctor of Laws was conferred upon M. Albert Linton, President of the Provident Mutual Life Insurance Company, by Miami University at its commencement in June of this year.

The Franklin Institute has awarded Franklin medals to Professor H. N. Russell, of Princeton University, for his "pioneer work in the application of physical theories to astronomical problems," and to Dr. Irving Langmuir, of the General Electric Company at Schenectady, "in recognition of his investigations in physics and chemistry."

Dr. W. F. Osgood, emeritus professor of mathematics at Harvard, has accepted an invitation to join the Department of Mathematics at the National University of Peking.

Professor J. C. Rogers, dean at Piedmont College, Demorest, Georgia, was made president of North Georgia College, at Dahlonega, in June of this year.

Professor H. S. Vandiver, of the University of Texas, and Professor Norbert Wiener, of the Massachusetts Institute of Technology, have been elected to membership in the National Academy of Sciences.

F. J. Brand of the University of British Columbia has been promoted to an assistant professorship.

Dr. S. S. Cairns of Lehigh University has been promoted to an assistant professorship.

Assistant Professors Jesse Douglas and R. D. Douglass of Massachusetts Institute of Technology have been promoted to associate professorships.

Dr. H. H. Ferns of the University of Toronto has been appointed assistant professor of mathematics at the University of Saskatchewan.

Dr. W. W. Flexner of Bryn Mawr College has been appointed assistant professor of mathematics at Cornell University.

Dr. C. M. Huber of Wilson Teachers College has been promoted to an assistant professorship.

Associate Professor C. C. MacDuffee of Ohio State University has been promoted to a professorship of mathematics.

Dr. E. J. McShane of Princeton University has been promoted to an assistant professorship.

Dr. H. W. Raudebush of Barnard College, Columbia University, has been appointed assistant professor of mathematics at Yale University.

Associate Professor E. E. Whitford of the College of the City of New York has been promoted to a professorship of mathematics.

Associate Professor A. H. Wilson of Haverford College has been promoted to a professorship of mathematics.

Dr. G. T. Whyburn of Johns Hopkins University has been appointed professor of mathematics at the University of Virginia.

The following appointments to instructorships in mathematics have been announced:

University of California at Davis: Dr. E. V. Roessler.
Johns Hopkins University: J. F. Wardwell.
Massachusetts Institute of Technology: Dr. J. L. Barnes.
Princeton University: H. G. Swain and Dr. R. J. Walker.
St. Bonaventure College: D. J. Colbert.

The following have been awarded National Research Fellowships in mathematics for the year 1934–35:

S. F. Barber, L. M. Blumenthal, R. H. Cameron, Ralph Hull, D. C. Lewis, Jr., W. T. Martin, Deane Montgomery, F. J. Murray, S. D. Myers, D. S. Nathan, M. I. S. Robertson, and G. C. Webber. This list includes renewals.

Dr. E. R. Lorch has been awarded a Cutting Fellowship at Columbia University.

Dr. W. C. Randels of Brown University has been awarded a Sterling Fellowship at Yale University.

Nicholas Rashevsky, formerly of the Westinghouse Company, has been awarded a General Education Board Scholarship in mathematical bio-physics for work at the University of Chicago.

J. T. Erwin, formerly professor at George Washington University, died October 15, 1933, at the age of 63.

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DIRECTORY

EDITORIAL CORRESPONDENCE should be addressed to the Editor-in-Chief, W. B. Carver, White Hall, Cornell University, Ithaca, N.Y.

BOOKS FOR REVIEW should be addressed to R. A. Johnson, Brooklyn College, 66 Court Street, Brooklyn, N.Y.

BUSINESS CORRESPONDENCE should be addressed to the Secretary-Treasurer of the Association, W. D. Cairns, 33 Peters Hall, Oberlin, Ohio.

CHANGE OF ADDRESS: Members should send notice of any change of address to the Secretary-Treasurer, W. D. Cairns, Oberlin, Ohio, before the 10th of each month.

MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Nineteenth Annual Meeting of the Association, Pittsburgh, Pa., Dec. 29, 1934—Jan. 1, 1935

The following is a list of the sections of the Association, with dates of those section meetings which have been scheduled for 1934 and reported to the Secretary.

Allegheny Mountain, Pittsburgh, Pa., Feb. 10; Washington, Pa., May 5.
Illinois, Jacksonville, May 4-5.
Indiana, La Fayette, May 11-12.
Iowa, Des Moines, April 20-21.
Kansas, Topeka, Mar. 17.
Kentucky, May.
Louisiana-Mississippi, Jackson, Miss., Mar. 23-24.
Maryland-District of Columbia-Virginia, Baltimore, Md., Dec. 8.
Michigan, Ann Arbor, Mar. 17.
Minnesota, Northfield, May 12.
Missouri.
Nebraska, Crete, Apr. 27.
Ohio, Columbus, Apr. 5.
Oklahoma, Oklahoma City, Feb. 9.
Philadelphia, Philadelphia, Dec. 1.
Rocky Mountain, Colorado Springs, Apr. 20-21.
Southeastern, University, Ala., Mar. 30-31.
Southern California, Riverside, Mar. 3.
Texas, College Station, May 5.
Wisconsin, Oshkosh, May 5.

Affiliated Organizations: The New England Association of Teachers of Mathematics, The National Council of Teachers of Mathematics.

GEORGE BANTA PUBLISHING COMPANY, MENASHA, WISCONSIN

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(INCORPORATED)

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THE EIGHTEENTH SUMMER MEETING OF THE MATHEMATICAL ASSOCIATION

The eighteenth summer meeting of the Mathematical Association of America was held, by invitation, at Williams College, Williamstown, Massachusetts, on Monday and Tuesday, September 3 and 4, 1934, in conjunction with the summer meeting and colloquium of the American Mathematical Society. Two hundred sixty-seven were present at the meetings, including the following one hundred forty-five members of the Association:

C. R. Adams, Brown University
R. P. Agnew, Cornell University
E. B. Allen, Rensselaer Polytechnic Institute
R. C. Archibald, Brown University
H. T. R. Aude, Colgate University

Ida Barney, Yale University
R. A. Beaver, New York State College for Teachers
J. A. Benner, Lafayette College
A. A. Bennett, Brown University
Theodore Bennett, University of Wisconsin
E. M. Berry, Lynchburg College
William Beverley, Lafayette College
Harry Birchenough, New York State College for Teachers
G. D. Birkhoff, Harvard University
L. M. Blumenthal, Institute for Advanced Study
Joseph Bowden, Adelphi College
Julia W. Bower, Connecticut College
J. G. Bowker, Middlebury College
H. W. Brinkmann, Swarthmore College
B. H. Brown, Dartmouth College
H. S. Brown, Hamilton College
R. E. Bruce, Boston University
J. A. Bullard, University of Vermont
L. H. Bunyan, Rutgers University
W. H. Bussey, University of Minnesota

S. S. Cairns, Lehigh University
W. D. Cairns, Oberlin College
R. H. Cameron, Princeton University
B. H. Camp, Wesleyan University
C. C. Camp, University of Nebraska
G. A. Campbell, Bell Telephone Laboratories
W. B. Carver, Cornell University
John Cawley, Lafayette College
Abraham Cohen, Johns Hopkins University
Hollis Cooley, New York University
Lennie P. Copeland, Wellesley College
H. B. Curry, Pennsylvania State College
D. R. Curtiss, Northwestern University
John Curtiss, Harvard University

H. H. Dalaker, University of Minnesota
D. R. Davis, New Jersey State Teachers College
F. F. Decker, Syracuse University
L. L. Dines, Carnegie Institute of Technology
H. A. DoBell, New York State College for Teachers
H. L. Dorwart, Williams College
Arnold Dresden, Swarthmore College
L. A. Dye, Cornell University

J. N. Eastham, Nazareth College
H. T. Engstrom, Yale University
G. W. Evans, Swampscott, Massachusetts
H. P. Evans, University of Wisconsin
H. S. Everett, University of Chicago

D. A. Flanders, New York University
L. R. Ford, Rice Institute
W. B. Ford, University of Michigan
R. M. Foster, Bell Telephone Laboratories
Philip Franklin, Massachusetts Institute of Technology
T. C. Fry, Bell Telephone Laboratories

C. A. Garabedian, Bard College
H. M. Gehman, University of Buffalo
B. P. Gill, College of the City of New York
D. C. Gillespie, Cornell University

J. G. Hardy, Williams College
L. A. Hazeltine, Stevens Institute of Technology
Olive C. Hazlett, University of Illinois
E. R. Hedrick, University of California at Los Angeles

34. Mrs. W. B. Carver
35. J. G. Hardy
36. W. B. Carver
37. Elizabeth Wilson
38 Mrs. E. C. Molina
39. Mrs. Perez
40. H. S. White

15. G. D. Birkhoff
16. R. C. Archibald
17. J. R. Kline
18. W. E. Roth
19. W. D. Cairns
20. E. V. Huntington

54. Mrs. Joseph Bowden
55. Mrs. Virgil Snyder
56. J. A. Shohat
57. Mrs. Philip Franklin
58. Philip Franklin
59. Mrs. V. B. Bagnall
60. Harry Levy

74. S. S. Cairns
75. G. M. Merriman
76. Marguerite Lehr
77. T. R. Hollcroft
78. J. L. Walsh
79. Virgil Snyder
80. C. R. Adams

94. R. P. Bailey
95. N. H. McCoy
96. Mrs. W. H. McEwen
97. Alan Hazeltine
98. R. L. Jeffery
99. Ruth E. Morris
100. Richard Morris

114. C. H. Yeaton
115. J. H. Kusner
116. Saunders MacLane
117 W. H. McEwen
118. E. R. Lorch
119. Mrs. A. E. Landry
120. E. C. Molina

135. T. Y. Thomas
136. Lennie P. Copeland
137. G. B. Price
138. F. J. Murray
139. E. P. Northrop
140. B. P. Gill

154. V. B. Bagnall
155. B. W. Jones
156. G. A. Campbell
157. H. B. Phillips
158. J. F. Randolph
159. D. C. Gillespie
160. H. W. Brinkmann

161. H. J. Miles
162. Garrett Birkhoff
163. Helen G. Russell

T. R. Hollcroft, Wells College
E. V. Huntington, Harvard University
W. A. Hurwitz, Cornell University

M. H. Ingraham, University of Wisconsin

R. L. Jeffery, Acadia University
B. W. Jones, Cornell University

L. S. Kennison, Brooklyn College
J. R. Kline, University of Pennsylvania
J. H. Kusner, University of Florida

W. D. Lambert, U. S. Coast and Geodetic Survey
A. E. Landry, Catholic University of America
R. E. Langer, University of Wisconsin
Marguerite Lehr, Bryn Mawr College
Harry Levy, University of Illinois
S. B. Littauer, Boston, Massachusetts

L. A. MacColl, Bell Telephone Laboratories
N. H. McCoy, Smith College
W. H. McEwen, Mount Allison University
Morris Marden, University of Wisconsin, Extension Division
A. E. Meder, Jr., Rutgers University
G. M. Merriman, University of Cincinnati
H. J. Miles, University of Illinois
W. M. Miller, Tufts College
E. C. Molina, Bell Telephone Laboratories
C. N. Moore, University of Cincinnati
Richard Morris, Rutgers University
Marston Morse, Harvard University
David Moskovitz, Carnegie Institute of Technology
L. T. Moston, Waynesburg College
E. J. Moulton, Northwestern University
J. R. Musselman, Western Reserve University

J. H. Neelley, Carnegie Institute of Technology

C. O. Oakley, Haverford College
Oystein Ore, Yale University
F. W. Owens, Pennsylvania State College
Mrs. F. W. Owens, State College, Pennsylvania

F. W. Perkins, Dartmouth College
L. R. Perkins, Middlebury College
G. B. Price, University of Rochester

Tibor Radò, Ohio State University
Susan M. Rambo, Smith College
O. J. Ramler, Catholic University
J. F. Randolph, Cornell University
W. R. Ransom, Tufts College
H. W. Raudenbush, Jr., Yale University
Mina S. Rees, Hunter College
W. T. Reid, University of Chicago
C. E. Rhodes, University of Cincinnati
R. G. D. Richardson, Brown University
Robin Robinson, Dartmouth College
W. E. Roth, University of Wisconsin

S. A. Schelkunoff, Bell Telephone Laboratories
Caroline E. Seely, American Mathematical Society
R. S. Shaw, College of the City of New York
I. M. Sheffer, Pennsylvania State College
J. A. Shohat, University of Pennsylvania
L. L. Silverman, Dartmouth College
W. G. Simon, Western Reserve University
T. M. Simpson, University of Florida
A. W. Smith, Colgate University
Clara E. Smith, Wellesley College
W. M. Smith, Lafayette College
Virgil Snyder, Cornell University
I. S. Sokolnikoff, University of Wisconsin
E. R. Stabler, Harvard University
Ellen C. Stokes, New York State College for Teachers

T. Y. Thomas, Princeton University
J. I. Tracey, Yale University

F. E. Ulrich, Union College

J. L. Walsh, Harvard University
L. E. Ward, University of Iowa
J. F. Wardwell, Johns Hopkins University
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C. W. Watkeys, University of Rochester
V. H. Wells, Williams College
A. H. Wilson, Haverford College
Ruth G. Wood, Smith College
Euphemia R. Worthington, University of California at Los Angeles

C. H. Yeaton, Oberlin College

The meetings were held in Jesup Hall, on the lower floor of which were the rooms for the registration of members. The mathematicians were conveniently housed in the college dormitories and had their meals at the Williams Inn or the Greylock Hotel. These arrangements made possible the happy personal contacts that have characterized the summer meetings in recent years. The committee on local arrangements, under the chairmanship of Professor J. G. Hardy, had made very complete arrangements for the meetings with the result that all the plans for the week proceeded very smoothly and efficiently. The Faculty Club close by on the campus was used constantly, especially during the evening and at tea time when the ladies of the mathematics faculty served refreshments. An organ recital was given on Wednesday evening at Chapin Hall by Mr. C. H. Safford, director of music, with a group of songs by Mrs. Safford; this was attended by most of the visitors and afforded a very pleasing feature of the week. Wednesday afternoon was devoted to an excursion to the new Bennington College for Women and the Bennington Battle Field, returning over the Taconic Trail; transportation was provided for those who did not have their own automobiles. Many short trips were made during the week to nearby points, including trips on foot and by automobile to Mount Greylock, 3,505 feet high, whence a wide and magnificent view is to be had.

One hundred eighty-one attended the joint dinner on Thursday evening at the Greylock, with Professor Huntington as toastmaster. President Dennett of Williams College welcomed the mathematicians to Williams College and Williamstown, and spoke of the continuing worth of the training for the Ph.D. degree in that it served very well as a character diploma; he emphasized also how far the education of the present day is ahead of that of thirty years ago. President Dresden of the Mathematical Association said that the conviction has been growing recently that no country is safe from the distress that has fallen upon Germany the past year. Thinkers must retain and assert their sanity by exercising some insight into causal relations. Many have proposed solutions of the present economic difficulties and it is as well necessary to inquire whether the present situation has a solution. It is a time when mathematicians must adhere to their mathematical points of view and stay by their *alma mater*, Mathematics. Speaking for the Society, Professor Hedrick noted that in these troublous times the Society stands primarily for research and now that everything seems to be haled into some court of investigation to be examined for its *raison d'être*, we ourselves must in one way or another show the world the values of mathematics. This can be done the more effectively by training young people properly in the schools, actuarial offices, etc., so that they too may make its importance evident. We must all aid through our special talents in studying what the outcome of the present political measures will be. Professor Virgil Snyder voiced the sentiments of the visitors in recognition of the hospitality afforded in the comfortable living quarters and the manifold ways in which the comfort and convenience of the visitors were enhanced; this was adopted by hearty and unanimous applause.

The American Mathematical Society held its fortieth summer meeting and seventeenth colloquium from Tuesday to Friday. The colloquium lectures were given by Professor Norbert Wiener of the Massachusetts Institute of Technology on "Fourier transforms in the complex domain," as originally planned by Professor Wiener and Doctor R. A. C. Paley, now deceased. On Thursday afternoon Professor J. A. Shohat of the University of Pennsylvania gave an address entitled "On the expansion of functions in series of orthogonal polynomials." Sessions of the Society for the presentation of papers were held on Tuesday and Thursday afternoons and Wednesday, Thursday and Friday mornings.

The Mathematical Association held sessions on Monday afternoon and Tuesday morning, Vice-President A. A. Bennett presiding at the first session and President Arnold Dresden at the second session. The Association is indebted to the program committee consisting of Professors L. L .Silverman (chairman), C. R. Adams, V. H. Wells, and Ruth G. Wood for the organization of the program. Due to the unavoidable absence of one of the speakers, the program was somewhat rearranged from the form at first announced.

First Session of the Association

1. "Semilinear equations" by Professor C. O. Oakley, Haverford College.
2. "Instruction in advanced preparatory mathematics in England, France and Italy" by Professor W. D. Cairns, Oberlin College. Discussion, led by Professor C. W. Watkeys, University of Rochester.
3. "The aesthetic element in mathematics as related to its teaching" by Professor C. N. Moore, University of Cincinnati.

All of the above papers will appear in full or in slightly modified form in early issues of the Monthly. In connection with his paper, Professor Cairns exhibited during the week an extensive collection of the textbooks currently used in the English "public schools," the French lycées, and the Italian licei, together with numerous examination papers and syllabi of courses. This collection will be shown at the Pittsburgh meetings of the National Council of Teachers of Mathematics and the Mathematical Association of America in December.

Second Session of the Association

1. "The historical and mathematical development of world maps" by Professor B. H. Brown, Dartmouth College.
2. "The place of rigor in mathematics," Retiring presidential address, by Professor E. T. Bell, California Institute of Technology, read by Professor H. W. Brinkmann, Swarthmore College.
3. "The life of Leonhard Euler" by Professor R. E. Langer, University of Wisconsin.

The papers by Professors Brown and Bell will appear in early issues of this Monthly. The paper by Professor Langer will be published in *Scripta Mathematica*.

Meetings of the Board of Trustees

Eight members of the Board were present at the Williamstown meetings. The following forty-seven persons were elected to membership on applications duly certified:

E. B. Allen, Ph.D. (Rensselaer) Prof., Head of Dept., Math. and Astr., Rensselaer Poly. Inst., Troy, N.Y.

Florence L. Axen, A.M. (Wisconsin) Instr., Univ. of Wisconsin, Extension Div., Milwaukee, Wis.

Amos Barksdale, A.M. (Southern Methodist) Dir. of Dept., State Teachers Coll., Denton, Texas.

J. A. Benner, A.M. (Lafayette) Asso. Prof., Lafayette College, Easton, Pa.

J. G. Black, Ph.D. (Michigan) Head of Dept., Math. and Physics, State Teachers Coll., Morehead, Ky.

J. G. Bowker, Ed.M. (Harvard) Asst. Prof., Middlebury College, Middlebury, Vt.

Richard Brauer, Ph.D. (Berlin) Visiting Prof. Univ. of Kentucky, Lexington, Ky.

C. E. Caldwell, A.M. (Ohio State) Instr., State Teachers Coll., Richmond, Ky.

John Cawley, M.S. (Lafayette) Asso. Prof., Lafayette Coll., Easton, Pa.

J. C. Cothran, Ph.D. (Cornell) Prof., Physical Science, State Teachers Coll., Duluth, Minn.

Mildred Dolezal, M.S. (Chicago) Asst., Univ. of Oklahoma, Norman, Okla.

O. E. Eggert, Postal Clerk, U. S. Railway Mail Service, Camden, N.J.

L. A. Fair, A.M. (Peabody) Instr., Math. and Physics, State Teachers Coll., Morehead, Ky.

L. F. Free, A.M. (Lafayette) Instr., Lafayette Coll., Easton, Pa.

J. P. Gill, A.M. (Alabama) Instr., Univ. of Alabama, University, Ala.

C. D. Gregory, A.M. (Yale) Asso. Prof., Coll. of William and Mary, Williamsburg, Va.

Georgia Haswell, A.B. (Ohio Wesleyan) Instr., Math. and Physics, Union Coll., Barbourville, Ky.

W. H. James, A.M. (Boston) Instr. and Head of Dept. of Science and Math., St. Philip's Jr. Coll., San Antonio, Texas.

Brother Josephus Edward, Head of Dept., St. Mary's Coll., Winona, Minn.

C. C. Kiplinger, M.S. (Ohio State) Head of Dept., Math. and Physics, State Teachers Coll., West Liberty, West Va.

R. C. Lamb, A.M. (Virginia) Asso. Prof., U. S. Naval Acad., Annapolis, Md.

Sister Mary Daniel, A.M. (Fordham) Prof., Marywood Coll., Scranton, Pa.

J. S. McNair, M.S. (Chicago) Teacher, Sr. High School, Sheboygan, Wis.

Joseph Milkman, B.S. (Brooklyn) Brooklyn, N.Y.

E. D. Mouzon, Jr., Ph.D. (Illinois) Asso. Prof., Acting Head of Dept., Southern Methodist Univ., Dallas, Texas.

Alexander Oppenheim, Ph.D. (Chicago) Prof., Raffles Coll., Singapore, Straits Settlements.

E. R. Ott, Ph.D. (Illinois) Instr., Univ. of Buffalo, Buffalo, N.Y.

Lois E. Pollard, A.M. (Columbia) Instr., Dean of Women, Jr. Coll., Eveleth, Minn.

O. M. Rogers, M.S. (Southern Methodist) Dean, Head of Dept., Wesley Coll., Greenville, Texas.

(Miss) Evon Ryan, A.M. (Columbia) Teacher, State Teachers Coll., Mankato, Minn.

J. M. Rysgaard, A.B. (North Dakota) Prof., Head of Dept., Physics and Math., Hamline Univ., St. Paul, Minn.

Roméo Savary, B.Surv. (Laval Univ.) Prof., Laval Univ.; Bridge Engineer, Dept. of Roads, Provincial Government, Quebec, P.Q., Canada.

M. G. Scherberg, Ph.D. (Minnesota) Instr., Univ. of Minnesota, Minneapolis, Minn.

L. W. Sheridan, Ph.D. (Catholic) Prof., Saint Thomas Coll., Scranton, Pa.

D. E. South, A.M. (Michigan) Asst. Prof., Univ. of Kentucky, Lexington, Ky.

C. E. Springer, A.M. (Oklahoma) Asst. Prof., Univ. of Oklahoma, Norman, Okla.

Dorothy I. Stephenson, A.B. (Colorado) Corning, Iowa.

Ruth W. Stokes, Ph.D. (Duke) Asso. Prof., N. Texas State Teachers Coll., Denton, Texas.

R. G. STURM, M.S. (Illinois) Research Engineer, Physicist, Aluminum Research Labs., New Kensington, Pa.
V. B. TEMPLE, A.M. (Texas) Head of Dept., Louisiana Coll., Pineville, La.
H. M. TENNEY, B.S. (Greenville Coll.) Instr., Chem. and Math., Greenville Coll., Greenville, Ill.
S. B. TOWNES, A.M. (Oklahoma) Asst. Prof., Univ. of Oklahoma, Norman, Okla.
J. F. WARDWELL, A.B. (Hamilton) Jr. Instr., Johns Hopkins Univ., Baltimore, Md.
MARGARET C. WEEBER, A.M. (Chicago) Asst. Prof., Hood Coll., Frederick, Md.
E. V. WHITE, A.M. (Baylor) Dean, Head of Dept., Texas State Coll. for Women, Denton, Texas.
G. H. WILSON, A.M. (Pennsylvania) Instr., Physics, Univ. of Delaware, Newark, Del.
W. F. WINN, Radio Operator, Sabine Transportation Co., Port Arthur, Texas.

Reports of progress were made on the two commissions of the Mathematical Association and a plan which might provide adequately for funds needed to carry out their plans as they are being developed.

W. D. CAIRNS, *Secretary-Treasurer*

THE APRIL MEETING OF THE ROCKY MOUNTAIN SECTION

The eighteenth regular meeting of the Rocky Mountain Section of the Mathematical Association of America was held at Colorado College, Colorado Springs, Colorado, on Friday and Saturday, April 20–21, 1934.

There were three sessions. Professor C. H. Sisam of Colorado College presided at each.

The attendance was forty-five, including the following twenty-four members of the Association: C. F. Barr, Jack Britton, A. G. Clark, I. M. DeLong, J. R. Everett, J. C. Fitterer, G. W. Gorrell, D. F. Gunder, I. L. Hebel, C. A. Hutchinson, L. Louise Johnson, A. J. Kempner, Claribel Kendall, A. J. Lewis, W. V. Lovitt, S. L. Macdonald, A. S. McMaster, W. K. Nelson, Greta Neubauer, E. D. Rainville, O. H. Rechard, A. W. Recht, Mary S. Sabin, C. H. Sisam.

The members and friends of the Section were guests of the College at a banquet on the evening of April 20. At the business session on Saturday, Professor J. C. Fitterer of Colorado School of Mines was elected Chairman for the coming year. Professor A. W. Recht of the University of Denver was elected Vice-Chairman.

The following ten papers were read:

1. "A note on polynomial curves" by Jack Britton, University of Colorado.
2. "A mathematical analysis of the hardening of copper" by J. D. Keyes, Montana School of Mines. (Read by the secretary in the absence of the author.)
3. "The Schwarz-Christoffel transformation as applied to the solution of certain problems in elasticity" by Professor D. F. Gunder, Colorado Agricultural College.
4. "A problem in magic squares" by Professor W. K. Nelson, University of Colorado.

5. "Fundamentals of the trial load method for stress analysis of arched dams" by R. E. Glover, United States Reclamation Bureau, by invitation.

6. "On the Schwarz-Christoffel transformation" by Professor C. A. Hutchinson, University of Colorado.

7. "Solution of a problem in heat conduction" by E. D. Rainville, Junior Engineer, United States Reclamation Bureau.

8. "A survey course in mathematics" by Professor O. H. Rechard, University of Wyoming.

9. "Complex numbers" by Professor A. J. Kempner, University of Colorado.

10. "Evaluation of certain expectancies" by Professor A. G. Clark, Colorado Agricultural College.

Abstracts of the papers follow, the numbers corresponding to the numbers in the list of titles:

1. The paper given by Mr. Britton is concerned with the following problem: Under what restrictions may we assign all the abscissas and two of the ordinates of the extremes (maxima and minima) of the polynomial curve $y=a_0x^n+\cdots+a_n$, which is to have the maximum number of extremes? This paper will appear in an early issue of this MONTHLY.

2. Recent investigations at Montana School of Mines by J. D. Keyes, assisted by C. L. Wilson, indicate that the natural law governing the precipitation hardening of copper may be expressed by the equation

$$R = a(T + b)^{-1} + c,$$

where R is the resistivity of the copper being hardened, T is the time of annealing, and a, b, and c are arbitrary constants. Theoretical approach to the problem resulted in failure, but an empirical approach based upon correspondence between certain arithmetical and geometrical series, resulted in the above equation. No effort was made to derive a formula that would fit data on the problem; the effort was made, however, to satisfy a fundamental requirement of a natural law: if corresponding values of the variables within a very limited range are known, and if the natural law is also known, then all other corresponding values of the variables may be found by extrapolation. Tests of extrapolation made with the above equation gave satisfactory results.

3. Professor Gunder gave a brief expository discussion of the value of the Schwarz-Christoffel transformation as applied to the solution of boundary value problems in which the area for which the solution is to be obtained is a polygon. He followed this with an example which gave the solution of the flexure problem for an elastic beam of rectangular cross section but with two symmetrical horizontal slits extending inward along the central line of the section.

4. Professor Nelson discussed a 3×3 magic square which has 9 as its center number. The remaining eight positions are to be filled so that the sum of the rows, columns and diagonals will be 27, with no element greater than 15. The solutions, four in number, were obtained by a graphical interpretation, transformations and inequalities.

5. In this paper, R. E. Glover described the methods devised by engineers of the U. S. Reclamation Bureau for calculating the stresses in such structures. The process is one of successive approximation involving the use of trial loads of various types which are applied for the purpose of satisfying the boundary conditions, and meeting the equilibrium and continuity conditions throughout the structure. The requirements for a satisfactory solution were examined with the aid of Kirchhoff's uniqueness theorem in the theory of Elasticity.

6. A brief outline of the application of the conformal transformation of Schwarz and Christoffel to problems of electric machine design was given.

7. Mr. Rainville considered the problem of the conduction of heat in a plane wedge with special reference to placing the formulas in a form suitable for computation of the temperature history of large dams. The boundary conditions were taken in such a manner that use can be made of the known fact that bedrock is at mean annual air temperature.

8. In this paper Professor Rechard suggested the need for a course in mathematics for students who do not wish to specialize in mathematics but who do wish to know something about the rôle this science has played and is playing in the history of the race. As a basis for discussion he outlined a course given during the winter quarter at the University of Wyoming. The two main points in the outline were the development of elementary mathematics as a human interest story, and the presentation of the foundations of mathematics in such a way as to emphasize the fact that mathematics is an invention of the human intellect.

9. Professor Kempner made some remarks, partly of a pedagogical character, concerning the distinction between "absolute value" and "distance" in the theory of complex numbers.

10. In this discussion, Professor Clark developed some of the most important methods for finding bounds for the expectancies of various functions with reference to distribution functions to which but very mild restrictions are applied. The methods were extended to cover cases of more than one variable and various examples of their application and interpretation were given.

A. J. LEWIS, *Secretary*

THE SECOND ANNUAL MEETING OF THE WISCONSIN SECTION

The second annual meeting of the Wisconsin Section of the Mathematical Association of America was held at the Oshkosh State Teachers College on Saturday, May 5, 1934. The Chairman of the Section, Professor G. A. Parkinson, presided.

The attendance was sixty-eight, including the following twenty-five members of the Association: Leon Battig, Ethelwynn R. Beckwith, May M. Beenken, Theodore Bennett, H. H. Conwell, L. A. V. DeCleene, Margaret C. Eide, H. P. Evans, M. L. Hartung, R. C. Huffer, M. H. Ingraham, Elizabeth E.

Knight, R. E. Langer, Morris Marden, Sister Mary Felice, J. S. McNair, G. A. Parkinson, H. P. Pettit, Irene Price, W. E. Roth, E. B. Skinner, H. L. Turrittin, J. I. Vass, Margarete C. Wolf, W. R. Woodmansee.

A luncheon was held at noon at the Colonial Inn. At the business meeting which followed, a committee was appointed to consider the organization of a state-wide association of undergraduate students of mathematics. A resolution was adopted instructing the secretary to express to the editors of THE AMERICAN MATHEMATICAL MONTHLY a desire for the publication of papers on the value of mathematics and the danger of restricting the subject in the schools. The following officers were elected for the year 1934–35: Chairman, H. P. Pettit, Marquette University; Secretary, May M. Beenken, Oshkosh State Teachers College; Program Committee, E. B. Skinner, University of Wisconsin, and Irene Price, Oshkosh State Teachers College. The third annual meeting was set for May 1935, to be held at Marquette University.

The following papers were read during the morning and afternoon sessions:

1. "Odd perfect numbers" by Professor W. E. Roth, University of Wisconsin Extension Division, Milwaukee.

2. "The unmathematical side of mathematics" by Professor Ethelwynn R. Beckwith, Milwaukee Downer College.

3. "Some plane linkages and their applications" by Professor H. P. Pettit, Marquette University.

4. "Modern trends and perils in mathematics" by Professor E. B. Skinner, University of Wisconsin.

Abstracts of these papers follow, the numbers corresponding to the numbers of the titles:

1. The present status of perfect numbers was discussed with emphasis upon the work of amateurs in that field. The speaker pointed out that at present no odd perfect number is known and that the proof of their non-existence has foiled the attempts of many eminent mathematicians.

2. This paper supported the thesis that mathematics is a definite cause of the distinctive character of any major culture. It presented as evidence changes in the social philosophy and in the art of western civilization which have followed the transition from classical to modern mathematical concepts.

3. This paper reviewed briefly the history of curve tracing instruments, and the attempts at converting circular into rectilinear motion. Some of the more interesting linkages were described, and their applications in building machines to construct the conchoids, the epicycloid and the ellipse were explained. The use of linkages was suggested as a possible means of increasing interest in mathematics on the part of students.

4. That the social sciences must form the heart and core of the high school curriculum is a fine idea, but in the opinion of the speaker it has been so seriously over-emphasized as to constitute a real danger to sound education. There is grave danger that the adoption of a purely social science course shall become a medium of propaganda for political and semi-political minorities. A knowl-

edge of mathematics is necessary to the understanding of modern science, economics or sociology, and therefore must exist side by side with the social science curricula.

Restricting the study of algebra and geometry in the high school means that the teacher of arithmetic in the grades will be asked to teach to the limit of her knowledge. This will increase the poor teaching of arithmetic and give business men all the more reason to complain that their employees cannot do even the elementary processes of arithmetic.

Professor Skinner emphasized his belief that the making of curricula, while it must have the services of men who are students of education in the broadest sense and students of public affairs, must also have the services of men who are specialists in their departments. Education will inevitably suffer if the present day tendency to exclude the specialist from all part in the work of making curricula is not corrected.

MAY M. BEENKEN, *Secretary*

THE ANNUAL MEETING OF THE TEXAS SECTION

The 1934 meeting of the Texas Section of the Mathematical Association of America was held at the Agricultural and Mechanical College of Texas on May 5th. Among the forty-seven people attending the meeting were the following seventeen members of the Association: P. M. Batchelder, J. H. Binney, A. A. Blumberg, H. E. Bray, Myrtle C. Brown, Alice C. Dean, Mary E. Decherd, C. H. Dix, Nat Edmonson, G. C. Evans, L. R. Ford, A. W. Randall, P. K. Rees, W. A. Rees, C. R. Sherer, Laura N. Turner, and C. L. Wilson. Those attending the meeting were the guests of the A. and M. College at a luncheon served in the banquet room of the College dining hall.

At the business meeting following the presentation of the papers, Professor W. L. Porter of the A. and M. College of Texas and Professor F. W. Sparks of the Technological College were elected Chairman and Vice-Chairman, respectively, for the coming academic year. These were the only offices to be filled at this time. The secretary reported eight applications for membership or for renewal of membership as the result of a campaign for more members of the Association carried out previous to the meeting.

A committee was appointed to formulate plans for a mathematics prize contest in the secondary schools of the state. In connection with this the members of the Texas Section decided upon a policy of maintaining closer relations with the teachers of secondary mathematics in the state. A new standard for departmental recommendation of prospective teachers of secondary school mathematics was set up for the mathematics departments of Texas colleges represented in the Texas Section. This standard requires that such prospective teachers must take at least thirty semester hours of pure and applied mathe-

matics, eighteen hours of which must be above sophomore grade, before departmental recommendation can be obtained.

The meeting extended a rising vote of thanks to Professor G. C. Evans of The Rice Institute, who leaves Texas at the end of the present academic year to join the staff of the University of California in Berkeley, for his active and valuable participation in the affairs of the Texas Section during past years.

The following papers were presented:

1. "Subharmonic functions and the differential geometry of surfaces" by Dr. E. F. Beckenbach, The Rice Institute, by invitation.

2. "Some properties of minimal surfaces" by J. H. Hahn, The Rice Institute, by invitation.

3. "Waring's Problem" by Professor P. M. Bachelder, University of Texas.

4. "Dirac negative energy transformations" by Dr. C. H. Dix, The Rice Institute.

5. "On a certain contact transformation of differential equations" by Professor Hillel Halperin, A. and M. College of Texas, by invitation.

6. "An elliptic system of integral equations on summable surfaces" by Professor J. H. Binney, A. and M. College of Texas.

7. "Preparation for graduate study" by Professor C. R. Sherer, Texas Christian University.

8. "Frozen credits" by Miss Mary E. Decherd, University of Texas.

1. This paper, the presentation of which occupied one hour, gave some of the results obtained by the author in this field. Greater detail may be obtained from papers by the author in the Bulletin of the American Mathematical Society, in the Transactions of the American Mathematical Society, in the Annals of Mathematics, and in the American Journal of Mathematics.

2. A theorem due to Weierstrass states: A necessary and sufficient condition that a surface S: $x_1 = x_1(u, v)$, $x_2 = x_2(u, v)$, $\cdots$, $x_n = x_n(u, v)$, (u, v) in a domain D, which is given in terms of isothermic parameters, be minimal is that the coordinate functions be harmonic. In this paper it is assumed that the surface is given in terms of isothermic parameters. Then the coordinate functions are expanded in Fourier series and the conditions on the coefficients implied by $E = G$, $F = 0$ are obtained. These conditions are used to prove the following theorems:

(A) If the zeros of $\sum_{j=1}^{n}(x_j - c_j)^2$ have a limit point interior to D, then $x_j \equiv c_j$.

(B) If the zeros of $\sum_{j=1}^{3}(X_j - C_j)^2$ have a limit point interior to D, then $X_j \equiv C_j$, where $X_j = \cos(x_j, n)$, n being the normal to the surface.

(C) If S' is some other minimal surface given in terms of isothermic parameters for (u, v) in D with coordinate functions y_j, and if the zeros of $\sum_{j=1}^{3}(x_j - y_j)^2$ and the zeros of $\sum_{j=1}^{3}(X_j - Y_j)^2$ have a common limit point interior to D, then $x_j \equiv y_j$.

3. The problem of representing integers by sums of squares or higher powers has occupied the attention of many mathematicians. In 1770 Waring

conjectured that for every power k there exists a finite integer N_k such that every integer can be expressed as the sum of N_k kth powers. This was proved by Hilbert in 1909. Much work has been and is being done on determining the values of N_k and other related problems.

4. In Dirac's relativistic theory of the electron there are two series of energies differing only in sign. The continuity considerations that exclude the negative energies in classical relativity cannot be used in Dirac's theory so the negative levels must be included. Dirac's interpretation is ambiguous. We propose to alter the point charge electron slightly by splitting it into two parts each of charge $e/2$ and placing these parts at $(a_1+ib_1, a_2+ib_2, a_3+ib_3)$, $(a_1-ib_1, a_2-ib_2, a_3-ib_3)$. The field is then real in the real space and the potential is double-valued. There is a singular circular ring of center (a_1, a_2, a_3) and radius $\sqrt{b_1^2+b_2^2+b_3^2}$. If a test charge, e, is carried over a closed path looping through this singular ring the initial potential energy is eV and the terminal potential is $-eV$. This is interpreted as a change in the sign of the test charge.

5. The reciprocal transformation

$$x = \frac{p'}{v - up'}, \qquad y = \frac{-1}{v - up'},$$

$p = -u/v$, is useful if $f(x, y, p)$ is a polynomial of degree m in p with coefficients homogeneous and of the same degree n, in x and y. If $n < m$, the transformation will lower the degree of the equation. If $f(x, y, p) \equiv \sum A_i f_i(p)$, where A_i are homogeneous and of the same degree in x and y and free of p, the transformed equation will involve p algebraically although in the original equations $f_i(p)$ are any functions. Again, equations of the form $(y-xp)^m = x^n f(p)$ or $(y-xp)^m = y^n f(p)$ are transformed into $\phi(u/v)\, d(u/v) = \psi(v)dv$ and $\phi(v/u)d(v/u) = \psi(u)du$ respectively. It is also useful in the case of equations of a higher degree than the first, and can be extended to partial differential equations.

6. Given $A(e)$; $B(e)$ two additive functions of point sets e, measurable Borel and contained in a simply connected plane bounded open region T. We consider the pair of elliptic integral equations

$$\int_s \phi dy + \theta dx = A(\sigma); \quad \int_s -\theta dy + \phi dx = B(\sigma), \tag{1}$$

where s is a variable simple closed rectifiable curve contained in T and σ its interior points.

The following results are proved. Let ψ be an arbitrary solution of Laplace's equation in T. The functions

$$\begin{aligned} \phi_0 &= (2\pi)^{-1} \int_T (MP)^{-1}[\cos(MP, y)dB(e_P) - \cos(MP, x)dA(e_P)] + \frac{\partial\psi}{\partial y} \\ \theta_0 &= (2\pi)^{-1} \int_T (MP)^{-1}[\cos(MP, x)dB(e_P) + \cos(MP, y)dA(e_P)] + \frac{\partial\psi}{\partial x} \end{aligned} \tag{2}$$

form a system of solutions of the pair of equations (1) for all simple closed rectifiable curves s in T on which the integrals

$$\int_T (MP)^{-1} | dA(e_P) | ; \quad \int_T (MP)^{-1} | dB(e_P)$$

represent summable functions with respect to arc length; in particular these functions are solutions for almost all rectangles in T with sides parallel to x and y directions.

Conversely, if ϕ and θ are summable superficially over any closed region in T and are solutions of (1) on almost all rectangles in T, they may be expressed in the form (2) except for a possible set of points in T of measure zero at most.

7. In preparation for graduate study a course should be given in the junior or senior year covering the following topics: (1) Fundamental theorems of matrices and determinants, including linear systems, linear independence, Jacobians, etc. (2) Necessary and sufficient conditions for convergence and divergence of infinite series. (3) Introduction to real and complex function theory. (4) Résumé of the theory of equations. (5) Use of the mathematical library.

8. Attention was called to an increasing appreciation of the usefulness of mathematics in carrying on the business of the world; also to the alarming neglect of mathematics in the elementary and secondary schools and by administrators. College teachers are handicapped by the poor preparation of their students, due in some measure to lack of thorough scholarship on the part of teachers in the lower schools. On the other hand the college professor often shirks responsibility for active participation in solution of problems of lower schools. Statistics covering five years of freshman examinations at the University of Texas showed the superior preparation of students who reviewed Algebra in the last year of high school and who also studied Trigonometry there.

NAT EDMONSON, *Secretary*

ANALYTIC CURVES FOR WHICH THE CHORD EQUALS THE ARC

By J. M. FELD, New York City

In a paper bearing the same title, read before the American Mathematical Society,[1] Professor Kasner discussed the problem of finding all analytic curves through a given point O such that the arc OP equals the chord OP for every point P on the curve. In that paper he stated that in ordinary three-space the curves lie in minimal planes. Since no proof of this result has appeared in the literature, we offer one here.

[1] An abstract appeared in the Bulletin of the American Mathematical Society, vol. 31 (1925), p. 214.

Let C be the arc of an analytic curve in three-space passing through the origin in a rectangular Cartesian coordinate system. The equations of the curve are $x_1 = f_i(t)$, $i = 1, 2, 3$, with $f_i(t_0) = 0$ and the functions $f_i(t)$ analytic in a region containing t as an interior point. The chord and arc lengths of C measured from the origin to any point P on C are equal. Therefore we have

$$(1) \qquad (x_1^2 + x_2^2 + x_3^2)^{1/2} = \int_{t_0}^{t} (x_1'^2 + x_2'^2 + x_3'^2)^{1/2} dt.$$

Differentiating and then squaring equation (1) we obtain

$$\frac{(x_1x_1' + x_2x_2' + x_3x_3')^2}{x_1^2 + x_2^2 + x_3^2} = x_1'^2 + x_2'^2 + x_3'^2$$

if $x_1^2 + x_2^2 + x_3^2 \neq 0$, and

$$x_1'^2 + x_2'^2 + x_3'^2 = 0,$$

if $x_1^2 + x_2^2 + x_3^2 = 0$. In the latter case we have $x_1x_1' + x_2x_2' + x_3x_3' = 0$. Hence, using the identity

$$\begin{aligned}(x_1x_2' - x_2x_1')^2 &+ (x_2x_3' - x_3x_2')^2 + (x_3x_1' - x_1x_3')^2 \\ &\equiv (x_1^2 + x_2^2 + x_3^2)(x_1'^2 + x_2'^2 + x_3'^2) - (x_1x_1' + x_2x_2' + x_3x_3')^2,\end{aligned}$$

we get for both cases

$$(2) \qquad (x_1x_2' - x_2x_1')^2 + (x_2x_3' - x_3x_2')^2 + (x_3x_1' - x_1x_3')^2 = 0.$$

Hence the curves sought must satisfy equation (2).

Equation (2) is satisfied if the three equations

$$(3) \qquad x_1x_2' - x_2x_1' = 0, \; x_2x_3' - x_3x_2' = 0, \; x_3x_1' - x_1x_3' = 0$$

are satisfied. If one of the equations (3) is satisfied, let us say $x_1x_2' - x_2x_1' = 0$, then

$$x_2x_3' - x_3x_2' = \pm i(x_1x_3' - x_3x_1').$$

Dividing these equations by x_2^2 and x_3^2 respectively, we can integrate and for result we obtain

$$x_1 = ax_2, \; x_2 = \pm ix_1 + bx_3.$$

Hence $x_2x_3' - x_3x_2' = x_2x_1' - x_1x_2' = 0$. If, then, *one* of the equations (3) is satisfied the other two are also satisfied, and we have as solutions *all* straight lines through the origin.

By virtue of the identity given above, equation (2) is equivalent to

$$(2') \qquad (x_1^2 + x_2^2 + x_3^2)(x_1'^2 + x_2'^2 + x_3'^2) = (x_1x_1' + x_2x_2' + x_3x_3')^2$$

Equation (2'), and therefore also equation (2), is evidently satisfied by the equation of the isotropic cone

$$x_1^2 + x_2^2 + x_3^2 = 0. \tag{4}$$

By the result in the preceding paragraph we know that the straight lines on (4) satisfy equation (2). We proceed to show that the straight lines are the *only* curves on the isotropic cone which satisfy (2).

We have $x_1^2+x_2^2+x_3^2=0$, hence from (1) $x_1'^2+x_2'^2+x_3'^2=0$, and by differentiation $x_1x_1'+x_2x_2'+x_3x_3'=0$. Using the identity

$$(x_1x_2' - x_2x_1')^2 = (x_1^2 + x_2^2)(x_1'^2 + x_2'^2) - (x_1x_1' + x_2x_2')^2,$$

and replacing $x_1^2+x_2^2$ by $-x_3^2$, $x_1'^2+x_2'^2$ by $-x_3'^2$, and $x_1x_1'+x_2x_2'$ by $-x_3x_3'$, we have at once $x_1x_2'-x_2x_1'=0$. Hence the only curves on the isotropic cone satisfying the conditions of the problem are the straight lines already treated.

So far we have considered two special cases: (a) that one of the equations (3) is satisfied, and (b) that $x_1^2+x_2^2+x_3^2=0$. In each case the solutions are straight lines only. To find the other solutions we may now assume that none of the equations (3) is satisfied and that $x_1^2+x_2^2+x_3^2\neq 0$.

Let us write

$$A = \begin{vmatrix} x_ix_i & x_ix_i' \\ x_i'x_i & x_i'x_i' \end{vmatrix}$$

where a repeated subscript means summation over 1, 2, 3. Thus $x_ix_i=x_1^2+x_2^2+x_3^2$. Therefore[1]

$$A = \sum_{i,j} \begin{vmatrix} x_i & x_j \\ x_i' & x_j' \end{vmatrix}^2$$

which by virtue of (2) equals zero. Differentiating A with respect to t we get

$$A' = \sum_{i,j} \frac{d}{dt}\begin{vmatrix} x_i & x_j \\ x_i' & x_j' \end{vmatrix}^2 = 2\sum \begin{vmatrix} x_i & x_j \\ x_i' & x_j' \end{vmatrix}\begin{vmatrix} x_i & x_j \\ x_i'' & x_j'' \end{vmatrix}$$
$$= 2\begin{vmatrix} x_ix_i & x_ix_i' \\ x_i''x_i & x_i''x_i' \end{vmatrix} = 0.$$

Let B represent the determinant

$$\begin{vmatrix} x_ix_i & x_ix_i' & x_ix_i'' \\ x_i'x_i & x_i'x_i' & x_i'x_i'' \\ x_i''x_i & x_i''x_i' & x_i''x_i'' \end{vmatrix}$$

Since $A=0$, we have

$$K_1x_ix_i + K_2x_ix_i' = 0, \quad K_1x_i'x_i + K_2x_i'x_i' = 0$$

where K_1 and K_2 are two functions of t not both zero. Because A' is also zero and $x_ix_i\neq 0$, it follows that

[1] O. Perron, *Algebra*, vol. I (1927), p. 117.

$$K_1 x_i'' x_i + K_2 x_i'' x_i' = 0.$$

Therefore

$$\begin{vmatrix} x_i' x_i & x_i' x_i' \\ x_i'' x_i & x_i'' x_i' \end{vmatrix} = 0.$$

Thus the minors of all the elements of the last column of B vanish and consequently $B=0$. However,

$$B = \begin{vmatrix} x_1 & x_2 & x_3 \\ x_1' & x_2' & x_3' \\ x_1'' & x_2'' & x_3'' \end{vmatrix}^2$$

which is the square of the wronskian of $x_i(t)$, and since the wronskian is identically zero, the $x_i(t)$ are linearly dependent. Therefore

$$c_1 x_1 + c_2 x_2 + c_3 x_3 = 0 \tag{5}$$

where the c_i are constants not all zero. From (5) we obtain

$$c_1 x_1' + c_2 x_2' + c_3 x_3' = 0$$
$$c_1 x_1'' + c_2 x_2'' + c_3 x_3'' = 0.$$

Consequently since none of the expressions $x_i x_j' - x_j x_i'$ vanish, we have

$$\frac{c_1}{x_2 x_3' - x_3 x_2'} = \frac{c_2}{x_3 x_1' - x_1 x_3'} = \frac{c_3}{x_1 x_2' - x_2 x_1'} = \mu$$

and therefore, by virtue of (2),

$$c_1^2 + c_2^2 + c_3^2 = \mu^2 \sum_{i,j} (x_i x_j' - x_j x_i')^2 = 0.$$

Therefore (5) is a minimal plane passing through the origin. Since every straight line through the origin lies in some minimal plane, it follows that all the curves satisfying the conditions of our problem lie in minimal planes. Conversely, all analytic curves in minimal planes have the property that the chord and arc joining any two points of the curve are equal.[1] Consequently we have the theorem: *A necessary and sufficient condition that an analytic curve in euclidean three-space have the property that the chord and the arc joining any two of its points be equal, is that it lie in a minimal plane.*

By methods precisely similar to those used in the three-dimensional case, one may prove: *In order that an analytic curve in a euclidean S_n have the property that its arc OP be equal in length to its chord OP for any point P on the curve, it is necessary that the curve lie in a minimal hyperplane containing the point O.*

[1] Beck, *Zur Geometrie in der Minimalebene*, Sitzungsberichte der Berliner mathematischen Gesellschaft, vol. 12 (1912).

A NEW METHOD FOR UNIVERSAL WARING THEOREMS WITH DETAILS FOR SEVENTH POWERS

By L. E. DICKSON, University of Chicago

1. *Introduction*

In 1920 Hardy and Littlewood obtained remarkable asymptotic results on sums of n-th powers. Thereupon I conceived the idea to supplement their results and obtain universal Waring theorems (all positive integers P are sums of $s(n)$ positive or zero integral n-th powers). This would necessitate proving that fact for every $P \leqq C$, where $C = C(n)$ is the limit beyond which the asymptotic results apply. When later $C(n)$ was correctly computed for certain values of n, it was found to be inconceivably large. Moreover, the methods which had been used to make tables were too crude to permit of serious use; they required the finding of all decompositions of each integer and the subsequent selection of a minimum decomposition.

My first progress was to devise a method based on leaders (see §4), used as a modular system, which enabled me to find at once a minimum decomposition without considering the others. For fifth powers my table to 300,000 has just been published by the British Association for the Advancement of Science. By that table and extensive supplements far out, I have proved that every positive integer is a sum of 54 fifth powers (the best algebraic result was 58). My students have made similar, but much shorter, tables for fourth and sixth powers, and reduced the former universal limits from 37 to 35 and from 478 to 180, respectively.

But for seventh and higher powers, such arithmetical tables of sufficient extent would require prohibitive calculation, and prevent reproduction however condensed. Moreover an arithmetical table prevents our detecting periodicity properties, and utilizing them to condense the table.

After much meditation and experiment, I have finally devised what I believe to be the perfect method of constructing a table of minimum decompositions for all integers between assigned limits. The new type of table effects maximum economy both in construction and condensation.

The table given here implies that the 110893 consecutive integers beginning with 436,186 are all sums of fifty seventh powers. From this fact we conclude that every positive integer is a sum of 258 integral seventh powers $\geqq 0$. The earlier result was that 3806 powers suffice.

2. *The general theory*

We write $a = 2^7 = 128$, $b = 3^7$, $c = 4^7$, $d = 5^7$, $e = 6^7$. Then

$$(1) \qquad b = 17a + 11, \quad c = 51 + 8a + 7b, \quad d = -10 + 13a + 5b + 4c.$$

By a *resolution* of n we mean one of the ways of expressing n as a linear function of $a, b, c, \cdots$ whose coefficients are positive or negative integers. When all the coefficients are integers $\geqq 0$, the resolution is a *decomposition*, and the sum of

the coefficients is the *weight*. Thus the first two in (1) are decompositions, while the third is not, but is a resolution. A decomposition of n is *minimal* if there is no other decomposition of n of smaller weight.

Our final table gives minimum decompositions of integers between $2d+e$ and $3d+e$. Such an integer can be expressed in the form $r+N$, where $0 \leqq r < a$,

$$N = Aa + Bb + Cc + 2d + e, \quad A < 18, B < 8, \quad C < 5. \tag{2}$$

Let A, B, C have arbitrarily assigned integral values $\geqq 0$. All homogeneous resolutions, involving e but not d, of $r+N$ $(r=0, \cdots, 127)$ are evidently obtained by adding N to all the resolutions of $0, 1, \cdots, 127$ of the form $L-2d$, where L is linear and homogeneous in a, b, c. One such resolution is obtained by doubling (1_3):

$$26a + 10b + 8c - 2d = 20. \tag{3}$$

Evidently all such resolutions are found by adding to (3) all homogeneous resolutions into a, b, c of $-20, \cdots, 107$. The latter resolutions are obviously obtained by repeated additions of

$$18a - b = 117, \qquad 17a - b = -11, \tag{4}$$

$$9a + 7b - c = 77, \qquad 8a + 7b - c = -51, \tag{5}$$

and the negatives of these four.

To obtain all homogeneous decompositions of $r+N$ we restrict attention to those resolutions which involve no one of $-18a$, $-8b$, $-5c$, by the inequalities in (2).

By successive substractions of (4_2) from (3), we get

$$9a+11b+8c-2d=31, \quad -8a+12b+8c-2d=42,$$

$-25a+\cdots$, the last of which is excluded. If we add (4_2) to (3), we get $43a+9b+8c-2d=9$. But if either this or (3) be added to N, we obtain sums which involve the leader $11a+b+e$, whose weight 13 exceeds that of the equal function $4+3c+3d$, whence the leader and therefore the two sums involving it are not minimum decompositions.

After a few repetitions of this process, we reach decompositions all of which involve $30b$, whence their sums with N involve the leader $30b+e$ and are not minimal.

It is an important fact that the process is independent of A, B, C. It yields the 51 functions in §3, and the following:

27	−5	10	−2	0	111	−6	−7	−1	4	−1	103
3	−7	18	0	−1	51	2	15	−4	4	−1	89
−2	−9	4	3	−1	36	−15	16	−4	4	−1	100
15	−10	4	3	−1	25						

The 58 functions would suffice for a complete table from $2d+e$ to $3d+e$. The small part of the table actually made suffices for Waring's problem, and for this part the above seven functions are not needed.

3. *List of linear functions*

No.	a	b	c	wt.	
1	18	−1	0	17	117
2	9	7	−1	15	77
3	−8	8	−1	−1	88
4	0	15	−2	13	37
5	−17	16	−2	−3	48
6	9	22	−3	28	114
7	−8	23	−3	12	125

No.	a	b	c	d	wt.	
20	5	13	3	−1	20	98
21	−4	6	4	−1	5	21
22	−4	21	2	−1	18	58
23	−12	14	3	−1	4	109
24	−13	29	1	−1	16	18
25	−12	−1	5	−1	−9	72
26	5	−2	5	−1	7	61
27	22	−3	5	−1	23	50
50	9	11	8	−2	26	31
51	1	4	9	−2	12	82
52	1	19	7	−2	25	119
53	−8	12	8	−2	10	42
54	−16	5	9	−2	−4	93
55	−8	−3	10	−2	−3	5
56	10	−4	10	−2	14	122

No.	a	b	c	d	e	wt.	
100	2	19	5	2	−1	27	43
101	24	1	12	1	−1	37	56
102	6	28	−1	3	−1	35	59
103	15	5	2	3	−1	24	62
104	7	2	12	1	−1	21	67
108	−2	10	11	1	−1	19	27
109	−2	6	2	3	−1	8	73
110	−2	21	0	3	−1	21	110
111	−6	16	15	0	−1	24	48
112	−6	12	6	2	−1	13	94
113	−10	3	12	1	−1	5	78
114	−10	18	10	1	−1	18	115
115	−11	14	1	3	−1	6	33
116	−14	9	16	0	−1	10	99
117	−15	5	7	2	−1	−2	17
118	−15	20	5	2	−1	11	54
119	−1	−1	22	−1	−1	18	72
120	−10	−1	3	3	−1	−6	124
121	7	−2	3	3	−1	10	113
122	−14	−2	27	−2	−1	8	16
123	−6	−3	8	2	−1	0	57
124	16	−6	13	1	−1	23	107
125	−1	−5	13	1	−1	7	118
126	−14	−6	18	0	−1	−3	62
127	2	0	−2	4	−1	3	52
128	−7	8	−3	4	−1	1	12
129	−11	29	−1	3	−1	19	70
130	−15	1	−2	4	−1	−13	63
131	11	−4	8	2	−1	16	46

4. *List of leaders*

$13a+5b+4c=10+d,$

$20b+7c=2+17a+2d,$

$5a+2b+14c=15+3d,$

$11a+17c=e,$

$b+17c=11+6a+e,$

$3a+27c=5+3b+2d+e,$

$30c=4+7a+10b+6d,$

$30a+d=8+15b+3c,$

$32b+d=13+5a+9c,$

$2a+4b+7c+2d=6+e,$

$12a+12b+3d=11+16c,$

$6a+13b+c+3d=22+e,$

$19a+4d=4+16b+e,$

$10a+7b+4d=1+3c+e,$

$41a+e=12+14b+6c+2d,$

$11a+b+e=4+3c+3d,$

$2a+24b+e=1+6c+3d,$

$30b+e=22+2a+2c+4d,$

$21b+4c+e=6+6a+5d.$

5. *Conditions for exclusion*

1, $A \geqq 12$ or $B \geqq 2$. 2, $A \geqq 2$. 4, $A \geqq 11$. 6, $A \geqq 2$ or $B \geqq 2$. 7, $A \geqq 10$. 20, $A \geqq 6$, 21, $A \geqq 15$. 22, $A \geqq 15$, or $A \geqq 6$, $B \geqq 3$, or $C \geqq 2$. 24, $A \geqq 15$ or $B \geqq 1$. 26, $A \geqq 6$. $B \geqq 3$. 27, $A \geqq 8$ or $B \geqq 4$. 50, $A \geqq 2$. 51, $A \geqq 10$. 52, $A \geqq 10$ or $B \geqq 1$. 56, $A \geqq 1$, $B \geqq 5$. 100, $A \geqq 4$ or $C \geqq 2$. 101, $A \geqq 6$ or $B \geqq 3$ or $C \geqq 2$. 102, $A \geqq 4$ or $C \geqq 2$. 103, $A \geqq 4$ or $B \geqq 2$ or $C \geqq 2$. 104, $B \geqq 2$ or $C \geqq 2$. 108, $A \geqq 4$. 109, $A \geqq 12$, $B \geqq 1$, or $A \geqq 15$, $C \geqq 2$. 110, $A \geqq 12$, or $A \geqq 8$, $C \geqq 1$. 111, $A \geqq 8$ or $B \geqq 4$. 112, $A \geqq 16$, or $A \geqq 12$, $B \geqq 1$, or $A \geqq 8$, $C \geqq 1$. 113, $A \geqq 12$, $B \geqq 1$, or $A \geqq 15$, $C \geqq 2$. 114, $A \geqq 12$ or $B \geqq 2$. 116, $A \geqq 16$, or $A \geqq 8$, $C \geqq 1$. 118, $C \geqq 2$. 119, $A \geqq 12$ or $B \geqq 2$. 121, $A \geqq 12$. 122, $B \geqq 3$. 124, $A \geqq 14$. 129, $A \geqq 15$, $C \geqq 2$, or $C \geqq 3$. 131, $A \geqq 8$.

6. *Construction of the table*

Each of the 51 functions in §3 is written on a narrow card along with the conditions, if any, for its exclusion (§5). White cards were used for the functions in which the coefficients of b and c are both $\geqq 0$; cards of two colors were used for the others.

The tablette $C=B=A=0$ is merely an arrangement according to the constant terms r (third column) of the functions in which the coefficients of a, b, c are all $\geqq 0$.

fcn.	r	diff.	wts.
50	31	12	26–37
100	43	13	27–39
101	56		

The final weight 37 is the sum of the weight 26 of function 50 and $12-1$. Thus 37 is the weight for the largest (unprinted) r between 31 and 43.

Since no white card has $-a$, our tablette applies for $A=1$ as well as $A=0$. The largest weight 42 is opposite $r=56$ (followed by $r=62$). We omit tablette $C=B=0$, $A=2$, 3 for the following reason. When we employ also the white cards with $-2a$ and $-3a$, we see by inspection that no new function is inserted between $r=46$ and $r=62$, nor is function 101 (with $r=56$) now excluded by the insertion of a new card with $r \leqq 56$. Hence 42 remains the greatest weight for the omitted tablette with $A=2$, 3.

7. *Table of minimum decompositions of* $r+Aa+Bb+Cc+2d+e$

Case $C=0$

$B=0$, $A=0, 1$

0	30	0	101	42	56	51	27	82
50	37	31	103	28	62	20	40	98
100	39	43	104	35	67	52	33	119

$B=0,\ A=4,\ 5$

0	20	0
21	39	21
101	38	56
22	26	58
104	26	67
109	16	73
51	27	82
20	31	98
110	29	110
52	33	119

$B=0,\ A=6,\ 7$

0	20	0
21	31	21
111	33	48
22–109, $A=4$		
51	23	82
112	28	94
110	29	110
52	33	119

$B=0,\ A=10$

0	20	0
21	25	21
53	25	42
22	26	58
104	26	67
109	12	73
113	20	78
112	28	94
110	25	110
114	30	115

$B=0,\ A=12$

0	20	0
21	16	21
115	14	33
53–113, $A=10$		
112	27	94
23	22	109

$B=0,\ A=14$

0	17	0
24	18	18
21–113, $A=12$		
112	17	94
116	19	99
23	22	109

$B=1,\ A=10$

0–22, $B=0,\ A=10$		
104	25	67
119	18	72
109–110, $B=0,\ A=10$		
114	19	115
1	23	117
120	-3	124

$B=1,\ A=12$

21–22, $B=0,\ A=12$		
104	25	67
25	27	72
23	18	109
120	-3	124

$B=1,\ A=14$

21–104, $A=12$		
25	17	72
116	19	99
23	18	109
120	-3	124

$B=1,\ A=15$

117	13	17
115	14	33
53	21	42
118	23	54
104–, $A=14$		

$B=2,\ A=0,\ 1$

0–100, $B=A=0$		
101	41	56
26	27	61
51	27	82
20	34	98
121	24	113

$B=2,\ A=4,\ 5$

0–101, $B=0,\ A=4$		
22	20	58
26	18	61
109–20, $B=0,\ A=4$		
110	23	110
121	24	113

$B=2,\ A=6,\ 7$

0–111, $B=0,\ A=6$		
22	20	58
26	18	61
109–112, $B=0,\ A=6$		
110	23	110
121	24	113

$B=2,\ A=8,\ 9$

0	20	0
21	25	21
53	25	42
22–, $A=6$		

$B=2,\ A=12$

21–53, $B=0,\ A=12$		
22	20	58
26	17	61
25–120, $B=1,\ A=12$		

$B=2,\ A=14$

22	12	16
21–53, $B=0,\ A=12$		
22	20	58
26	17	61
25–, $B=1,\ A=14$		

$B=2,\ A=15$

122	8	16
117–53, $B=1,\ A=15$		
118	17	54
26–, $A=14$		

$B=3,\ A=0,\ 1$

0	30	0
50	37	31
100	33	43
27	33	50
26–, $B=2,\ A=0$		

$B=3,\ A=2,\ 3$

0	26	0
108	34	27
100, 27, $A=0$		
26–, $B=2,\ A=4$		

$B=3,\ A=4,\ 5$

0	20	0
21	33	21
27	30	50

22–, $B=2,\ A=4$

$B=3,\ A=6,\ 7$

0	20	0
21	31	21
111	25	48
27	29	50
123	15	57

109–, $B=2,\ A=6$

$B=3,\ A=8,\ 9$

0	4	0
55	12	5
21	25	21
53	24	42

123–, $A=6$

$B=3,\ A=12,\ 13$

55	12	5
21	16	21
115	14	33
53	24	42
123	14	57

25–, $B=1,\ A=12$

$B=3,\ A=14$

55–123, $A=12$

25–, $B=1,\ A=14$

$B=3,\ A=15$

55	8	5

117–53, $B=1,\ A=15$

118	13	54
123	14	57

25–, $B=1,\ A=14$

$B=6,\ A=11$

55–53, $B=3,\ A=12$

123	15	57
109	12	73
113	20	78
112	25	94
124	25	107
110	23	110
121	14	113
125	12	118
120	−3	124

$B=6,\ A=14$

55–53, $B=3,\ A=12$

123	4	57
126	6	62
25	17	72
116	19	99
23	12	109

125–, $A=11$

$B=6,\ A=15$

55–118, $B=3,\ A=15$

123–, $A=14$

Case $C=1,\ 2$

$C=B=1,\ A=2,\ 3$

0	26	0
108	34	27
100	39	43
101	39	56
102	37	59
103	28	62
104	25	67

119	18	72

109–20, $B=0,\ C=0,\ A=4$

110	27	110
1	27	117

$C=B=1,\ A=4,\ 5$

0–22, $B=0,\ C=0,\ A=4$

104–, $A=2$

$C=B=2,\ A=4,\ 5$

0	20	0
21	20	21
4	27	37
127	11	52

26–, $C=0,\ B=2,\ A=4$

Case $C=3$

$B=0,\ A=0,\ 1$

0	30	0
50	31	31
4	27	37
127	27	52
2	19	77
51	27	82
20	35	98
6	32	114
52	33	119

$B=0,\ A=2,\ 3$

0	26	0

108	28	27
4	27	37
127	23	52
109	16	73
51	27	82
20	31	98
110	29	110
52	33	119

$B=0,\ A=8,\ 9$

0	11	0
128	9	12
21	20	21

4	17	37
53	19	42
127	23	52
109	16	73
51	17	82
3	29	88
52	30	119
7	14	125

$B=0,\ A=10$

0–127, $A=8$

109	12	73
113	14	78

3	25	88
114	27	115
7	14	125

$B=0,\ A=12$

0	11	0
128	9	12
21	16	21
115	14	33
53	19	42
127	20	52
129	21	70
109	12	73
113	14	78
3	19	88
23	19	109
7	14	125

$B=0,\ A=15$

0	11	0
128	5	12
117	13	17
115	14	33
53	19	42
127	13	52
130	11	63

3–, $A=12$

$B=4,\ A=0,\ 1$

0	30	0
50	31	31
4	21	37
131	21	46
127	11	52
26	22	61
2	19	77
51	27	82
20	34	98
121	18	113
56	19	122

$B=4,\ A=2,\ 3$

0	26	0
108	28	27

4–127, $A=0$

26	18	61

109–20, $B=0, A=2$

110	23	110
121	18	113
56	19	122

$B=4,\ A=6$

0	21	0
21	20	21
4	21	37
131	21	46
127	7	52
123	15	57
109	16	73
51	23	82
112	28	94

110–, $A=2$

$B=4,\ A=8,\ 9$

0	4	0
55	3	5

128–53, $B=0,\ A=8$

127–109, $A=6$

51	17	82
3	23	88
121	18	113
56	16	122
7	14	125

$B=4,\ A=12$–14

55	3	5

128–53, $B=0,\ A=12$

127	7	52
123	14	57
25	6	72
3	19	88
23	16	109
56	15	122
120	−3	124

8. *Conclusions from the table $r+Aa+Bb+Cc+2d+e$*

Let G denote the greatest weight in a tablette (fixed A, B, C). Our table for $B=C=0$ gives

A	0–3	4–5	6–9	10–11	12–13	14–16
G	42	39	33	30	27	26
$A+G$	45	44	42	41	40	42

where $A+G$ uses the largest A. If $A=17$, $r=11$, and $A+G=27$. Including 3 for $2d+e$, we see that $3+45=48$ powers suffice for every A. For $B=1$, $C=0$, the only altered items are $G=28$ if $A=10, 11$, $G=25$ if $A=15, 16$, whence $48+1=49$ powers suffice for every A. For $B=2$, $C=0$,

A	0–3	4–5	6–7	8–11	12–13	14	15–16
G	41	39	33	28	27	25	21
$A+G$	44	44	40	39	40	39	37,

whence 49 powers suffice for every A. For $B=3$, $C=0$,

A	0–1	2–3	4–5	6–7	8–11	12–13	14	15–16
G	37	34	33	31	28	27	24	21
$A+G$	38	37	38	38	39	40	38	37,

whence $6+40=46$ powers suffice for every A (and 45 except when $A=13$). Hence 48 powers suffice if $B=4$ or 5.

Let $B=6$, $C=0$. By the table, $G=25$ if $A=11$–13, $G=24$ if $A=14$, $G=21$ if $A=15$–16, whence $6+3+38=47$ powers suffice if $A\geqq 11$. But if $B=3$, we saw that $A+G\leqq 38$ if $A\leqq 10$, whence $6+3+38=47$ powers suffice if $B=6$. Hence if $B=6$, $C=0$, 47 powers suffice for every A.

Since $B<8$, this proves

LEMMA 1. *If $C=0$, 49 powers suffice for every A, B, while 49 are necessary only when $B=1$, $A=3$; $B=2$, $A=3$, $A=5$.*

Let $C=1$. The table gives $G=39$ if $B=1$, $A=2$–5, whence $5+1+1+3+39=49$ powers suffice. This with Lemma 1 yields

LEMMA 2. *If $C=1$, 49 powers suffice for every A, B, except for $A=5$, $B=2$, when 50 powers are necessary.*[1]

Let $C=2$. The tablette $C=B=2$, $A=4$, 5 shows that $G=31$, whence $5+2+2+3+31=43$ powers suffice. This with Lemma 2 yields

LEMMA 3. *If $C=2$, 50 powers suffice for every A, B.*

Let $C=3$. If $B=0$, the table gives

A	0–1	2–7	8–9	10–11	12–14	15–17
G	35	33	30	27	21	19
$A+G$	36	40	39	38	35	36,

whence 46 powers suffice. Hence if $B=3$, 48 powers suffice except for $A=7$. But for $B=3$, $C=0$, $A=7$, we saw that $A+G=38$, whence $9+38$ powers suffice if $C=B=3$, $A=7$. When $B=4$, the table gives

A	0–1	2–5	6–7	8–11	12–17
G	34	31	28	23	19
$A+G$	35	36	35	34	36,

whence $4+3+3+36=46$ powers suffice and are necessary only for $A=5$, 17. Hence if $B=5$, 6, 48 suffice. Let $B=7$. We stop at $8a+51$ by (1_2). Hence 48 powers suffice except for $A=5$, viz.,

$$11+5a+7b+3c+2d+e+\text{fcn. }20=11+10a+20b+6c+d+e,$$

which has the decomposition $2+21a+b+16c+3d$ of lower weight 43. This proves

[1] But only for $34+5a+2b+c+2d+e+\text{fcn. }21=34+a+8b+5c+d+e$.

LEMMA 4. *If* $C=3$, 48 *powers suffice for every* A, B.

Thus 50 powers suffice if $C=4$ or 5 and hence from $4c+2d+e$ to $6c+2d+e$. We saw that 46 powers suffice if $C=3$, $B=0$, whence 50 powers suffice if $C=6$. $B=0, 1$, and hence to

$$2b + 6c + 2d + e = 72 + 12a + 3b + c + 3d + e.$$

From this number on to $2c+3d+e$ we see that 50 powers suffice by adding d to the numbers in Lemma 2 with $B>2$. Our results for $C=4$–6 and Lemmas 1–4 yield

THEOREM 1. *The* 110893 *integers from* $j=2d+e=436186$ *to* $k=2c+3d+e$ *are sums of* 50 *seventh powers.*

By adding d twice we see that 52 powers suffice from j to $k+2d$. Adding d and e in turn, we see that 53 suffice both from j to $k+3d$ and from $j+e<k+3d$ to $k+2d+e$. Hence 53 suffice from j to 983, 265. The next ascents are similar but may be made without trial.[1] We find that 66 powers suffice from j to $L_0 = 6{,}974{,}455\times 10^7$.

The next ascents are made at one step.[2] Then $m+66$ powers suffice from j to L_m if

$$\log L_m = (7/6)^m(\log L_0 + w) - w, \quad w = 7\log 1/7,$$
$$w = \bar{6}.08431, \quad \log(\log L_0 + w) = .89915, \quad \log 7/6 = .0669467.$$

We take $m=192$. Then $\log\log L_m = 13\cdot 753$. The analytic theory (as refined by R. D. James for odd powers) shows that every integer $>C$ is a sum of $T+46$ seventh powers if $\log_e C = 25.7^3\, 2^y$, where

$$y = \frac{84.7T + 2645.045}{7T - 860},$$

and if T is even. We shall have $T+46=192+66=258$ if $T=212$. Then

$$y = 33.015,\ y\log 2 = 9.938,\ \log\log C = 13.509$$

to base 10. Hence $L_m > C$. Thus every integer $\geqq j$ is a sum of 258 seventh powers.

By (1) the integers $<b$ are $r+Aa(0\leqq r\leqq 127.\ A\leqq 16)$ and $r+17a(0\leqq r\leqq 10)$, and hence are all sums of 143 seventh powers. Adding b seven times, we see that every integer $<c<8b$ is a sum of 150 powers. Adding c 26 times, we conclude that every integer $<j<27c$ is a sum of 176 powers.

THEOREM 2. *Every positive integer is a sum of* 258 *integral seventh powers* $\geqq 0$.

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[1] Theorem 10, *Bull. Amer. Math. Soc.*, 1933, p. 710. The numbers added are n^7, $n=7$, 8, 9, 10, 12, 14, 17, 21, 26, 34, 46, 64, 94.

[2] Theorem 12, *loc. cit.*

SELECTIVE FUNCTIONS AND OPERATIONS[1]

By HARRY BATEMAN, California Institute of Technology

1. The power series

$$g(x) = c_0 + c_1\frac{x}{1!} + c_2\frac{x^2}{2!} + \cdots \tag{1}$$

may be written in the form[2]

$$g(x) = c_0 w(x) + c_1 I w(x) + c_2 I^2 w(x) + \cdots \tag{2}$$

where I denotes the operation of integrating from 0 to x and the function $w(x)$ is unity. The rule for determining the coefficients is

$$c_n = \lim_{x\to 0} D^n g(x), \tag{3}$$

where D denotes the operation of differentiation with respect to x. It is successful because we have the relation

$$\lim_{x\to 0} D^m I^n w(x) = \delta(m, n) \tag{4}$$

where $\delta(m, n)$ is zero when $m \neq n$ and is unity when $m = n$. When the function $w(x)$ is not unity but is represented by a power series

$$w(x) = 1 + a_1\frac{x}{1!} + a_2\frac{x^2}{2!} + \cdots \tag{5}$$

the coefficients c_n may be determined with the aid of an operator $F_n(D)$ defined by means of the generating function

$$\sum t^n F_n(D) = \frac{1}{W(t)(1 - Dt)}$$

where[3]

$$W(t) = 1 + a_1 t + a_2 t^2 + \cdots.$$

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[1] Read at the meeting of the Southern California Section of the Association at Riverside, California, Mar. 3, 1934.

[2] Series of this type in which the operation I is integration from x_0 to x and the function x is not unity were considered by H. Laurent, Journal de mathématique spéciale (5), vol. 21 (1897), p. 152. He determined the coefficients by means of a recurrence relation

$$c_n = \frac{d}{dx_0} c_{n-1} + (n-1)\, c_{n-1} \frac{d}{dx_0} [\log w(x_0)].$$

[3] If $C(x) = c_1 + c_2 x/1! + c_3 x^2/2! + \cdots$ the equation (2) may be written as an integral equation $g(x) = \int_0^x C(x-t)\, w(t)\, dt + c_0\, w(x)$ and may be solved by the method of V. Pareto, Crelle's Journal, vol. 110, (1892), p. 290. See T. Kubota, Tôhoku Math. Journal, vol. 22 (1922), p. 336. The convergence of the series (2) has been discussed by C. Guichard, Annales de l'École Normale Supérieure, ser. 3, vol. 4 (1887), p. 61; and by Y. Okoda, Tôhoku Math. Journal, vol. 22 (1922), p. 325.

This operator $F_n(D)$ possesses the property that

$$\lim_{x\to 0} F_m(D)I^n w(x) = \delta(m, n)$$

and the coefficient c_n is given by the rule

$$c_n = \lim_{x\to 0} [F_n(D)g(x)].$$

When $w(x)=e^x$, we have $W(t)=1/(1-t)$ and so $F_n(D)=D^n-D^{n-1}$. Also, when

$$w(x) = 1 + \frac{x}{(1!)^2} + \frac{x^2}{(2!)^2} + \frac{x^3}{(3!)^2} + \cdots = I_0(2\sqrt{x})$$

we have

$$W(t) = 1 + \frac{t}{1!} + \frac{t^2}{2!} + \cdots = e^t$$

$$F_n(D) = D^n - \frac{1}{1!}D^{n-1} + \frac{1}{2!}D^{n-2} - \cdots + \frac{(-1)^n}{n!}$$

$$I^n w(x) = x^{n/2} I_n(2\sqrt{x}).$$

A function $\delta(m, n, x)$ will be called selective when

$$\lim_{x\to 0} \delta(m, n, x) = \delta(m, n).$$

This definition can be generalized by replacing the limiting process by some other linear operation but the present definition is sufficiently general for the purposes we have in mind. In this definition we may either restrict $m+1$ and $n+1$ to be positive integers, as in the examples already considered, or we may allow m and n to take all integral values. An interesting example of the second case occurs when

$$\delta(m, n, x) = \frac{\sin \pi(x + m - n)}{\pi(x + m - n)}.$$

This function is used in the well known formula of interpolation

$$g(x + m) = \sum_{n=-\infty}^{\infty} \frac{\sin \pi\,(x + m - n)}{\pi(x + m - n)} g(n)$$

and the different possible generalisations or modifications of this formula should lead to other functions of type $\delta(m, n, x)$. Functions of type $\delta(m, n, x)$ may be formed in many ways. One way is to choose a set of distinct points in some metrical space. Let these points be numbered and let $d_{m,n}$ denote the distance between the points numbered m and n, then the function

$$\delta(m, n, x) = x^{d_{m,n}}$$

has the desired property.

In this paper we shall be interested chiefly in the case in which the function $\delta(m, n, x)$ has the form $F_m f_n w(x)$, where F_m and f_n are operators which operate on functions and $w(x)$ is a particular function. The operation F_m will be called a selective operation. The function which has just been considered is a δ-function of this particular type because if the operator E changes $f(x)$ into $f(x+1)$ and the operator H changes $f(x)$ into $f(x-1)$ we may write

$$\frac{\sin \pi(x+m-n)}{\pi(x+m-n)} = E^m H^n w(x), \text{ where } w(x) = \sin(\pi x)/\pi x.$$

A particularly interesting case occurs when the operation F_n is the same as f_n. If, in particular

$$w(x) = \sum_{n=0}^{\infty} w_n \frac{x^n}{n!}$$

$$f_m = \sum_{n=0}^{m} a_{m,n} D^n,$$

the coefficients $a_{m,n}$ are given by the equations

$$\begin{aligned}
0 &= a_{m,0}w_0 + a_{m,1}w_1 + \cdots + a_{m,m}w_m \\
0 &= a_{m,0}w_1 + a_{m,1}w_2 + \cdots + a_{m,m}w_{m+1} \\
&\cdots\cdots\cdots\cdots\cdots\cdots \\
&\cdots\cdots\cdots\cdots\cdots\cdots \\
0 &= a_{m,0}w_{m-1} + a_{m,1}w_m + \cdots + a_{m,m}w_{2m-1} \\
1 &= (a_{m,0}w_m + a_{m,1}w_{m+1} + \cdots + a_{m,m}w_{2m})a_{m,m}.
\end{aligned}$$

These equations are usually sufficient to determine the quantities $a_{m,n}$ uniquely. An exceptional case arises when $w(x)$ is the sum of a finite number of terms of type $A_2 \exp(\lambda_2 x)$ but this is not of much interest. The coefficients may often be determined by special devices as the following examples indicate.

1. Let

$$w(x) = \frac{\sin x}{x} = \frac{1}{2}\int_{-1}^{1} e^{-ix\mu} d\mu$$

then[1] $f_m = P_m(iD)$, where $D \equiv d/dx$ and

$$f_m f_n w(x) = \frac{1}{2}\int_{-1}^{1} e^{-ix\mu} P_m(\mu) P_n(\mu) d\mu = p_{m,n}(x)$$

$$p_{m,n}(0) = \frac{1}{2}\int_{-1}^{1} P_m(\mu) P_n(\mu) d\mu = \frac{1}{2n+1}\delta(m, n).$$

---

[1] $P_n(z)$ represents the Legendre polynomial.

Let

$$p_m(x) = f_m w(x) = \frac{1}{2}\int_{-1}^{1} e^{-ix\mu} P_m(\mu)d\mu = i^{-m}(\pi/2x)^{1/2} J_{m+1/2}(x)$$

then the coefficients in an expansion

$$g(x) = \sum_{n=0}^{\infty} c_n p_n(x)$$

are given by the rule

$$c_m = (2m + 1) \lim_{x\to 0} f_m g(x).$$

In particular

$$p_n(x + y) = \sum_{m=0}^{\infty} (2m + 1) p_m(x) p_{m,n}(y).$$

2. Let

$$w(x) = (2\pi)^{+1/2} e^{-1/2x^2} = \int_{-\infty}^{\infty} e^{-1/2\tau^2 - ix\tau} d\tau.$$

Then, if[1] $f_m = H_m(iD)$, we have

$$f_m f_n w(x) = \int_{-\infty}^{\infty} e^{-1/2\tau^2 - ix\tau} H_m(\tau) H_n(\tau) d\tau = h_{m,n}(x)$$

$$h_{m,n}(0) = \int_{-\infty}^{\infty} e^{-1/2\tau^2} H_m(\tau) H_n(\tau) d\tau = (2\pi)^{1/2} n!\delta(m, n).$$

Now let

$$h_n(x) = f_n w(x) = \int_{-\infty}^{\infty} e^{-1/2\tau^2 - ix\tau} H_n(\tau) d\tau;$$

then a rule that is sometimes useful for the determination of the coefficients in an expansion

$$g(x) = \sum_{n=0}^{\infty} c_n h_n(x)$$

is

$$c_n = (2\pi)^{-1/2}(n!)^{-1}[f_n g(x)]_{x=0}.$$

In particular,

$$h_m(x + y) = (2\pi)^{-1/2} \sum_{n=0}^{\infty} \frac{1}{n!} h_n(x) h_{m,n}(y).$$

3. Let

[1] We use $H_n(z)$ for the polynomial of Hermite $\exp(x^2/2)\, D^n[\exp(-x^2/2)](-)^n$.

$$w(x) = J_0(x) = \frac{1}{\pi}\int_0^\pi e^{-ix\cos\theta}d\theta.$$

Then, if $f_n = T_n(iD)$, where[1] $T_n(u) = \cos n\theta$ when $u = \cos\theta$, we have

$$f_m f_n w(x) = \frac{1}{\pi}\int_0^\pi e^{-ix\cos\theta}\cos m\theta\cos n\theta d\theta = j_{m,n}(x)$$

$$j_{m,n}(0) = \frac{1}{\pi}\int_0^\pi \cos m\theta\cos n\theta d\theta = \tfrac{1}{2}\delta(m, n) \quad m > 0$$

$$= \delta(m, n) \quad m = 0.$$

If

$$j_n(x) = f_n w(x) = \frac{1}{\pi}\int_0^\pi e^{-ix\cos\theta}\cos n\theta d\theta = (-i)^n J_n(x)$$

the coefficients in an expansion

$$g(x) = c_0 J_0(x) + 2\sum_{n=1}^{\infty} c_n J_n(n)$$

are given by the rule[2]

$$c_n = \lim_{x\to 0} i^n f_n g(x).$$

In particular,

$$j_m(x + y) = j_{m,0}(y)J_0(x) + 2\sum_{n=1}^{\infty} i^n j_{m,n}(y)J_n(x).$$

4. Let

$$w(x) = \operatorname{sech} x = \frac{1}{2}\int_{-\infty}^{\infty} e^{-ixu}\operatorname{sech}(\pi u/2)du.$$

Then, if $f_n = T_n(\tanh \pi i D/2)$, we have

$$f_m f_n w(x) = \frac{1}{2}\int_{-\infty}^{\infty} e^{-ixu}\operatorname{sech} t T_m[\tanh t]T_n[\tanh t]du, \quad t = \pi u/2$$

$$= t_{m,n}(x), \text{ say};$$

$$t_{m,n}(0) = \frac{1}{\pi}\int_{-1}^{1}(1-\theta^2)^{-1/2}T_m(\theta)T_n(\theta)d\theta = \begin{matrix}\tfrac{1}{2}\delta_{m,n} & m > 0 \\ \delta_{m,n} & m = 0;\end{matrix}$$

---

[1] We use here the notation employed by Balth van der Pol and Th. J. Weijers in their paper on *Tchebycheff Polynomials*, Physica, vol. 1, (1933), p. 78.

[2] This is equivalent to the rule given by Watson for the case in which the Maclaurin series of the function $g(x)$ is known; G. N. Watson, *Bessel Functions*, Chapter XVI, p. 523. Another rule was given by Neumann, *Theorie des Bessel'schen Functionen*, Leipzig, 1867, pp. 33–35.

and if

$$t_n(x) = f_n w(x) = \frac{1}{2}\int_{-\infty}^{\infty} e^{-ixu} \operatorname{sech} t \, T_n[\tanh t] du,$$

the coefficients in an expansion

$$g(x) = c_0 t_0(x) + 2\sum_{n=1}^{\infty} c_n t_n(x)$$

are given by the rule

$$c_n = \lim_{x\to 0} [f_n g(x)].$$

In particular, with suitable restrictions,

$$t_n(x + y) = t_0(x)t_n(y) + 2\sum_{m=1}^{\infty} t_m(x)t_{m,n}(y)$$

$$e^{ax} = t_0(x) + 2\sum_{m=1}^{\infty} t_m(x)T_m\left[\tanh\frac{\pi i a}{2}\right].$$

5. Let

$$w(x) = \Gamma(x + 1) = \int_0^{\infty} t^x e^{-t} dt \qquad R(x) > -1.$$

Then, if $f_n = L_n(E)$, where $Eh(x) = h(x+1)$, we have

$$f_m f_n w(x) = \int_0^{\infty} t^x e^{-t} L_m(t) L_n(t) dt = l_{m,n}(x)$$

$$l_{m,n}(0) = \int_0^{\infty} e^{-t} L_m(t) L_n(t) dt = \delta(m, n),$$

and if $L_n(t)$ is the polynomial of Laguerre

$$l_n(x) = f_n w(x) = \int_0^{\infty} t^x e^{-t} L_n(t) dt = \sum_{s=0}^{n} (-)^s \binom{n}{s} \frac{\Gamma(x + s + 1)}{\Gamma(s + 1)}$$

$$= (-)^n \Gamma(x + 1)\binom{x}{n},$$

the coefficients in the expansion

$$g(x) = \sum_{n=0}^{\infty} c_n l_n(x) = \Gamma(x + 1)\sum_{n=0}^{\infty} (-)^n \binom{x}{n} c_n$$

are given by the rule

$$c_n = \lim_{x\to 0} f_n g(x).$$

In particular, if $R(x)>0$,

$$t^x = \Gamma(x+1)\sum_{n=0}^{\infty}(-)^n\binom{x}{n}L_n(t).$$

6. Let

$$w(x) = x^{-1}\pi^x = \int_0^{\pi} t^{x-1}dt.$$

Then, if $f_n=\cos(nE)$, we have

$$f_m f_n w(x) = \int_0^{\pi} t^{x-1}\cos(mt)\cos(nt)dt = C_{m,n}(x),$$

$$\begin{aligned} C_{m,n}(1) = \int_0^{\pi} \cos(mt)\cos(nt)dt &= 0 & m &\neq n \\ &= (\pi/2) & m &= n > 0 \\ &= \pi & m &= n = 0. \end{aligned}$$

Let

$$C_n(x) = \int_0^{\pi} t^{x-1}\cos nt\, dt = \pi^x\sum_{s=0}^{\infty}(-)^s\frac{(n\pi)^{2s}}{(2s)!(x+2s)}$$

then the coefficients in an expansion

$$g(x) = \sum_{n=0}^{\infty} c_n C_n(x)$$

are given by the rule

$$\epsilon_n c_n = \lim_{x\to 0} f_n g(x), \quad \epsilon_n = \pi/2, \quad n > 0, \quad \epsilon_0 = \pi.$$

In particular,

$$\begin{aligned} \pi C_m(x+y) &= C_0(x)C_m(1+y) + 2\sum_{n=1}^{\infty}C_n(x)C_{m,n}(1+y) \\ &= \sum_{n=-\infty}^{\infty}C_n(x)C_{m+n}(1+y). \end{aligned}$$

This is easily confirmed with the aid of Parseval's theorem.

QUESTIONS, DISCUSSIONS AND NOTES

EDITED BY R. E. GILMAN, Brown University, Providence, Rhode Island

The department of Questions, Discussions, and Notes in the Monthly is open to all forms of activity in collegiate mathematics, including the teaching of mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.

The Numerical Evaluation of a Class of Trigonometric Series

By Morgan Ward, California Institute of Technology

1. In a recent problem arising in the design of an X-ray tube, it was necessary to sum two slowly convergent trigonometric series

$$\sum_1^\infty n^{-3/2}\cos 2n\pi x,\quad \sum_1^\infty n^{-3/2}\sin 2n\pi x,\ 0 \leqq x \leqq 1,$$

to a fair degree of accuracy. It was thought that the method devised to transform these series into more rapidly converging ones might be useful to others confronted with a similar task.

2. Let us consider quite generally a trigonometric series of the form

$$F(x) = \sum_1^\infty \phi(n)e^{2n\pi i x}, \tag{2.1}$$

where $\phi(n)$ is real and such that the series converges in the interval $0 \leqq x \leqq 1$.

Let $\Delta\phi(n)$, $\Delta^2\phi(n)$ and so on, represent the successive differences $\phi(n+1)-\phi(n)$, $\Delta\phi(n+1)-\Delta\phi(n)$, of $\phi(n)$; and let us write for brevity $\Delta^r\phi(a)$ for the value of the rth difference of $\phi(n)$ when n equals a. We then have the following

Theorem. *If a is any positive integer, and if the $(s+1)$th difference of $\phi(n)$ is of invariable sign for all positive integral values of n greater than a, then*

$$F(x) = \sum_1^a \phi(n)e^{2n\pi i x} + \sum_{r=0}^{s-1}\Delta^r\phi(a+1)\left(\frac{\csc \pi x}{2}\right)^{r+1} e^{2\pi i a x+\pi i(r+1)(x+1/2)} + R, \tag{2.2}$$

where

$$|R| \leqq 2\left(\frac{\csc \pi x}{2}\right)^{s+1} |\Delta^s\phi(a+1)|. \tag{2.3}$$

If we consider the real and imaginary parts of $F(x)$ separately, we can annunciate two precisely similar theorems where the exponentials appearing in (2.2) are replaced by the corresponding cosine and sine terms.

As regards our hypothesis about the sign of the $(s+1)$th difference of $\phi(n)$, we may remark that if $\phi(t)$ may be considered as a function of the continuous variable t for all values of $t \geqq a$, and if the $(s+1)$th derivative of $\phi(t)$ exists and is of invariable sign for $t \geqq a$, then the $(s+1)$th difference of $\phi(n)$ is also of invariable sign for $n > a$. These conditions will be satisfied for example for any positive integers a and s if $\phi(n) = n^{-\beta}$, $\beta > 1$.

To demonstrate the utility of our result for practical computation, take $\phi(n) = n^{-3/2}$, $a = 5$, and $s = 2$. Then we have

$$\sum_1^\infty n^{-3/2}\cos 2n\pi x = \sum_1^5 n^{-3/2}\cos 2n\pi x - \frac{\csc \pi x}{2}6^{-3/2}\sin 11\pi x$$

$$-\left(\frac{\csc \pi x}{2}\right)^2 \Delta 6^{-3/2} \cos 12\pi x + R,$$

where

$$| R | \leqq 2\left(\frac{\csc \pi x}{2}\right)^3 \Delta^2 6^{-3/2} \leqq 2\left(\frac{\csc \pi x}{2}\right)^3 (.0023).$$

Thus in the range $1/6 \leqq x \leqq 5/6$ where $1/2 \leqq (\csc \pi x)/2 \leqq 1$, $|R| < .005$. A comparable degree of accuracy from the series itself would require several hundred terms.

3. The above theorem may be proved as follows. Let

$$F_{(a)}(x) = \sum_{a+1}^{\infty} \phi(n) e^{2n\pi i x} \tag{3.1}$$

be the remainder of the series (2.1) after a terms, so that

$$F(x) = \sum_{n=1}^{a} \phi(n) e^{2n\pi i x} + F_{(a)}(x). \tag{3.2}$$

Then if s is any positive integer,

$$\begin{aligned}(1 - e^{2\pi i x})^{s+1} F_{(a)}(x) &= \sum_{r=0}^{s} (1 - e^{2\pi i x})^{s-r} \Delta^r \phi(a + 1) e^{2\pi i (a+r+1)x} \\ &\quad + \sum_{n=a+1}^{\infty} \Delta^{s+1} \phi(n) e^{2\pi i (n+s+1)x}.\end{aligned} \tag{3.3}$$

For this formula is easily seen to be true when $s = 0$. Assume that it is true when $s = k - 1$:

$$\begin{aligned}(1 - e^{2\pi i x})^k F_{(a)}(x) &= \sum_{r=0}^{k-1} (1 - e^{2\pi i x})^{k-1-r} \Delta^r \phi(a + 1) e^{2\pi i (a+r+1)x} \\ &\quad + \sum_{n=a+1}^{\infty} \Delta^k \phi(n) e^{2\pi i (n+k)x}.\end{aligned}$$

Multiply this expression by $1 - e^{2\pi i x}$. Then

$$\begin{aligned}(1 - e^{2\pi i x})^{k+1} F_{(a)}(x) &= \sum_{r=0}^{k-1} (1 - e^{2\pi i x})^{k-r} \Delta^r \phi(a + 1) e^{2\pi i (a+r+1)x} \\ &+ \Delta^k \phi(a + 1) e^{2\pi i (a+k+1)x} + \sum_{n=a+2}^{\infty} \Delta^k \phi(n) e^{2\pi i (n+k)x} - \sum_{n=a+1}^{\infty} \Delta^k \phi(n) e^{2\pi i (n+k+1)x}.\end{aligned}$$

The term $\Delta^k \phi(a+1) e^{2\pi i (a+k+1)}$ can be incorporated into the first summation, on changing its upper index from $k-1$ to k. In the second summation, we replace n by $n+1$, and then combine the resulting expression with the third summation. On recalling that by definition, $\Delta^k \phi(n+1) - \Delta^k \phi(n) = \Delta^{k+1} \phi(n)$, we obtain in this manner (3.3) with $s = k$. Hence (3.3) is true when $s = 0$, and if it is true for $s = k - 1$, it is true for $s = k$. Therefore, by induction, it is true generally.

Now assume that $\Delta^{s+1}\phi(n)$ is of invariable sign. Then in (3.3)

$$\left|\sum_{n=a+1}^{\infty}\Delta^{s+1}\phi(n)e^{2\pi i(n+s+1)x}\right| \leqq \sum_{n=a+1}^{\infty}\left|\Delta^{s+1}\phi(n)e^{2\pi i(n+s+1)x}\right|$$

$$= \pm\sum_{n=a+1}^{\infty}\Delta^{s+1}\phi(n) = \pm\sum_{n=a+1}^{\infty}\Delta^{s}\phi(n+1) - \Delta^{s}\phi(n) = \left|\Delta^{s}\phi(a+1)\right|, \text{ or}$$

$$\left|\sum_{n=a+1}^{\infty}\Delta^{s+1}\phi(n)e^{2\pi i(n+s+1)x}\right| \leqq \left|\Delta^{s}\phi(a+1)\right| \cdot \tag{3.4}$$

Finally, $1-e^{2\pi ix}$ may be written $2\sin\pi x e^{\pi i(x-1/2)}$. Therefore, if we divide both sides of (3.3) by $(1-e^{2\pi ix})^{s+1}$, we obtain

$$F_{(a)}(x) = \sum_{r=0}^{s}\Delta^{r}\phi(a+1)\left(\frac{\csc\pi x}{2}\right)^{r+1} e^{2\pi iax+\pi i(r+1)(x+1/2)}$$
$$+\left(\frac{\csc\pi x}{2}\right)^{s+1}\sum_{n=a+1}^{\infty}\Delta^{s+1}\phi(n)e^{2\pi inx+\pi i(s+1)(x+1/2)}.$$

It follows then from (3.4) that

$$F_{(a)}(x) = \sum_{r=0}^{s-1}\Delta^{r}\phi(a+1)\left(\frac{\csc\pi x}{2}\right)^{r+1} e^{2\pi iax+\pi i(r+1)(x+1/2)} + R, \tag{3.5}$$

where

$$|R| < 2\left(\frac{\csc\pi x}{2}\right)^{s+1}\left|\Delta^{s}\phi(a+1)\right| .$$

On combining (3.2) and (3.5), we obtain the result stated in the theorem.

An Application of Stirling's Numbers

By H. J. Goldstein, New York City

1. J. Ginsburg has called attention in the February, 1928, issue of this Monthly, to the varied history of the Stirling numbers. These numbers, while perhaps as interesting and useful as the allied Bernoulli numbers, have received relatively little attention. Their properties have, however, been discussed at considerable length by N. Nielsen[1] and C. Tweedie.[2] We shall consider an elementary application, and derive, *en passant*, a relation among the numbers themselves. The latter is probably not new, but does not appear in the works cited, or in others consulted by the author.

The Stirling numbers of the first and second species, designated respectively by C_n^r and Γ_n^r ($n>0$, $r\geqq 0$) are the coefficients in the expansions

[1] *Theorie der Gamma Funktion* (1904), pp. 66–78; and *Recherches sur les Nombres et les Polynomes de Stirling*, Annali di Matematica, vol. 10 (1904), pp. 287–318.

[2] Proc. Edinburgh Math. Soc., vol. 37 (1919), pp. 11–25.

$$(1) \qquad x(x+1)\cdots(x+n-1) = \sum_{s=0}^{n-1} C_n^s x^{n-s}$$

and

$$(1') \qquad x^n = \sum_{s=1}^{n} \Gamma_{s+1}^{n-s} x(x-1)\cdots(x-s+1).$$

In particular,

$$C_n^0 = 1,\ C_n^1 = \Gamma_n^1 = \binom{n}{2},\ C_n^{n-1} = (n-1)!,\ C_n^r = 0(n \leqq r),$$
$$\Gamma_n^0 = 1,\ \Gamma_1^r = 0(r > 0),\ \Gamma_2^r = 1\ \ \Gamma_3^r = 2^{r+1} - 1.$$

The numbers satisfy the following difference equations, from which successive values may be computed:

$$(2) \qquad C_{n+1}^r = C_n^r + nC_n^{r-1}$$
$$(2') \qquad \Gamma_{n+1}^r - n\Gamma_{n+1}^{r-1} = \Gamma_n^r.$$

Their properties need not be further developed here, and we shall assume that the reader has access to the papers mentioned above.

2. Consider the operation $\theta_n = x^n D^n$, $(D = d/dx,\ n \geqq 0)$; and put $\theta_1 = \theta$. The Stirling numbers enable us to express θ^n ($=\theta$ performed n times) and θ_n in terms of each other. We have:

$$\theta^1 = xD = \theta_1$$
$$\theta^2 = \theta(\theta^1) = xD(xD) = xD + x^2D^2 = \theta_1 + \theta_2$$
$$\theta^3 = xD(xD + x^2D^2) = xD + x^2D^2 + 2x^2D^2 + x^3D^3 = \theta_1 + 3\theta_2 + \theta_3$$

etc., and the coefficients are soon seen to be the Stirling numbers of the second species, a fact easily demonstrated by induction. Assume:

$$(3) \qquad \theta^n = \Gamma_{n+1}^0\theta_n + \Gamma_n^1\theta_{n-1} + \cdots + \Gamma_2^{n-1}\theta_1.$$

Then

$$\begin{aligned}\theta^{n+1} = \theta\theta^n &= \Gamma_{n+1}^0[\theta_{n+1} + n\theta_n] + \Gamma_n^1[\theta_n + (n-1)\theta_{n-1}] + \cdots + \Gamma_2^{n-1}[\theta_2 + \theta_1] \\ &= \Gamma_{n+1}^0\theta_{n+1} + [n\Gamma_{n+1}^0 + \Gamma_n^1]\theta_n + \cdots + [2\Gamma_3^{n-2} + \Gamma_2^{n-1}]\theta_2 + \Gamma_2^{n-1}\theta_1 \\ &= \Gamma_{n+2}^0\theta_{n+1} + \Gamma_{n+1}^1\theta_n + \Gamma_n^2\theta_{n-1} + \cdots + \Gamma_2^n\theta_1,\end{aligned}$$

by (2'). The result therefore holds generally. If we add the term $\Gamma_1^n\theta_0$, which equals 0 unless $n=0$, (3) is true also for $n=0$.

But the two systems of equations

$$\left.\begin{aligned} b_n &= \sum_{s=0}^{n-1} \Gamma_{n-s+1}^s a_{n-s} \\ a_n &= \sum_{s=0}^{n-1} (-1)^s C_n^s b_{n-s} \end{aligned}\right\} \qquad (n = 1, 2, \cdots, r)$$

are interchangeable.[1] And since θ_n and θ^n are associative and distributive, we may put $a_n=\theta_n$, $b_n=\theta^n$. It follows that

$$(3') \qquad \theta_n = C_n^0\theta^n - C_n^1\theta^{n-1} + \cdots (-)^{n-1}C_n^{n-1}\theta^1.$$

We have, moreover, from (1),

$$(4) \qquad C_n^0 x^n - C_n^1 x^{n-1} + \cdots (-)^{n-1}C_n^{n-1}x = x(x-1)\cdots(x-n+1).$$

If θ is substituted for x in the right-hand member, the factors are commutative, for it is easily verified that $(\theta-a)\ [(\theta-b)f(x)]=(\theta-b)\ [(\theta-a)f(x)]$. Equation (4) therefore holds equally, if x is replaced throughout by θ. This gives:

$$(5) \qquad \theta_n = \theta(\theta-1)(\theta-2)\cdots(\theta-n+1),$$

where the order of the factors is immaterial. That is,

$$(6) \qquad \theta_{n+1} = (\theta-n)\theta_n = \theta_n(\theta-n),$$

an identity which may be obtained directly, and the other results derived from it. Equation (3) then follows from (1′).

The operations θ_n and θ^n have many interesting properties analogous to those of D^n. The analogues of Leibniz' Theorem are perhaps worth noting. I omit the proofs, which are simple. We have:

$$(7) \qquad \theta^n uv = \sum_{s=0}^{n}\binom{n}{s}\theta^{n-s}u\theta^s v$$

$$(7') \qquad \theta_n uv = \sum_{s=0}^{n}\binom{n}{s}\theta_{n-s}u\theta_s v$$

with the usual multinomial expansion for products of r factors. Also,

$$(8) \qquad \theta_r x^n f(x) = x^n\sum_{s=0}^{r} n_s\theta_{r-s}f(x),\ n_s = n!/s!,$$

and

$$(9) \qquad \theta_a\theta_b f(x) = \sum_{s=0}^{a}\binom{a}{s}b_s\theta_{a+b-s}f(x).$$

These results may be expressed symbolically as follows:

$$(10) \qquad \theta^n uv = (\theta u + \theta v)^n$$

$$(10') \qquad \theta_n uv = (\theta u + \theta v)_n$$

$$(11) \qquad \theta_r\cdot x^n = x^n(\theta+n)_r$$

$$(12) \qquad \theta_a\theta_b = (\theta_b+b)_a.$$

3. Let $f(x)=\sum_{n=0}^{\infty}a_nx^n$ be convergent for $|x|<R$, and let $\phi(x)=\sum_{s=0}^{r}p_sx^s$ be a polynomial of degree r in x, with real coefficients. Then both $\sum_{n=0}^{\infty}n^sa_nx^n$ and $\sum_{n=0}^{\infty}\phi(n)a_nx^n$ have the same radius of convergence as $f(x)$; for

[1] Nielsen, *Gamma Funktion*, p. 69, *Recherches*, p. 302; Tweedie, op. cit., p. 12.

$$\lim_{n \doteq \infty} (n+1)^s/n^s = \lim_{n \doteq \infty} \phi(n+1)/\phi(n) = 1,$$

so long as r is finite. Also, since the convergence is uniform, we have, for $|x| < R$: $\theta^s f(x) = \sum_{n=0}^{\infty} n^s a_n x^n$, whence:

$$(13)\quad \phi(\theta) f(x) = \sum_{s=0}^{r} p_s \theta^s f(x) = \sum_{s=0}^{r} p_s \sum_{n=0}^{\infty} n^s a_n x^n = \sum_{n=0}^{\infty} \sum_{s=0}^{r} p_s n^s a_n x^n = \sum_{n=0}^{\infty} \phi(n) a_n x^n.$$

Or, symbolically:

$$(13')\qquad \phi(\theta) \cdot \sum_{n=0}^{\infty} = \sum_{n=0}^{\infty} \cdot \phi(n).$$

That is: *The operation $\phi(\theta)$ performed on $f(x)$ has the effect of multiplying the nth term by $\phi(n)$.*

Replacing every θ^s by its equal in (3), we obtain:

$$\phi(\theta) = \sum_{s=0}^{r} p_s \theta^s = \sum_{s=0}^{r} q_s \theta_s,$$

where

$$(14)\qquad q_s = \overset{0}{\Gamma}_{s+1} p_s + \overset{1}{\Gamma}_{s+1} p_{s+1} + \cdots + \overset{r-s}{\Gamma}_{s+1} p_r.$$

Therefore, from (13):

$$(15)\qquad \sum_{n=0}^{\infty} \phi(n) a_n x^n = \sum_{s=0}^{r} q_s \theta_s f(x) = q_0 f(x) + q_1 x f'(x) + \cdots + q_r x^r f^{(r)}(x).$$

For example, if $f(x) = e^x$, we have

$$(16)\qquad \sum_{n=0}^{\infty} \phi(n) \frac{x^n}{n!} = e^x \sum_{s=0}^{r} q_s x^s.$$

If we take $\phi(n) = (n+1)^r$, then

$$p_s = \binom{r}{s},$$

and

$$q_s = \overset{0}{\Gamma}_{s+1} \binom{r}{s} + \overset{1}{\Gamma}_{s+1} \binom{r}{s+1} + \cdots + \overset{r-s}{\Gamma}_{s+1} \binom{r}{r},$$

which[1] equals $\overset{r-s}{\Gamma}_{s+2}$; so that

$$\sum_{n=0}^{\infty} (n+1)^r a_n x^n = \sum_{s=0}^{r} \overset{r-s}{\Gamma}_{s+2} \theta_s f(x),$$

[1] Nielsen, *Recherches*, p. 299.

and

$$\begin{aligned}\sum_{n=0}^{\infty}\phi(n+1)\,a_nx^n &= \sum_{s=0}^{r}p_s\sum_{n=0}^{\infty}(n+1)^s a_n x^n\\ &= \sum_{s=0}^{r}p_s\sum_{i=0}^{s}\Gamma_{i+2}\theta_i f(x)\\ &= \sum_{s=0}^{r}Q_s x^s f^{(s)}(x),\end{aligned}\tag{17}$$

where

$$Q_s = \Gamma_{s+2}^{0}p_s + \Gamma_{s+2}^{1}p_{s+1} + \cdots + \Gamma_{s+2}^{r-s}p_r.\tag{18}$$

We may choose, instead of $\phi(n)$, a polynomial of degree r in $1/n$:

$$\psi(n) = \sum_{s=0}^{r}p_s'\,n^{-s},\tag{19}$$

and obtain from the relation[1]

$$\frac{1}{z^n} = \sum_{i=n-1}^{\infty}C_i^{i-n+1}/z(z+1)\cdots(z+i),$$

the result

$$\sum_{n=0}^{\infty}\psi(n+1)a_nx^n = \sum_{s=0}^{\infty}q_s'\,\theta_{-s}f(x) = \sum_{s=0}^{\infty}q_s'\,x^{-s}\int_0^x\cdots\int_0^x f(x)dx^s,\tag{20}$$

where

$$\begin{aligned}q_0' &= p_0'\\ q_s' &= C_{s-1}^{s-1}p_1' + C_{s-1}^{s-2}\,p_2' + \cdots + C_{s-1}^{0}p_s'.\end{aligned}\tag{21}$$

If, in (20), we put $f(x)=e^x$, as in (17) above, we have:

$$\sum_{n=0}^{\infty}\frac{\psi(n+1)x^n}{n!} = e^x\sum_{s=0}^{2r-1}q_s'\,x^{-s}.\tag{22}$$

We may combine (20) with (17) to give an expression for

$$\sum_{n=0}^{\infty}\Phi(n+1)a_nx^n$$

where

$$\Phi(n) = p_{-r}n^{-r} + p_{-r+1}n^{-r+1} + \cdots + p_0 + \cdots + p_sn^s.$$

---

[1] Nielsen, op. cit., p. 302; Tweedie, op. cit., p. 10.

Other results similar to (17) and (20) are readily obtained. All follow from the fact that successive differentiation or integration of a power-series introduces into the nth term a factor involving factorial expressions in n.

4. Tweedie has derived[1] numerous identities among the Stirling numbers by the use of the expansion (1′). However, the following relation is not, I believe, included in any he has given. By (1′) and (4),

$$(23) \qquad (x-1)^m = \Gamma_2^{m-1}(x-1) + \Gamma_3^{m-2}(x-1)(x-2) + \cdots + \Gamma_{m+1}^{0}(x-1)\cdots(x-m),$$

and

$$(24) \quad (x-1)(x-2)\cdots(x-r) = C_{r+1}^{0}x^r - C_{r+1}^{1}x^{r-1} + \cdots (-)^r C_{r+1}^{r}.$$

Substituting for the factorials in (23) their equals in (24), and equating the coefficients of x^n on the two sides of the resulting equation, we have:

$$(-1)^{m-n}\binom{m}{n} = \Gamma_{n+1}^{m-n}C_{n+1}^{0} - \Gamma_{n+2}^{m-n-1}C_{n+2}^{1} + \cdots (-1)^{m-n}\Gamma_{m+1}^{0}C_{m+1}^{m-n},$$

i.e., dividing by $(-1)^{m-n}$, and replacing m by $m-1$ and n by $n-1$,

$$(25) \quad \Gamma_m^{0}C_m^{m-n} - \Gamma_{m-1}^{1}C_{m-1}^{m-n-1} + \cdots (-1)^{m-n}\Gamma_n^{m-n}C_n^{0} = \binom{m-1}{n-1}, \quad m > n > 0.$$

In particular, for $n=1$:

$$(26) \qquad (m-1)!\Gamma_m^{0} - (m-2)!\Gamma_{m-1}^{1} + \cdots (-1)^{m-2}1!\Gamma_2^{m-2} = 1, \quad m > 1.$$

I take this occasion to note an error in Ginsburg's article. The values he gives for ${}_nS_r'$ (i.e., the sum of the r-products of the first n integers, repetitions being included), are actually those of ${}_{n-1}S_r'$. In fact, ${}_nS_r' = \Gamma_{n}{}^{r}{}_{+1}$, while ${}_nS_r$ (i.e., the corresponding sum without repetitions) $= C_{n+1}^{r}$. The latter values are given correctly.

RECENT PUBLICATIONS

EDITED BY R. A. JOHNSON, Brooklyn College of the City of New York

All books for review should be sent directly to the editor of this department, at Brooklyn College, 66 Court Street, Brooklyn, N. Y., and not to any of the other editors or officers of the Association.

REVIEWS

Science and Sanity. An Introduction to Non-Aristotelian Systems and General Semantics. By Alfred Korzybski. Lancaster, The Science Printing Company, 1933. xx+798 pages. \$7.00; with discount, \$5.50.

In the limits of a short notice it is impossible even to indicate the wealth of material in Korzybski's introduction to general semantics,[2] ranging as it does

[1] Op. cit., pp. 11–14.

[2] From σημαίνειν "to signify," "to mean."

from a general discussion of "structure," through "the non-Aristotelian language called mathematics" and "the foundations of psychophysiology" to "the sematics of the differential calculus" and "the structure of matter"; so we shall attempt merely to indicate a few of the high spots which readers of this MONTHLY will find of particular interest. Much of the book represents pioneering work by the author, who insists that the greatest value of his new approach is in its experimental and practical possibilities for sane behaviour rather than its philosophical importance. For a more adequate review the reader is referred to an article by Professor Keyser.[1]

Mathematical readers will probably best make their way into the book by reading first Supplement III, pp. 747–761, where Korzybski orients part of his work with respect to the four leading schools of current mathematical thought, namely the logistic (Peano, Whitehead, Russell), the axiomatic (Hilbert), the intuitional (Brouwer, Weyl), and the Polish non-Aristotelian schools (Łucasiewicz, Tarski, Leśniewski, Skarżeński, Chwistek). To these Korzybski adds a fifth school: "The average prevalent mathematical technician, who does not realize that he belongs to the numerically large class which may be called the 'Christian Science' school of mathematics, which proceeds by faith and disregards entirely any problems of the epistemological foundations of their supposed 'scientific' activities." This sounds rather unkind. But is it wholly undeserved? Let him that is without sin among us cast the first stone.

Korzybski's system is definitely non-Aristotelian in several respects, only one of which can be noted here. Korzybski rejects outright Aristotle's first law, the so-called law of identity, which is quoted in Jevons' form: "Whatever is, is." Another statement of the law is "A is A." Readers familiar with Russell's "theory of types" will recognize an isomorphism between it and Korzybski's sharply clear account of the different "levels of abstraction" by which mathematicians and others verbalize brute objects, like bricks, at the *unspeakable* level, into symbols, like the word b-r-i-c-k-s, which in their turn are "named," or otherwise raised to a higher level of abstraction, and so on, apparently indefinitely. Blunders in reasoning and common, plain thinking multiply when the levels are confused, and the "is" of identity appears as the most prolific source of such confusions. From this point of view it would seem that the famous "axiom of reducibility" could never even come into sight. Whether or not this is so, it will be clear to any reader who takes the pains to understand Korzybski's position that the author has raised an issue of the greatest interest to all those who seek to understand the foundations of mathematics.[2]

[1] *Scripta Mathematica*, vol. 2 (1934), pp. 247–260.

[2] Professor Keyser (loc. cit., p. 246) remarks that Korzybski, "in proposing to eliminate the 'is of identity' completely from all linguistic structure, has gone far beyond all other critics, Aristotle included. In fairness, however, to Aristotle, it must be said that he did not fail to note that peculiar sense, among the several senses, of the term 'is' and did not fail to indicate the danger of employing it uncritically." Aristotle, no less than the devil, must be given his due. But it seems to the reviewer that Professor Keyser, as shown by the quotations from Aristotle on p. 254 of his article, gives Aristotle considerably more than his due. Korzybski's rejection, in its relation to

Mathematicians will get a good idea of Korzybski's program by reading (pp. 93–94) the explicit statements of certain of the things which Korzybski either accepts or rejects. Among the acceptances are relations, structure, and order (concepts undefined in the system). Readers acquainted with attempts to define these concepts in mathematics[1] may be willing to grant the author that these three are as well left undefined at present. However, all of them, and in particular "structure," which is all-important for Korzybski's position, are sufficiently explained—explanation of course is not definition by a set of postulates.

General semantics itself is described as the science of significant behaviour. This may remind some of Clive Bell's definition of art as "significant form," and the experience of Bell's readers in learning from his book on the subject that "significant form" is nothing more nor less than "significant form." Was there not some difficulty in *Principia Mathematica* over the question of "significant" statements— the presumed insight on the part of the reader which would enable him to judge whether a given statement was "significant" or just a "meaningless" jumble of words or symbols? Anyhow, Wittgenstein appears to have seen through this particular difficulty, and to have disposed of it (temporarily) with his decree that "mathematical truths" shall be analytic. However, it will probably be agreed by all readers that Korzybski has given them a detailed description of what he means by general semantics, and that he has illuminated mathematics by placing it as a detail, but a highly important one, in his vaster picture.

Another innovation is the wholesale exploitation of what Korzybski calls "non-elementalism," already classical in theoretical physics through the fusion of "space" and "time" into "space-time" by Einstein and Minkowski. Closely allied to this is the insistence upon the organism-as-a-whole point of view of certain biologists, which Korzybski also exploits. Thus (p. 30), "I must construct a non-elementalistic language in which 'senses' and 'mind,' · · · are no longer to be verbally split, because a language in which they *are* split is not similar in structure to the known empirical facts · · · ." It may be recalled in passing that Whitehead, some years ago, raised similar objections to what he called "bifurcation" theories of nature. How these extremely general and far-reaching ideas are applied to mathematics and to mathematical physics, must be seen in the book itself.

Although it is a minor point in the sweep of the general development, there were some things in the chapter on "linearity" (pp. 603–614) which the reviewer found difficult to understand. Thus it is stated (p. 613) that "*approximation* · · · is strictly connected with *linearity* or *additivity*." From its context,

mathematics, is of a different kind from Aristotle's. It is difficult (at least for the reviewer) to see how Aristotle's rather naïve distinctions between the "senses borne by the term 'Sameness' " could apply to Korzybski's *levels of abstraction*, or to Russell's *types*. It is not clear, from the quotations (which see), that Aristotle's position is relevant for Korzybski's.

[1] For relations, the Whitehead-Russell definition; for order, projective geometry, where it is assumed that *abc* and *acb* are distinguishable and distinct orders of the three letters involved; for structure, any of the attempts to define it in fairly recent work (including some of Russell's).

this seems to imply that in *all* kinds of approximative work in mathematics it is sufficient to consider only terms of order not higher than the first. This is contradicted by so simple a thing as the indicatrix. If by "strictly connected" is meant "semantically connected," then again the statement seems to say too much. There is nothing sacrosanct about the linearity of certain differential equations (and hence the additivity of their solutions) that makes most of mathematical physics as we know it a possibility; a more competent generation may find that linearity is a gratuitous concession to present mathematical disabilities. It has been conjectured (although possibly not in print) by Einstein that some of our failures to give a coherent (="semantic," in Korzybski's sense) account of some physical phenomena may be rooted in the traditional demand for linearity. This is not the place to go into the history of this demand, but a consideration of it from Huyghens to the present might show that it is on a par with any other postulate, sufficient so long as it serves, but not necessarily "significant" at any time or in any place (or in any "time-place," to be non-elementalistic). The existence of doubts as to its sufficiency seem to indicate that a revision of its "semantic" status is about due. But, as already stated, this is a minor point, and we shall ignore others of a similar character, since, to dwell upon them, would only give a false impression of the book.

For students of mathematics, probably the most illuminating of the many new points of view in the book will be those which re-value mathematics in the light of human experience as a whole. Korzybski emphasizes that mathematics and mathematical physics have succeeded better in their self-appointed tasks than some other human enterprises because the *structure* of both is more closely patterned than is that of any other "language" to the thing which is to be undertaken. To indicate the basis for this claim we must refer to the book itself. Further, mathematics is here brought down from the celestial void of pure disembodied thought. Like Brouwer, Korzybski regards mathematics as a form of human behaviour or, differently expressed, as a social activity of human beings. Unlike his predecessors, Korzybski backs his claim with a mass of evidence drawn from practically the entire range of science—including the biological sciences—such as has not been assembled in any one place before. Mathematicians will find their estimates of their activities both inflated and deflated by a reading of this remarkable book.

E. T. BELL

Einführung in die Differentialrechnung und Integralrechnung. By Edmund Landau, Groningen, P. Noordhoff, 1934. 368 pages.

This treatise on the calculus is based on lecture courses given repeatedly by Professor Landau at Göttingen. It is not a faithful reproduction of his lectures. For instance, Professor Landau devoted considerable time, in his courses, to geometric applications. Not wishing to presuppose a knowledge of the foundations of geometry, and being anxious at the same time that his book possess

complete rigor, Professor Landau restricted himself to purely analytic aspects of the calculus.

This work is thus not to be compared with ordinary calculus texts. Except for the fact that it leaves the general theory of point sets almost untouched, it may be regarded as an exposition of the theory of functions of a real variable. As a reference work in courses on the real variable, it will be found very valuable.

There is presupposed, on the part of the reader, such an acquaintance with the properties of real numbers as can be gained from Landau's recent monograph *Grundlagen der Analysis*.

In the first part, which deals with the differential calculus, the general topics studied are sequences, functions and continuity, derivatives, infinite series, the Taylor expansion, functions of two variables and implicit functions. There are chapters on the elementary transcendental functions. Special topics of interest are the fundamental theorem of algebra and the decomposition of rational functions into partial fractions. The spirit of the work is shown by such details as the proof of the representability of a continuous function as a limit of polynomials, and the proof, given almost immediately after the definition of derivative, of the existence of functions which lack a derivative everywhere.

In the second part, on the integral calculus, the general topics are the indefinite integral, the integral as a limit of a sum, the integration of infinite series and improper integrals. The integrals of rational functions are examined in detail. A brief treatment is given of the gamma function. The book closes with a chapter on Fourier series, in which an expansion theorem sufficient for the ordinary applications is obtained.

Edmund Landau is a great mathematician and a great expositor, of whom the whole mathematical world may be justly proud. His writings perpetuate brilliantly the traditions of Gauss and of Weierstrass. This latest book will survive, in its crystalline beauty, an inspiration to students of all nations, long after the chauvinistic anthropological twaddle which has assailed mathematical ears in recent months has passed on to its appropriate limbo.

J. F. RITT

MATHEMATICS CLUBS

EDITED BY F. M. WEIDA, The George Washington University, Washington, D. C.

All reports of club activities, suggestions and topics for club programs should be sent to F. M. Weida, The George Washington University, Washington, D. C. All manuscripts should be typewritten with double spacing, and with margins at least one inch wide.

CLUB ACTIVITIES

1933–1934

THE PI MU EPSILON MATHEMATICAL FRATERNITY

Pi Mu Epsilon is an academic fraternity in institutions of university grade. Its primary aim is

the advancement of mathematics and scholarship. It is a living, active, working fraternity of scholars in which the members are actively engaged in study and research and the exchange of ideas in the field of mathematical science.

Pi Mu Epsilon of the University of Missouri

The Missouri Alpha Chapter of Pi Mu Epsilon met on the second and fourth Tuesdays of each month. We started the year with a total membership of fifty-one in Columbia, about thirty of whom were active in the work of the fraternity. At most of the regular meetings we had a lecture by a faculty member or student on some topic from pure or applied mathematics and all persons interested in mathematics were invited to attend. On five of these occasions light refreshments were served and the business meeting was held after the adjournment of the open meeting.

We initiated three new members on December 14, 1933 and fourteen on May 12, 1934.

For a number of years we have offered two prizes of ten dollars each to be awarded annually to the sophomore or junior student in the calculus and to the freshman or sophomore student in analytic geometry who attains the highest grade in a special examination set for the purpose. The prize examinations were held May 5, 1934. The winners were announced and the prizes were awarded at the annual banquet following the initiation ceremony on May 12, 1934.

All the officers except the corresponding secretary were student members, the Director and Vice Director both being undergraduates.

GEORGE M. EWING, *Corresponding Secretary*

Pi Mu Epsilon of the University of California

The year 1933–1934 was a very prosperous and interesting one for the California Beta Chapter of Pi Mu Epsilon under the leadership of the following officers: John Oxtoby, Director; Rafael Robinson, Vice Director; Myra Waddell, Secretary; Willis Lamb, Jr., Treasurer; Gabriel G. Bejarano, Librarian.

At the Fall initiation on October 2, 1933, twelve new members were initiated, seven in mathematics, two in physics, and three in chemistry. Mr. James W. Hoge acted as toastmaster. The initiates entertained the fraternity with a debate on the question: "Resolved that the law of gravitation should be repealed."

At the Spring initiation on March 3, 1934, five new members were initiated, four in mathematics, and one in physics. Dr. C. B. Morrey acted as toastmaster. The neophytes entertained the fraternity with some amusing skits.

During the year the following interesting papers were presented:

September 20, 1933: "Separation theorem for Jordan curves" by Dr. C. B. Morrey.
October 18, 1933: "Integrals" by Professor Wm. F. Osgood.
November 7, 1933: "Converses" by Mr. Mugele.
November 28, 1933: "Projective differential geometry" by Wm. Hutchings.
January 24, 1934: "Dimensional analysis" by John Oxtoby.
February 15, 1934: "Algebraic ideals" by Dr. A. L. Foster.
March 15, 1934: "Hilbert space" by Professor V. F. Lenzen.
April 2, 1934: "Automorphic functions" by Professor E. T. Whittaker.
April 25, 1934: "Continued fractions" by Eleanor Lazansky; Election of officers for 1934–1935.

The annual picnic was held Wednesday evening, May 9, 1934, at Cordoneces Park. Games, bridge, and dancing were enjoyed by the members.

VIRGINIA WOOD, *Secretary*

LOCAL MATHEMATICAL CLUBS

The Junior Mathematics Club of the University of Chicago

The club is conducted by graduate students of mathematics, and membership is open to students specializing in mathematics and mathematical astronomy.

The officers for 1933–1934 were: G. Cuthbert Webber, President; Herman H. Goldstine, Treasurer; Mary K. Landers and Kenneth S. Ghent, Social Committee; L. Roy Wilcox, Program Chairman.

The meetings and programs were as follows:

October 25, 1933: "What is geometry" by Professor E. P. Lane.
November 8, 1933: "A problem in determinants" by R. Oldenburger.
November 22, 1933: "The uses of mathematics in economics and business" by Professor T. O. Yntema.
December 6, 1933: "Rapid computation" by Dr. S. J. Krieger.
January 3, 1934: "On pure Riemann matrices" by D. M. Dribin.
January 17, 1934: "Some topics in the theory of quasi-analytic functions" by I. E. Perlin.
January 31, 1934: "The Waring problem for cubic functions" by G. C. Webber.
February 14, 1934: "Some graphic methods of representing group properties" by Dr. J. K. Senior.
February 28, 1934: "The theory of matric differential equations" by H. H. Goldstine.
March 14, 1934: "Representation of integers by sets of forms" by O. K. Sagen.
April 4, 1934: "Constants of nature" by Sir Arthur Eddington.
April 11, 1934: "Some elementary notions of topology" by Professor M. I. Logsdon.
April 25, 1934: "The mechanics of double stars" by E. Johnson.
May 9, 1934: "Nilpotent algebras in four units" by K. S. Ghent.
May 23, 1934: "Algebraic matric equations" by Miss M. G. Humphreys.

G. CUTHBERT WEBBER, *President*

The Mathematics Club of Milwaukee-Downer College

The club exists for the purpose of creating an interest in mathematics by the discussion of subjects not usually included in the curriculum courses. Membership is open to all who are willing to contribute to the value of the programs.

The meetings and programs were as follows:

October 17, 1933: Supper meeting: "Study of mathematical bibliography."
November 21, 1933: "Mathematics and citizenship."
December 19, 1933: "Methods of reckoning time"; "History and reform of the calendar."
January 16, 1934: "Aesthetic measure"; "Dynamic symmetry."
February 20, 1934: "Mazes and labyrinths, their history and theory."
March 20, 1934: "Linkages and string figures."
April 26, 1934: "The mathematics of the honeycomb."
May 15, 1934: Closing dinner at the College club.

RUTH MIKULA, *Secretary*

The Junior Mathematics Club of the Extension Division of the University of Wisconsin

The purpose of the club is to promote interest in mathematics and to afford an opportunity to study interesting matters connected with mathematics that do not find a place in the usual classroom discussion. Membership is open to all students interested in mathematics.

The officers for 1933–1934 were: Paul Guenther, President; Florence Meyer, Secretary-Treasurer; Florence Axen, Club Adviser.

The meetings and programs were as follows:

October 26, 1933: "Magic squares" by Paul Guenther.
November 23, 1933: "Aerial gunning and bombing" by Professor J. E. Case.
December 19, 1933: Christmas party at which mathematical puzzles were solved.
January 28, 1934: "The story of American mathematics" by Professor Morris Marden.

February 15, 1934: "How to draw a straight line" by Professor H. P. Pettit of Marquette University.

March 15, 1934: "Astronomical coordinates" by Amron Katz.

April 19, 1934: "Elementary theory of lenses" by Professor Martin of the Physics department.

May 10, 1934: Spring exhibit to which all High School students were invited. The exhibit consisted of mathematical models and curves, calculating machines, telescopic view of the sky, Lissajou's figures and short talks. The talks were as follows: "How a ship finds its position at sea" by Radcliffe Park; "Magic squares" by Paul Guenther; "An extra dimension or two" by Robert Feinstein; "The history of algebraic equations" by Florence Meyer.

FLORENCE MEYER, *Secretary*

PROBLEMS AND SOLUTIONS

EDITED BY OTTO DUNKEL, H. L. OLSON AND W. F. CHENEY, JR.

Send all communications about Elementary Problems and Solutions to W. F. Cheney, Jr., Dept. Box 35, Storrs, Connecticut.

The Department of Elementary Problems welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. Problems may be submitted unaccompanied by their solutions.

PROBLEMS FOR SOLUTION

E 118. *Proposed by V. Thébault, Le Mans, France.*

Determine the largest and smallest multiples of 63 which can be written with the ten digits, 0, 1, 2, $\cdots$ 9, used once each in each multiple.

E 119. *Proposed by Maud Willey, Long Beach, Mississippi.*

Nine equal squares, five marked with the letter X and four with the letter O, are arranged at random in a square array. What is the probability that some row, column or diagonal of the array contains only squares bearing the letter O?

E 120. *Proposed by L. S. Johnston, University of Detroit.*

Given the perpendicular distances, a and b, from a point P to the arms of a known angle, θ, within which P lies; it is required to compute the lengths of the radii of the two circles, each of which passes through P and is tangent to both arms of θ.

E 121. *Proposed by W. F. Cheney, Jr., Connecticut State College.*

The sides of the real triangle, ABC, are three different positive integers, no two of which have a common factor. AD is tangent to the circumscribed circle at A, and meets BC produced at D. Prove that AD, BD and CD are each always rational, but that none of them can ever be an integer.

E 122. *Proposed by C. A. Rasmussen, University of Alabama.*

The lines joining the three vertices of a given triangle, ABC, to a point O in its plane, cut the sides opposite the vertices A, B and C in the points K, L and M respectively. A line through M parallel to KL cuts BC at V and AK at W. Prove that $VM=MW$.

E 123. *Proposed by W. R. Ransom, Tufts College, Massachusetts.*

Prove that if the integer, $1111 \cdots 12222 \cdots 24$, has one more 2 than 1's, then it is the product of two factors whose digits are all 3's except for a terminal 4 in one factor and a terminal 6 in the other.

E 124. *Proposed by B. W. Jones, Cornell University.*

Show that the volume generated by revolving a cube of edge a about one of its space diagonals is $\pi a^3/\sqrt{3}$.

SOLUTIONS

E 87 [1934, 264]. *Proposed by William Douglas, Courtenay, British Columbia.*

Find the radius of the circle in which a thirty foot chord subtends a thirty-two foot arc.

Solution by Paul Herget, Cincinnati Observatory.

The radian measure of half the arc subtended by the chord is $16/r$ and its sine is $15/r$, so that

$$\frac{15}{r} = \frac{16}{r} - \frac{1}{3!}\left(\frac{16}{r}\right)^3 + \frac{1}{5!}\left(\frac{16}{r}\right)^5 - \frac{1}{7!}\left(\frac{16}{r}\right)^7 + \cdots .$$

This is intrinsically an equation in powers of $1/r^2$. Multiplying by r^3 and transposing, we have

$$r^2 = 16^3\left[\frac{1}{3!} - \frac{1}{5!}\left(\frac{256}{r^2}\right) + \frac{1}{7!}\left(\frac{256}{r^2}\right)^2 - \frac{1}{9!}\left(\frac{256}{r^2}\right)^3 + \cdots\right].$$

Now we may substitute an approximate value of r^2 on the right and obtain a new value of r^2 which is accurate to two more places of approximation. If we begin with 683, the successive results are 670, 669.74, 669.7377, 669.737683, 669.7376787, etc., and thence $r = 25.87929054$ feet.

Also solved by W. E. Buker, J. E. Burnam, Hansraj Gupta, E. L. Harp, Jr., W. W. Johnson, Elmer Latshaw, F. L. Manning, D. M. Perkins, C. W. Trigg, Simon Vatriquant and the proposer.

E 88 [1934, 264]. *Proposed by R. K. Morley, Worcester Polytechnic Institute.*

Show that m arctan x is greater or less than arctan mx according as m is greater or less than unity, providing m and x are positive and the angles are taken in the first quadrant.

Solution by E. P. Starke, Rutgers University.

The given functions may be put into the respective forms,

$$F_1 = \int_0^x m dx/(1 + x^2), \text{ and } F_2 = \int_0^x m dx/(1 + m^2x^2).$$

The theorem is now apparent, either by interpreting F_1 and F_2 as areas under the curves, $y = m/(1+x^2)$ and $y = m/(1+m^2x^2)$, or as the limits of sums.

Also solved by W. E. Buker, C. W. Trigg, Simon Vatriquant and the proposer.

E 89 [1934, 265]. *Proposed by E. P. Starke, Rutgers University.*

It is required to find two integers, one the square of the other, which together contain each of the nine digits from one to nine just once. Show that there are just two solutions.

Solution by J. E. Burnam, Simmons University, Abilene, Texas.

Let A be the integer. Then A must consist of three digits and A^2 of six. A is less than 987 and A^2 is greater than 123456, so that A is greater than 351.

Let S and T be the digit sums of A and A^2 respectively. Then $S+T=45$, and $S+T\equiv 0$ (mod. 9). Also, $S^2-T\equiv 0$ (mod. 9) in accordance with the rules for casting out nines. Adding the two congruences gives $S(S+1)\equiv 0$ (mod. 9). But since A is less than 987, S is less than 24, and may only equal 8, 9, 17 or 18.

We note further that A can not have 1, 5 or 6 as units digit. The hundreds digit of A is 3 or more. If 4 appears among the first two digits of A, the units digit cannot be 2 or 8. If the digit 9 appears in the hundreds or tens place of A, the units digit can not be 3 or 7. If 6 appears as either of the first two digits, the units digit can not be 4. If 1 appears in the first or second place of A, the units digits can not be 9.

Examining all numbers between 351 and 987 with these restrictions, there are found only two numbers which meet the required conditions. These are 567, whose square is 321489, and 854, whose square is 729316.

Editor's Note. This problem has been published before. See V. Thébault, Education Mathématique, Paris, 1921, question 4224; also the Journal of the Indian Mathematical Society, Part II, Vol. 19, No. 7, (1932) pp. 174–175; and again as problem 2? on page 79 of Mr. A. A. Krishnaswami Ayengar's Mathematical Problem Papers.

Also solved by W. E. Buker, M. L. Constable, Paul Rosa, Jr., C. W. Trigg, Simon Vatriquant and the proposer.

E 90 [1934, 265]. *Proposed by W. B. Campbell, Rangoon, Burma, India.*

Show that all the planes which cut the tetrahedron $ABCD$ and are parallel to AB and CD, cut it in parallelograms which are equiangular to each other. In case $AB=CD=r$, these parallelograms are of constant perimeter $2r$.

Solution by Simon Vatriquant, A. R. d'Ixelles, Brussels, Belgium.

Let us denote respectively by K, L, M and N, the points where the planes

cut the edges AD, DB, BC and CA. The sides KL and MN of the section quadrilateral are parallel to AB, and similarly LM and NK are parallel to CD. Thus the section is a parallelogram whose angles are equal to the angles of the skew lines AB and CD.

Now, if we draw through LM a plane parallel to the face ACD, cutting AB at P, then the tetrahedra $ABCD$ and $PBML$ are similar, and if $AB=CD$, then $BP=LM$. But $KL=AP$, hence $KL+LM=AP+PB=r$, and the perimeter of $KLMN$ is $2r$.

Also solved by E. P. Starke and C. W. Trigg.

E 91 [1934, 265]. *Proposed by Morgan Ward, California Institute of Technology.*

Let d be the greatest common divisor of the two positive integers, a and b, with $a=a'd$ and $b=b'd$. Now if n is any integer greater than unity, show that (n^a+1) and (n^b-1) can not have any common factor greater than 2 as long as b' is odd.

Solution by Hansraj Gupta, Hoshiarpur, India.

A common factor of (n^a+1) and (n^b-1) must also divide their sum, $(n^a+n^b)=n^b(n^{a-b}+1)$, and hence be a common factor between $(n^{a-b}+1)$ and (n^b-1). (It is here assumed that a is greater than b. If the reverse were true, $n^{b-a}+1$ would be used instead.)

Proceeding thus it can be easily shown that a factor common to (n^a+1) and (n^b-1) must also divide (n^d+1).

Now let $n^d=x$, so that $n^b-1=x^{b'}-1$. Also let b' be odd. Then, dividing n^b-1 by n^d+1 is the same as dividing $x^{b'}-1$ by $x+1$, and gives a remainder of -2. Hence (n^a+1) and (n^b-1) can not have any common factor greater than 2, and that only when n is odd.

Also solved by E. P. Starke and the proposer.

E 92 [1934, 265]. *Proposed by L. S. Johnston, University of Detroit.*

It is desired to make a rectangular box by cutting square corners out of a rectangular sheet of cardboard and turning up the sides and ends. If the depth of the box is to be a inches, find the dimensions of the sheet of cardboard such that the box made in the manner described is the largest rectangular box which can be made from the sheet. This is the converse of an elementary problem which appears in most calculus texts.

Solution by Roy MacKay, Eastern New Mexico Junior College.

Let x and y be the dimensions of the cardboard. Then the volume of the box constructed is $V=a(x-2a)\ (y-2a)$. For a maximum volume, the relationship which must hold between x and y is

$$dV/da = 12a^2 - 4a(x + y) + xy = 0$$

which, when solved for y, gives the equation

$$y = 4a(3a - x)/(4a - x)$$

which is true for any pair of values of x and y on the hyperbola represented by this equation.

While this equation is indeterminate, it is physically necessary that x and y each exceed $2a$, which consideration rules out the lower branch of the hyperbola. On the upper branch, x and y are each larger than $4a$. Now if we set $x=4a+b$, we find that $y=4a+c$, where b and c are any two positive numbers whose product is $4a^2$.

Also solved by W. E. Buker, J. E. Burnam, E. P. Starke, Simon Vatriquant, Maud Willey and the proposer.

Advanced Problems

Send all communications about Advanced Problems and Solutions to Otto Dunkel, Washington University, St. Louis, Mo. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide.

Problems containing results believed to be new or extensions of old results are especially sought. The editorial work would be greatly facilitated, if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known textbooks or results found in readily accessible sources, will not be proposed as problems for solution in this department. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

PROBLEMS FOR SOLUTION

3705. *Proposed by Raphael Robinson, University of California at Berkeley.*

Show that when the quadratic form

$$\sum_1^n |\, i - j \,|\, x_i x_j, \; n > 1,$$

is reduced to the sum of squares by a real linear transformation, one of the terms will be positive, the other $n-1$, negative. The determinant of the form has been proposed for evaluation in problem 3667 [1934, 193].

3706. *Proposed by G. R. Livingston, State Teachers College, San Diego.*

Using compasses only, find the points which divide a given circle into five equal arcs.

3707. *Proposed by Bernard Friedman, Brooklyn, New York.*

If p is a prime of the form $3n+1$, it can be expressed as

$$p = A^2 + 27B^2,$$

where A and B are positive integers, if and only if 2 is a cubic residue of p.

3708. *Proposed by Albert A. Bennett, Brown University.*

Let n denote an arbitrary given positive integer, p any rational prime, $q\neq 0$, any rational integer for which q^2-1 is not divisible by p. Show that the system

of square matrices of order n of the form given below constitutes a field, where the n independent arguments $x_1, x_2, \cdots, x_n$, of the generic matrix, range over the rational field. The element in the rth row and sth column is defined as x_{s-r} (for $r<s$), as px_{n-r+1} (for $s=1$), as $px_{n-r+s}+qx_{n-r+s-1}$ (for $r\geqq s>1$).

3709. *Proposed by E. B. Escott, Oak Park, Ill.*

Determine the values of A in the trinomial

$$x^{12}+Ax^6y^6+y^{12}$$

so that it will have two polynomial factors of the sixth degree with rational coefficients.

3710. *Proposed by Harry Langman, Brooklyn, N.Y.*

If the C's represent binomial coefficients, show that

$$\begin{vmatrix} C_2^2 & C_3^3 & C_4^4 & \cdots C_{n-1}^{n-1} & C_n^n & C_{n+1}^{n+1} \\ -(n-1) & C_2^3 & C_3^4 & \cdots C_{n-2}^{n-1} & C_{n-1}^n & C_n^{n+1} \\ 0 & -(n-2) & C_2^4 & \cdots C_{n-3}^{n-1} & C_{n-2}^n & C_{n-1}^{n+1} \\ 0 & 0 & -(n-3) & \cdots C_{n-4}^{n-1} & C_{n-3}^n & C_{n-2}^{n+1} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots -2 & C_2^n & C_3^{n+1} \\ 0 & 0 & 0 & \cdots \;\;0 & -1 & C_2^{n+1} \end{vmatrix} = (n!)^2.$$

SOLUTIONS

272 [1917, 427], *Proposed by C. C. Yen, Tangshan, North China.*

How many integers prime to n are there in each of the sets:

(*a*) $1\cdot 2,\quad 2\cdot 3,\quad 3\cdot 4, \cdots, n(n+1);$

(*b*) $1\cdot 2\cdot 3,\quad 2\cdot 3\cdot 4,\quad 3\cdot 4\cdot 5, \cdots, n(n+1)(n+2);$

(*c*) $\frac{1\cdot 2}{2}, \frac{2\cdot 3}{2}, \frac{3\cdot 4}{2}, \cdots, \frac{n(n+1)}{2};$

(*d*) $\frac{1\cdot 2\cdot 3}{6}, \frac{2\cdot 3\cdot 4}{6}, \frac{3\cdot 4\cdot 5}{6}, \cdots, \frac{n(n+1)(n+2)}{6}?$

Solution by E. P. Starke, Rutgers University.

This problem in exactly this form is given in *Theory of Numbers* by Carmichael (1914), page 36.

Let the numbers of set (*a*) be represented by

$$a_j, \quad j=1, 2, 3, \cdots, n.$$

The necessary and sufficient condition that a_j be divisible by a prime, p, is

$$j \equiv p - 1 \text{ or } p \bmod p.$$

Hence, if a_j is prime to p, we must have

$$j \equiv 1, 2, \cdots, p - 2 \bmod p.$$

Let n be represented as $p_1^{r_1}p_2^{r_2}p_3^{r_3} \cdots p_s^{r_s}$, where the p's are the distinct prime factors of n. If a_j is prime to n, j must satisfy a set of s congruences,

$$(1a) \qquad j \equiv c_i \bmod p_i, \; i = 1, 2, \cdots, s,$$

where c_i has a value selected from the set

$$1, 2, \cdots, p_i - 2.$$

There are $(p_1-2)(p_2-2) \cdots (p_s-2)$ distinct systems of congruences (1a), whose solutions, if less than n, give suitable values of j.

By the "Chinese Remainder Theorem" there exists for each such system a unique solution $j \leqq p_1p_2 \cdots p_s$. Hence there are in all $(p_1-2)(p_2-2) \cdots (p_s-2)$ distinct values of j, $1 \leqq j \leqq p_1p_2 \cdots p_s$, for which a_j is prime to n.

The numbers from 1 to n inclusive divide up into $n/p_1p_2 \cdots p_s$ sets such that each number in any set is congruent, mod $p_1p_2 \cdots p_s$, to one of the numbers from 1 to $p_1p_2 \cdots p_s$ and conversely. The total number of values of j, and hence the number of integers in set (a) prime to n, is then $(p_1-2)(p_2-2) \cdots (p_s-2)$ times $n/p_1p_2 \cdots p_s$, which reduces immediately to

$$n(1 - 2/p_1)(1 - 2/p_2) \cdots (1 - 2/p_s).$$

Following the same line of argument, we have for set (b) the condition that a_j be divisible by p is

$$j \equiv p - 2 \text{ or } p - 1 \text{ or } p \bmod p.$$

So then congruences (1a) become here

$$(1b) \qquad j \equiv c_i \bmod p_i, \; i = 1, 2, \cdots, s,$$

where c_i now has a value selected from the set $1, 2, \cdots, p-3$. Continuing as for set (a), we find the number of integers in the set (b) prime to n is

$$n(1 - 3/p_1)(1 - 3/p_2) \cdots (1 - 3/p_s).$$

The solution for set (c) when n is any odd number, is the same as for set (a). But if n is even, 2 will divide a_j if and only if $j \equiv 3$ or 4 mod 4. That is, for a_j to be prime to n, j must satisfy, besides congruences (1a), the following,

$$(1c) \qquad j \equiv 1 \text{ or } 2 \bmod 4.$$

Now suppose 4 is a factor of n. Then congruence $(1c)$ behaves with respect to n in the same way as the other congruences $(1a)$, so that the number of integers a_j in set (c) prime to n is given by

$$n(1 - 2/4)(1 - 2/p_1)(1 - 2/p_2) \cdots (1 - 2/p_s),$$

where $p_1, p_2, \cdots, p_s$ are the distinct odd prime factors of n.

But if $n=2m$, m odd, for a_j to be prime to n, j must be an integer less than $2p_1^{r_1}p_2^{r_2}\cdots p_s^{r_s}$ which satisfies the congruences (1a) and (1c). The Chinese Remainder Theorem gives us the number of values of j between 1 and $4p_1p_2\cdots p_s$ for which a_j is prime to n, but as $n=2p_1^{r_1}p_2^{r_2}\cdots p_s^{r_s}$ is not divisible by $4p_1p_2\cdots p_s$ the rest of the previous arguments cannot be followed here.

We may show, however, that the number of integers prime to $2m$ (for $1\leqq j\leqq 2m$) is the same as the number of integers prime to m (for $1\leqq j\leqq m$). Let us put

$$k = 2m - j - 1. \tag{2}$$

Relation (2) establishes a one-to-one correspondence between the set of subscripts $1\leqq j\leqq m-2$ and the set $m+1\leqq k\leqq 2m-2$. Also, since $a_k=2m^2-(2j+1)m+a_j$, we have a one-to-one correspondence between the integers in the two sets a_j and a_k which are prime to m. Let us now separate into two classes the integers prime to m in each set:

(A)	$j\equiv 1$ or $2 \bmod 4$	(A′)	$k\equiv 1$ or $2 \bmod 4$
(B)	$j\equiv 3$ or $4 \bmod 4$	(B′)	$k\equiv 3$ or $4 \bmod 4$.

To integers in class (A) correspond those in (B'); to the integers in (B) correspond those in (A'), since relation (2) implies $j+k\equiv 1 \bmod 4$. Thus the number of terms prime to $2m$ is the number of integers in (A) and (A′), which is the same as the number of integers in (A) and (B).

We have then the results for set (c):

$$\text{If } n \text{ is odd, } n(1-2/p_1)(1-2/p_2)\cdots(1-2/p_s);$$
$$\text{If } n \text{ is even, } n(1-2/p_1)(1-2/p_2)\cdots(1-2/p_s)/2,$$

where $p_1, p_2, \cdots, p_s$ are the distinct odd prime factors of n.

The solution for set (d) when n is any number prime to 6, is the same as for set (b). Since a_j is odd only when j is 1,5, 9, $\cdots$, it follows that when 2 is a factor of n, we must include with the congruences (1b) the following,

$$j \equiv 1 \bmod 4. \tag{1d}$$

Similarly 3 is not a divisor of a_j unless the numerator contains a multiple of 9. Hence if 3 is a factor of n we must include also the condition,

$$i \equiv 1, 2, \cdots, 6 \bmod 9. \tag{1d'}$$

The cases where n is divisible by 4 but not 3, by 9 but not 2, or by both 4 and 9, are easily disposed of by the same arguments used in earlier cases. We have, the number of integers in the set (d) prime to n is

$$n(1-3/4)(1-3/9)(1-3/p_1)(1-3/p_2)\cdots(1-3/p_s),$$

where only those factors are to be included which correspond to the distinct prime factors of n, in the three cases above.

Suppose however $n=2m$, or $3m$, or $6m$, where m is prime to 6. a_j will be prime to n if besides the congruences $(1b)$ j satisfies $(1d)$, or $(1d')$, or both, respectively. Unfortunately there seems to be no formula or simple set of formulae which will give the number of solutions for $1 \leqq j \leqq n$ of these congruences.

Formulae for certain special cases are simple enough to be of some interest. Let $\psi(n)$ represent the number of integers of set (d) which are prime to n.

I. By an extension of the method used under (c), we can show that $\psi(2m)=\frac{1}{2}\psi(m)$, where m is prime to 6.

II. $\psi(3p)$, where p is a prime greater than 3, will equal $2p-4$, $2p-5$, $2p-6$, $2p-7$ according as p is congruent, mod 9, to 8, 1 or 5, 2 or 4, 7 respectively.

A table follows, showing the values of $\psi(n)$ for the values of n from 1 to 109.

	0	1	2	3	4	5	6	7	8	9
0		1	1	3	1	2	2	4	2	6
1	1	8	2	10	2	5	4	14	3	16
2	2	7	4	20	4	10	5	18	4	26
3	2	28	8	16	7	8	6	34	8	20
4	4	38	3	40	8	12	10	44	8	28
5	5	30	10	50	9	16	8	33	13	56
6	5	58	14	24	16	20	8	64	14	41
7	4	68	12	70	17	19	16	32	10	76
8	8	54	19	80	6	28	20	52	16	86
9	6	40	20	56	22	32	16	94	14	48
10	10	98	13	100	20	17	25	104	18	106

Proofs for the two special cases above may be derived as follows.

I. Suppose a_{j_1} prime to $m, j_1<2m$. It is easy to verify that $a_{j_i}-a_{j_1}$, $i=2, 3, 4$, is divisible by m for $j_2=m-2-j_1$, $j_3=m+j_1$, $j_4=2m-2-j_1=m+j_2$; so that each a_{j_i} is also prime to m. Since m is odd, we see that one and only one of the $j_i \equiv 1$ mod 4. Call this one j_1. For every a_{j_1} there are three a_{j_i} for which $j \not\equiv 1$ mod 4. Hence the a_j separate into four sets containing $\frac{1}{2}\psi(m)$ integers each, and such that all integers in the first set satisfy $j \equiv 1$ mod 4. There are then $\frac{1}{2}\psi(m)$ integers a_j (for $j \leqq 2m$) prime to $2m$.

II. Place the numbers $1 \leqq j \leqq 3p$ in three rows of p columns each. Consider the values of j for which a_j is prime to $3p$. By $(1b)$ the last three columns give no such values of j. The additional values of j to be excluded by $(1d')$ are easily reckoned as soon as we know the residue of p mod 9.

Similar special results are easily obtained for $3p^2$, for $3p^3$, etc. The results for $3p_1p_2 \cdots$ depend upon the possible combinations of residues mod 9 of $p_1, p_2, \cdots$. The results for $6p$, $6p^2$, etc. depend upon the twelve possible residues of p mod 36. Results for $6p_1p_2 \cdots$ depend upon the possible combinations of residues mod 36 of $p_1, p_2, \cdots$.

A Note by the Editors. In Carmichael's text, from which this problem appears to be taken, two methods are given for treating a similar and simpler problem; one on pages 30–32 and the other on pages 33, 34. The second method does not seem to be easily applied to our problem, but the first is easily applicable. This is the method of residue classes which is frequently used in such questions, and it is employed in parts of the above solution. The method may be stated thus: Let $f(x)$ be a polynomial in x with integral coefficients, and let m be a positive integer. Denote by $\psi(m)$ the number of integral values of x, $1 \leqq x \leqq m$, for which $f(x)$ is prime to m. We then prove that

$$\psi(mn) = \psi(m)\psi(n),$$

where m and n are relatively prime. The integral values of $x \leqq mn$ for which $f(x)$ is prime to m may be separated into classes $C_0, C_1, \cdots, C_{n-1}$, where in C_k are placed all those x's such that $mk+1 \leqq x \leqq (k+1)m$. It is easily shown that a 1-1 correspondence may be established between the elements of C_0 and C_k,

$$x = x_i + km,$$

where x_i is the ith element in C_0 and the corresponding x is the ith element in C_k. From the ith elements in each class we easily find that we can select $\psi(n)$ elements for which $f(x)$ is also prime to n. This gives a new class D_i of elements for which $f(x)$ is prime to both m and n. Since there are $\psi(m)$ values of i, the theorem above follows.

For the case of $f(x) = x(x+1)(x+2)$ and $m = p^r$, where p is a prime not 2 or 3, we have classes C_k with the elements

$$kp + 1,\ kp + 2, \cdots, (k + 1)p - 3.$$

There are p^{r-1} such classes, and hence there are $p^{r-1}(p-3)$ positive integral values of $x < p^r$ for which $f(x)$ is prime to p^r.

3551 [1932, 240]. *Proposed by Eugene M. Berry, Lynchburg College.*

Find the equation of a curve whose evolute is the same as one of its involutes but is not the same as the original curve.

Solution by the Proposer.

Let
$$s = f(\tau)$$

be the intrinsic equation of a curve where s is the length of arc measured from the origin and τ is the angle the tangent to the curve at any point makes with the tangent to the curve at the origin. We always have that $\tau = 0$ when $s = 0$.

The evolute of the above curve is given by

$$s = f'(\tau)\Big]_0^\tau$$

and the involute by

$$s = \pm \int_0^\tau [f(\tau) + l] d\tau,$$

where l is an arbitrary constant. Since the evolute is the same as one of the involutes we have

$$f'(\tau)\Big]_0^\tau = \pm \int_0^\tau [f(\tau) + l] d\tau.$$

Differentiating and putting s in place of $f(\tau)$ we get

$$\frac{d^2 s}{d\tau^2} \pm (s + l) = 0. \tag{1}$$

Since $s=0$ when $\tau=0$ the solutions of (1) are of the following types:

$$s = c \sin \tau \quad \text{or} \quad s = l(\cos \tau - 1), \tag{2}$$

$$s = l(e^\tau - 1), \tag{3}$$

$$s = c \sinh \tau, \tag{4}$$

$$s = l(\cosh \tau - 1). \tag{5}$$

In the first part of (2) and in equation (4) $l=0$. The constants c and l may always be taken positive since $s=f(\tau)$ and $s=-f(\tau)$ represent identically the same curve. In fact a change in the value of c or l merely changes the scale for the curve. The general solutions of (1) do not give additional curves as the general solutions can be shown to be identical with these except for a change of origin.

Either equation in (2) represents an ordinary cycloid. Equation (3) is a logarithmic spiral. Neither of these is the curve wanted for each is congruent to its evolute. Equations (4) and (5) are the curves desired.

The radius of curvature R of (4) is given by $R=c \cosh \tau$ which shows it to be a spiral since R increases continuously with τ. If τ is small we have $s=c\tau$, a circle, as an approximation to the curve. If τ is large we get $s=c(e^\tau-1)$, a logarithmic spiral, as an approximation to the curve. Equation (5) can be shown to be a spiral which approaches a logarithmic spiral when τ is large.

By using the formulae

$$x = \int_0^\tau f'(\tau) \cos \tau d\tau$$

$$y = \int_0^\tau f'(\tau) \sin \tau d\tau$$

we can change equations (4) and (5) to rectangular coordinates. Equation (4) becomes

$$x = \frac{c}{2}(\cosh \tau \sin \tau + \sinh \tau \cos \tau)$$

$$y = \frac{c}{2}(1 + \sinh \tau \sin \tau - \cosh \tau \cos \tau)$$

and (5) becomes

$$x = \frac{c}{2}(\sinh \tau \sin \tau + \cosh \tau \cos \tau - 1)$$

$$y = \frac{c}{2}(\cosh \tau \sin \tau - \sinh \tau \cos \tau).$$

Generalization. If the problem had read, "Find the equation of a curve whose evolute and one involute differ only in size but are not the same shape as the original curve," equation (1) would have been

$$\frac{d^2s}{d\tau^2} \pm k^2(s + l) = 0.$$

The solutions would be

(6) $$s = c \sin k\tau \quad \text{or} \quad s = l(\cos k\tau - 1),$$

(7) $$s = l(e^{k\tau} - 1),$$

(8) $$s = c \sinh k\tau,$$

(9) $$s = l(\cosh k\tau - 1).$$

The curves in (6) are epicycloids or hypocycloids according as k is less than or greater than unity. Equation (7) is a logarithmic spiral. The curves in (8) and (9) have the same general characteristics as (4) and (5).

Note by the Editors. In this solution a correspondence is set up between the point P of the original curve, P_e of the evolute, and P_i of the involute, where the three points correspond to the same value of τ. The curve (P_e) is then required to be the same as (P_i) so that the two are congruent by corresponding points being made to coincide. A more general treatment would be obtained by considering the case of congruence in which corresponding points do not necessarily fall upon each other when the curves are made to coincide. A problem of this general character is as follows: Find the curve such that its evolute is the same curve. The method used above would give as a solution curves of the type (3) but not the cycloid. In order to obtain the latter solution corresponding points in the above sense must not be made congruent.

3586 [1932, 608]. *Proposed by R. E. Gaines, University of Richmond.*

If while an ellipse is turned about in its plane it remains tangent to a fixed straight line at a fixed point, its foci trace a curve whose area is $2\pi a(a-b)$.

II. *Solution by F. E. Relton, Royal School of Engineering, Giza, Egypt.*

(1) If a curve carries a tracing point P while it rolls on a straight line, the

area under P's path is twice the area of the curve's pedal with respect to P. For example let the rolling curve be a circle, and let P be on its circumference. The pedal of the circle with respect to P is a cardioid. When the circle rolls, the path of P is a cycloid. The proposition then states the known result that the area of the cycloid is twice the area of the cardioid. Similarly, let the rolling curve be an ellipse and the tracing point a focus. The pedal curve is then the major circle; hence the proposition states that, if an ellipse rolls on a straight line, each focus describes a curve whose area is twice that of the major circle. A proof of this by analytical methods would probably be difficult, though it might lead to some interesting integrals.

(2) The area between a curve and its pedal equals the area of the pedal of its evolute. For example, take again a circle of radius a and a point P on its circumference. At any point A of the circle draw the tangent AT and the normal AN. From P drop the perpendiculars PT, PN. The locus of T is the pedal of the circle, a cardioid of area $3\pi a^2/2$. The locus of N is a circle of radius $a/2$, described twice. The proposition then gives $\pi a^2+2\pi a^2/4$ as the area of the cardioid.

If the proposition be applied to an ellipse, with the point P at a focus (so that the pedal curve is the major circle) it gives $\pi a(a-b)$ as the area of the rather complicated curve obtained by dropping perpendiculars from the focus upon the normals. It is distinctly tedious to obtain this result analytically.

(3) If a curve slides in contact with a fixed straight line at a given point, the motion can be generated by rolling the curve's evolute on the perpendicular at the point.

I refrain from giving proofs of these propositions; indeed they need scarcely more than stating to be admitted. Thus in No. 2, AT and PN are always equal and parallel, so that they sweep out equal areas, which is the substance of the proposition.

The problem 3586 can now be generalized as follows:—*If any oval curve carries a tracing point P whilst it slides in contact with a given straight line at a fixed point, P describes a curve whose area is twice the difference between the area of the oval and the area of its pedal with respect to P.*

A Note by the Editors. These beautiful theorems, which are ingeniously combined into a proof of a generalization of the problem, were sent as an answer to the last sentence of the note to the first solution [1933, 565]. The following extract from the note of the author to the Editor may interest readers: "You may be interested to know that your excellent little periodical penetrates into the land of the Pharaohs, the birthplace of Mathematics, where it is much appreciated."

Proofs will be given for the benefit of those to whom the theorems are not obvious. Let P be a point fixed in position with respect to a given curve C and in its plane; Q, the foot of the perpendicular from P to the tangent at T to C; T', a point on C near T; and Q', the pedal point on the tangent $T'Q'$. The ele-

ment of polar area of the pedal curve (Q) is the area of the triangle PQQ'. If the curve rolls on TQ so that T' and Q' fall at T_1' and Q_1' on TQ, then P is carried by C to P', a point on the perpendicular to TQ at Q_1'. The element of rectangular area under the locus of P is the area of the rectangle with base QQ_1' and altitude QP. It will suffice to obtain theorem (1) to show that Q' lies on $Q_1'P'$, or that the projection of QQ' on TQ is QQ_1', disregarding terms of higher order than the first. Considering C as stationary and P as the origin of vectors, and setting $PT=\mathbf{r}$, we have $PQ=\bar{\mathbf{r}}$, $TQ=l\mathbf{t}$, where $\mathbf{t}$ is the unit vector tangent at T in the direction of increasing arc length s of C, we have

$$\bar{\mathbf{r}} = \mathbf{r} + l\mathbf{t}, \tag{1}$$

$$d\bar{s}\bar{\mathbf{t}} = ds\mathbf{t} + dl\mathbf{t} + lKds\mathbf{n}, \tag{2}$$

where $\bar{\mathbf{t}}$ is the unit vector tangent at Q to curve (Q), $\bar{s}$ is the arc length on (Q), K, the curvature of C, and $\mathbf{n}$, the unit vector normal at T. Taking the scalar product of (2) and $\mathbf{t}$, we have

$$d\bar{s}\bar{\mathbf{t}}\cdot\mathbf{t} = ds + dl. \tag{3}$$

Then $QQ_1' = (ds+l+dl)-l = d\bar{s}\bar{\mathbf{t}}\cdot\mathbf{t}$, and this completes the proof.

A direct proof of theorem (3) may now be given. If instead of rolling, C slides so that it is tangent at T to TQ, then when T' is at T, Q' is at Q_2' on TQ, and the projection on TQ of the new position of P, P', is Q_2'. The element of rectangular area under the locus of P is the area of the rectangle with base QQ_2' and altitude QP; the triangles PTT' and PQQ' are elements of polar area for C and (Q). Writing (1) in the form

$$d\bar{\mathbf{r}} = d\mathbf{r} + \mathbf{t}dl + ld\mathbf{t},$$

we have $\bar{\mathbf{r}}\cdot\mathbf{t}=0$, $\mathbf{t}\times d\mathbf{r}=0$, $\mathbf{t}\cdot d\mathbf{t}=0$, $\bar{\mathbf{r}}\times d\mathbf{t}=0$. Hence

$$\begin{aligned} \bar{\mathbf{r}} \times d\bar{\mathbf{r}} &= (\mathbf{r} + l\mathbf{t}) \times d\mathbf{r} + (\mathbf{r} + l\mathbf{t}) \times \mathbf{t}dl + \bar{\mathbf{r}} \times ld\mathbf{t}. \\ &= \mathbf{r} \times d\mathbf{r} + \mathbf{r} \times \mathbf{t}dl \end{aligned} \tag{4}$$

Since $QQ_2' = dl$, this is an equation in elementary vector areas which says that

$$2(\text{area } PQQ' - \text{area } PTT') = QP\cdot dl$$

disregarding terms of higher order than the first.

NEWS AND NOTICES

Readers are invited to contribute to the general interest of this department by sending items to Professor J. H. Weaver, Ohio State University, Columbus, Ohio.

A new type of computing machine called the Trianalyst, has been invented by B. R. Wellington of Troy, N.Y. It consists essentially of a rigid right triangle with graduated sliding scales in each of the legs; a movable hypotenuse pivoted at one end on an indicator which slides along one of the legs of the triangle; and a square with its edges perpendicular to the legs of the triangle and

with its vertex sliding along the bisector of the right angle of the triangle. By means of this instrument one may add, subtract, multiply or divide numbers, raise numbers to powers or extract roots of numbers, find logarithms of numbers, find the trigonometric functions of an angle, solve quadratic, cubic and biquadratic equations in one unknown, and certain simultaneous equations in two unknowns.

The Board of Trustees of the University of Illinois has made an appropriation to the Graduate School for the publication of the first volume of the works of G. A. Miller, who retired as professor of mathematics in 1931. A committee has been appointed to collaborate with Professor Miller in preparing the manuscript and seeing it through the press. The first volume is expected to appear early in 1935. It will contain publications prior to 1900 and will contain about five hundred pages. It is estimated that six volumes with approximately five hundred pages each will be required for the collected works.

Professor Richard Courant, formerly director of the Mathematics Institute of the University of Göttingen, has been appointed visiting professor of mathematics at New York University.

Professor Gregory Breit, of New York University, has been appointed professor of physics at the University of Wisconsin.

At the University of Nebraska Professor W. C. Brenke has been made chairman of the Department of Mathematics. The retiring chairman, Professor A. L. Candy, will continue his teaching in the department.

H. B. Huntley, of Rutgers University, has been appointed dean of the Union County Junior College, Roselle, N.J.

T. L. Koehler has been promoted to an assistant professorship at Muhlenberg College.

Assistant Professor N. H. McCoy has been promoted to an associate professorship at Smith College.

Associate Professor V. S. Mallory has been promoted to a professorship of mathematics at the New Jersey State Teachers College, Montclair.

J. D. Mancill has been promoted to an assistant professorship at the University of Alabama.

Dr. L. C. Mathewson has been promoted to a professorship of mathematics at Dartmouth College.

Dr. A. F. Moursund has been promoted to an assistant professorship at the University of Oregon.

Dr. F. W. Perkins has been promoted to an assistant professorship at Dartmouth College.

Associate Professor Evelyn Carroll Rusk has been promoted to a professorship of mathematics at Wells College.

Dr. A. E. Staniland, of the University of Pittsburgh, has been promoted to an assistant professorship.

Dr. W. J. Trjitzinsky, of Northwestern University, has been appointed to an assistant professorship at the University of Illinois.

Assistant Professor Morgan Ward, of the California Institute of Technology, has been appointed to a professorship at the Institute for Advanced Study.

Professor R. L. Wilder has been appointed associate professor of mathematics at the Johns Hopkins University beginning September 1935.

Assistant Professor J. H. Zant, of the Oklahoma A and M College, has been promoted to an associate professorship.

The following appointments to instructorships in mathematics are announced:

University of Alabama: J. P. Gill
Brooklyn College: Walter Prenowitz
University of California (Berkeley), Dr. R. D. James
Harvard University: Dr. M. R. Hestenes and Dr. Saunders MacLane (Peirce instructors).
Johns Hopkins University: Dr. E. K. Haviland.
Princeton University: A. H. Taub.
University of South Dakota: W. E. Ekman.
Wayne University: Dr. C. H. Fischer
Wellesley College: Dr. Ruth G. Mason.
University of Wisconsin: Henry Scheffé.

Dr. Nathan Jacobson and Dr. J. B. Rosser have been awarded Proctor fellowships at Princeton University.

George Cary Comstock, director of the Washburn Observatory, died May 11, 1934. At the beginning of his career he served as professor of mathematics at the Ohio State University for two years (1885–87).

Professor W. H. Echols, professor of mathematics at the University of Virginia since 1891, died at his home in his seventy-fifth year on September 25, 1934. He was engaged in teaching until the time of his death. He was a charter member of the Mathematical Association.

John E. Stocker, associate professor of mathematics at Lehigh University, died July 5, 1934, at the age of sixty years.

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DIRECTORY

EDITORIAL CORRESPONDENCE should be addressed to the EDITOR-IN-CHIEF, W. B. CARVER, White Hall, Cornell University, Ithaca, N.Y.

BOOKS FOR REVIEW should be addressed to R. A. JOHNSON, Brooklyn College, 66 Court Street, Brooklyn, N.Y.

BUSINESS CORRESPONDENCE should be addressed to the SECRETARY-TREASURER of the Association, W. D. CAIRNS, 33 Peters Hall, Oberlin, Ohio.

CHANGE OF ADDRESS: Members should send notice of any change of address to the SECRETARY-TREASURER, W. D. CAIRNS, Oberlin, Ohio, before the 10th of each month.

MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Nineteenth Annual Meeting of the Association, Pittsburgh, Pa., Dec. 29, 1934—Jan. 1, 1935

The following is a list of the sections of the Association, with dates of those section meetings which have been scheduled for 1934 and reported to the Secretary.

ALLEGHENY MOUNTAIN, Pittsburgh, Pa., Feb. 10; Washington, Pa., May 5.
ILLINOIS, Jacksonville, May 4-5.
INDIANA, La Fayette, May 11-12.
IOWA, Des Moines, April 20-21.
KANSAS, Topeka, Mar. 17.
KENTUCKY, May.
LOUISIANA-MISSISSIPPI, Jackson, Miss., Mar. 23-24.
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Baltimore, Md., Dec. 8.
MICHIGAN, Ann Arbor, Mar. 17.
MINNESOTA, Northfield, May 12.
MISSOURI.
NEBRASKA, Crete, Apr. 27.
OHIO, Columbus, Apr. 5.
OKLAHOMA, Oklahoma City, Feb. 9.
PHILADELPHIA, Philadelphia, Dec. 1.
ROCKY MOUNTAIN, Colorado Springs, Apr. 20-21.
SOUTHEASTERN, University, Ala., Mar. 30-31.
SOUTHERN CALIFORNIA, Riverside, Mar. 3.
TEXAS, College Station, May 5.
WISCONSIN, Oshkosh, May 5.

AFFILIATED ORGANIZATIONS: THE NEW ENGLAND ASSOCIATION OF TEACHERS OF MATHEMATICS, THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS.

GEORGE BANTA PUBLISHING COMPANY, MENASHA, WISCONSIN

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THE AMERICAN MATHEMATICAL MONTHLY

By R. G. SANGER, The University of Chicago

PAPERS, REPORTS OF MEETINGS

QUESTIONS, DISCUSSIONS AND NOTES

RECENT PUBLICATIONS—NEW BOOKS RECEIVED

RECENT PUBLICATIONS—REVIEWS

MATHEMATICAL CLUBS—TOPICS

MATHEMATICAL CLUBS—ACTIVITIES

(Including Chapters of the Pi Mu Epsilon Mathematical Fraternity and other similar organizations).

PROBLEMS—AUTHORS

Numbers refer to pages, black-face type indicating a problem solved and solution published; italics, a problem solved but solution not published; ordinary type, a problem proposed.

PROBLEMS—SOLUTIONS

Numbers in black-face type refer to problems, those in light face to pages.

NOTES AND NEWS

PERSONAL MENTION

This section contains the names of members attending meetings, persons taking any active part in meetings, newly elected members, officers of the Association and of the various Sections, and persons mentioned in the department of News and Notices.

NECROLOGY

THE AMERICAN MATHEMATICAL MONTHLY

THE OFFICIAL JOURNAL OF THE
MATHEMATICAL ASSOCIATION OF AMERICA
(INCORPORATED)

DEVOTED TO THE INTERESTS OF COLLEGIATE MATHEMATICS

THE AMERICAN MATHEMATICAL MONTHLY, FOUNDED IN 1894 BY BENJAMIN F. FINKEL, WAS PUBLISHED BY HIM UNTIL 1913. FROM 1913 TO 1916 IT WAS OWNED AND PUBLISHED BY REPRESENTATIVES OF FOURTEEN UNIVERSITIES AND COLLEGES IN THE MIDDLE WEST

VOLUME XLI, 1934

NUMBER 10, DECEMBER

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THE ELEVENTH MEETING OF THE INDIANA SECTION

The eleventh meeting of the Indiana Section of the Mathematical Association of America was held on Friday and Saturday, May 11–12, 1934, at Purdue University, Lafayette, Indiana.

There were forty-one registered for the meetings on Saturday including the following twenty-seven members of the Association: Stanley Bolks, J. H. Butchart, H. T. Davis, C. S. Doan, J. E. Dotterer, W. E. Edington, C. H. Frick, E. D. Grant, G. H. Graves, W. R. Hardman, C. T. Hazard, F. H. Hodge, H. K. Hughes, E. L. Klinger, Juna M. Lutz, William Marshall, T. E. Mason, Gertrude I. McCain, Gladys (Banes) McColgin, H. A. Meyer, G. T. Miller, J. A. Reising, C. K. Robbins, L. S. Shively, R. B. Stone, Anna K. Suter, K. P. Williams.

On Friday evening a joint dinner meeting was held with the Purdue chapter of Sigma XI and the Indiana Association of College Physics Teachers. Following the dinner there was a public address by Professor R. D. Carmichael of the University of Illinois on the subject "Consonance of thought and things." Professor Carmichael's paper contains a contribution to a problem in the philosophy of science, namely, the problem of the extent of agreement which exists between the relations connecting phenomena in experience on the one hand, and on the other the relations connecting the elements of thought when presented in deductive form.

The sessions on Saturday were presided over by the retiring chairman, Professor Juna M. Lutz. At the business session the following officers were elected: Chairman, Professor T. E. Mason, Purdue University; Vice-Chairman, Professor H. A. Meyer, Hanover College; Secretary, Professor P. D. Edwards, Ball State Teachers College.

At the afternoon session a resolution was passed expressing the sorrow of the members because of the death of Professor E. N. Johnson of Butler University. Professor Johnson was one of the charter members of the Indiana Section and served as the Chairman in 1927. A second resolution expressed the appreciation of the members of the Section of the courtesy extended them by the members of the staff of Purdue University and to Professor Carmichael and to Professor Roys for their addresses.

The following papers were presented at the Saturday sessions.

1. "The need for mathematical training in the biological and social sciences" by the retiring chairman, Professor Juna M. Lutz, Butler University.

2. "An engineer looks at mathematics" by Professor Carl S. Roys, Department of Electrical Engineering, Purdue University, by invitation of the Program Committee.

3. "The bearing of college mathematics on the teaching of secondary mathematics" by Professor K. P. Williams, Indiana University.

4. "A vacation among mathematicians" by Professor T. E. Mason, Purdue University.

5. "An experiment with student criticism" by Professor G. H. Graves, Purdue University.

6. "Properties of the polygamma function" by Professor H. T. Davis, Indiana University.

7. "On the asymptotic development of analytic functions" by F. C. Smith, Fort Wayne, by invitation.

8. "Helices in Euclidean n-space" by Dr. J. H. Butchart, Butler University.

9. "Ghost waves and negative energy" by A. W. Pershing, Indiana University, by invitation.

Abstracts of the papers follow, the numbers corresponding to the numbers of the papers:

1. In the opinion of the mathematician, there are valuable applications of his theory in all fields. To what extent do others share his view? In particular, many examples in the biological and social sciences can be found where quantitative methods indicate underlying laws which were uncertain or obscure when the qualitative method of description only was used. Some specialists in these fields are convinced that mathematical training is desirable or even essential, but they seem to be in the minority. The result of a questionnaire which was sent to the heads of the departments of biology and social science in a number of the larger universities indicates that there is still no general need felt for such training. Few schools require any collegiate mathematics of their majors in these subjects, and graduate students are usually advised to take such courses only when their thesis work requires it. Some have advocated special mathematical courses in preparation for work in these sciences. Which topics should be included in these courses, which ones omitted, and whether these brief courses would be sufficient to supplant the conventional ones are open questions. The responsibility of proving the practical value of his subject rests partly upon the college teacher of mathematics. It is his duty to discover as many of the applications of his theory as possible and to show his students and associates in other fields that it is a necessary and vital factor in the advancement of all scientific knowledge.

2. By way of introduction, Professor Roys brought out the fact that the theory of Electrical Engineering has always been characterized by the gradual reduction of the older phases of the subject to an exact science, together with a rapid expansion in the field of application. This has resulted in a larger proportion of the electrical engineers requiring a training today that is at once more extensive and analytical than has been required in the past. The part played by the mathematician in training the engineer, together with his development of new operations many years in advance of possible applications to material problems was discussed.

Mathematics was treated as a "tool," "a system of shorthand notation," and as a research method following the lines of either inductive or deductive

reasoning. Many illustrations from engineering practice and teaching experience were cited that show the advantages of the analytical method as well as the possibilities of arriving at erroneous results if the work is too completely divorced from practical considerations. The ability to consider on paper the effect of varying a single factor at a time, in contrast to the futility, in many cases, of even attempting such a procedure in the laboratory, was especially emphasized.

In conclusion, a number of research topics were suggested whose solutions would require not simply a close cooperation between an engineer and a mathematician, but rather the combined efforts of an engineer-mathematician and a mathematician-engineer. This led to an advancement of the idea that a major line of study should be offered in engineering schools that would be known as Mathematical Engineering, corresponding to the present Mathematical Physics in schools of science.

3. The importance of an active interest on the part of the mathematicians of the state in the mathematics of the secondary schools was discussed and a committee is to be appointed to formulate plans for the Indiana Section of the Association to take the initiative in stimulating greater interest in mathematics.

4. Professor Mason related some incidents of his stay at Cambridge University and his attendance at the International Congress of Mathematicians at Zurich.

5. For the past nine years, Professor Graves has made a practice of asking students to criticize, under conditions insuring anonymity, the course they have had. This is in the belief that teachers need to know what students are thinking about their courses and to impress upon students the fact that their relation to their own education is much broader than merely following a prescribed series of lessons.

Since the experiment started, Professor Graves' method of administering cources has been extensively modified.

6. In this paper the author discussed the properties of the polygamma functions, namely the nth derivatives of $\psi(x)=d \log \Gamma(x)/dx$. Special attention was paid to the asymptotic expansion of the zeros of these functions on the negative real axis. The theorem of Gauss for the computation of values of $\psi(x)$ at rational points in the interval $(0,1)$ was extended to the polygamma functions. Announcement was made that the statistics laboratory of Indiana University has completed tables of the trigamma, tetragamma, pentagamma, and hexagamma functions from $x=-10$ to $x=100$ at intervals from .01 to .1 to 10 and 15 significant figures, eight thousand values in all. This computation required auxiliary tables from 16 to 18 significant figures of $1/x^n$, $n=2, 3, 4, 5$, for the first thousand integers. Work on the pentagamma and hexagamma tables was largely carried out by E. B. Morris and Lucy C. Kantz.

7. During the years 1900–1908, E. W. Barnes obtained the asymptotic developments of a large number of function types by means of highly specialized

methods. In more recent years, general methods of determining such asymptotic expansions have been developed by W. B. Ford and C. V. Newsom. In this paper F. C. Smith used the general theory in considering the asymptotic behavior of the following function types:

$$(1)\qquad f(z) = \sum_{n=0}^{\infty} \frac{z^n}{(n+\theta)^\beta}; \qquad\qquad (2)\qquad f(z) = \sum_{n=0}^{\infty} \frac{h(n)z^n}{\Gamma(n+P)}.$$

8. A necessary and sufficient condition for a curve in three dimensions to be a helix is that the ratio of the first and second curvatures be constant. This condition is generalized for a curve in n-space and the author then takes up some properties of helices and associated curves termed pseudo-helices.

9. Mr. Pershing discussed the mathematical theory and physical interpretation of deBroglie waves. He showed the connection between Fermat's principle in the classical optics and the principle of least action. He indicated that the maximum velocity c in free space as postulated in relativity is true only for the motion of real energy packets. A simple derivation of the Bohr atom from the deBroglie equation was given and the relativistic formulation of the Schrödinger hydrogen atom was converted to the Bohr form by a simple reduction to the equation of matter waves. His correlation of the orbital frequency of an electron, material wave frequency, and radiated frequency was described. The speaker discussed the quantization of the space time continuum and made application of the Fermi Dirac statistics to negative energy states. Holes in space were interpreted as real entities with apparent masses, and the extension made of ghost waves to virtual and negative masses.

P. D. EDWARDS, *Secretary*

THE ANNUAL MEETING OF THE MINNESOTA SECTION

The annual meeting of the Minnesota Section of the Mathematical Association of America was held at St. Olaf College, Northfield, Minnesota, on Saturday, May 12, 1934. Sessions were held at 11:00 o'clock and at 2:15 o'clock with a luncheon at 12:45 o'clock.

Professor C. S. Carlson, chairman of the Section, presided at the two sessions, except when Professor R. W. Brink relieved the chairman during the presentation of his own paper. Seventy-five persons attended the meeting including the following thirty members of the Association: Sister Mary Aloysius, C. J. Blackall, Jessie W. Boyce, R. W. Brink, L. E. Bush, W. H. Bussey, C. S. Carlson, L. L. Cruise, H. H. Dalaker, Margaret C. Eide, C. H. Fischer, Gladys Gibbens, C. H. Gingrich, Borghild Gunstad, W. L. Hart, H. E. Hartig, E. Marie Hove, Dunham Jackson, C. M. Jensen, W. H. Kirchner, Marie M. Ness, M. G. Scherberg, A. J. Strane, F. J. Taylor, Ella Thorp, A. L. Underhill, O. E. Walder, Marion B. White, Marian A. Wilder, G. L. Winkelmann; and Sister Thomas à Kempis, institutional member representative.

At the afternoon session a vote of thanks was adopted as a sign of appreciation of the cordial hospitality of St. Olaf College, and the efforts of its depart-

ment of mathematics. Officers for the following year were elected as follows: Chairman, Sister Thomas à' Kempis, College of St. Teresa; Secretary, A. L. Underhill, University of Minnesota; Members of the Executive Committee: H. E. Hartig, University of Minnesota; A. J. Strane, Duluth Junior College; L. E. Bush, College of St. Thomas, St. Paul.

The following nine papers were presented:

1. "On the theorem of Coriolis" by Dr. M. G. Scherberg, University of Minnesota.
2. "Curriculum revision at St. Olaf College" by Professor C. S. Carlson, St. Olaf College, Northfield.
3. "Nomographic methods applied to solution of problems" by Professor W. H. Kirchner, University of Minnesota.
4. "Insurance work as an outlet for the mathematically trained undergraduate" by Dr. C. H. Fischer, Northwestern National Life Insurance Company, Minneapolis.
5. "A system of orthogonal polynomials" by Mrs. Olive Rose Gardner, Minneapolis, by invitation.
6. "Orthogonal polynomials" by Professor Dunham Jackson, University of Minnesota.
7. "The walking polyglot (Maria Gaetana Agnesi)" by Sister Thomas a Kempis, College of St. Teresa.
8. "Mathematical trends in the Arts College of the University of Minnesota" by Professor William L. Hart, University of Minnesota.
9. "Some remarks on Rolle's Theorem and the generalized law of mean" by Professor H. H. Dalaker, University of Minnesota.

Abstracts of some of these papers follow, the numbers corresponding to the list of titles:

1. Many problems in the kinematics of points in plane motion arise out of the fact that the data describing the motion is distributed over two or more systems of reference. Where there are two systems of reference having a common origin the theorem of Coriolis enables one to translate acceleration data from one to the other of the systems.

The theorem may be demonstrated by a superposition of motions which shifts one from the point of view of an observer on one of the systems of reference to that on the other. This method gives one an insight into the elements of the theorem and suggests a simple method of applying it.

2. A study was made of the mathematics curriculum at St. Olaf College and an attempt was made to provide for the needs of five groups of students, i.e.: (1) the student majoring in mathematics, with graduate work in mind; (2) the student majoring in mathematics, seeking only a minimum major towards graduation; (3) the student majoring in the physical sciences, who desires a major in mathematics as a supporting subject; (4) the student majoring in

mathematics with secondary school teaching in view; (5) the group of students which for want of better name is called the general reader.

3. The graphic solution of simultaneous linear equations was shown by the use of point coordinates, line coordinates, a projective transformation involving the use of two radial pencils, and an affine construction. A nomogram constructed from the determinant form of a quadratic equation was used to show that any one of the conics could be used for the determination of the roots. The transformation of one figure to another and the methods of graduating the curves were reviewed briefly.

5. This was a brief discussion of the system of polynomials orthogonal over the interval $(-1, 1)$ with respect to the weight function $(1-x)$.

6. If $p_0(x)$, $p_1(x)$, $p_2(x)$, $\cdots$ with $p_k(x)=a_kx^k+b_kx^{k-1}+\cdots$ are the normalized orthogonal polynomials corresponding to a weight function $\rho(x)$ in the interval $(-1, 1)$, it is well known from a fundamental memoir of Darboux that any three successive polynomials of the set satisfy a recurrent relation of the form $xp_n(x)=A_np_{n+1}(x)+B_np_n(x)+C_np_{n-1}(x)$, where A_n and C_n are expressible in terms of the coefficients a_{n+1}, a_n, and a_{n-1}. It is pointed out in this note that B_n, having the value $(b_n/a_n)-(b_{n+1}/a_{n+1})$, is in particular equal to zero if $\rho(x)$ is an even function.

7. The paper on Maria Gaetana Agnesi by Sister Thomas à Kempis will be published in full in an early number of *Scripta Mathematica.*

8. Professor Hart discussed data concerning the registration in courses in mathematics in the College of Science, Literature, and the Arts at the University of Minnesota. On the basis of 1923–24 as a norm, the registration for 1933–34 in advance of sophomore calculus has increased by 115 percent; the total student load in college mathematics in advance of $2\frac{1}{2}$ units of high school mathematics has increased by 25 percent; the student load in the college as a whole is about the same as in 1922–23. In 1932–33, as contrasted with 1923–24, a much larger proportion of entering freshmen presented more than $2\frac{1}{2}$ units of high school mathematics. Professor Hart called attention to local conditions which may be causing the desirable features of these changes.

9. Professor Dalaker showed how to derive Cauchy's Generalized Law of Mean Value by setting up a linear function $\phi(x)$ of the two given functions $f(x)$ and $g(x)$ and applying to ϕ Rolle's Theorem, assuming that $\phi(a)=\phi(b)$ in the interval $a\leqq x\leqq b$.

A. L. UNDERHILL, *Secretary*

COLLEGIATE TRAINING IN MATHEMATICAL STATISTICS

The following statement, drafted by a special subcommittee, has been unanimously approved by the Committee on Research of the American Statistical Association:

The Committee endorses the proposition that colleges and universities should be urged to provide courses in statistical theory and technique under the direction of mathematically trained teachers.

The Committee recognizes the value of statistics courses in which mathematics plays a minor part. The field of statistics is many-sided, and mathematical theory represents only one aspect of the field; but in the opinion of the Committee it is an aspect of such importance that it is desirable to provide more adequate training in our colleges than has heretofore been available.

An important reason for this attitude is that it is inadvisable for one to deal with certain mathematical methods and formulae which are already in fairly common use, unless one understands fully the underlying conditions on which they rest. Many of these conditions are mathematical in nature and certainly cannot be presented to students who have not already had a course in analytic geometry; they can be presented much more easily to students who have also studied the calculus.

The subcommittee which drafted the report on which the above statement is based was composed of the following persons: B. H. Camp (chairman), Harold Hotelling, H. L. Rietz, W. Shewhart, and E. B. Wilson. The Committee on Research includes representation from the principal subject-matter fields.

THE PLACE OF RIGOR IN MATHEMATICS[1]

By E. T. BELL, California Institute of Technology

I. *A Mathematical Truth*

As the majority of the members of this Association are vitally interested in the presentation of college mathematics, I decided that a discussion of some topic directly related to mathematical education would be more acceptable than a technical paper on research. Accordingly I chose the question of rigor in mathematics, for that, it seems to me, is the one question in mathematical education which today is of the first importance to both the sanguine educators and the would-be educated.

There is not much to say. For that very reason, by the fundamental law of all public speaking, I shall probably take an interminable time to say it. As a matter of fact everything that it is both necessary and sufficient to say on the place of rigor in mathematics can be said by adding three words to the title. The three words are "is in mathematics." Thus we have the tautology, "The place of rigor in mathematics is in mathematics." This is all I have to say, al-

[1] Retiring presidential address presented at the meeting at Williamstown, Mass., Sept. 3. 1934.

age that will have any abiding significance for future generations of mathematicians. The brilliant analysis of our own day, as hard, as sharp and as clear as a well cut diamond, may be a beautiful thing to look at, but others before us did relatively as well, if not better. The new light and shade on *all* mathematical reasoning that our generation has seen is something that our predecessors did *not* see. Why ignore it all merely to be in fashion?

A NEW THEOREM CONCERNING THE RANK OF A MATRIX

By W. H. METZLER, New York State College for Teachers

Let $\alpha \equiv |\alpha_{1\lambda}|$, $\lambda = (n)_m$, be the mth compound of the determinant $A \equiv |a_{1n}|$, and let the groups of numbers $1, 2, \cdots, m-2$ and $1, 2, \cdots, m-1$ be represented by g_1 and g_2 respectively. Let $z_i \{ i = 1, 2, \cdots, (n-m+1) \}$ be the combinations of the numbers $m, m+1, \cdots, n$ taken two at a time.

THEOREM. *If*

$$M_i \equiv \begin{vmatrix} g_1 & x_i \\ g_1 & x_i \end{vmatrix} \neq 0, \quad x_i = m - 1, m + 1, m + 2, \cdots, n \tag{1}$$

$$\begin{vmatrix} g_2 & y_i \\ g_2 & y_i \end{vmatrix} = 0, \quad y_i = m, m + 1, \cdots, n \tag{2}$$

and

$$\begin{vmatrix} g_2 & z_i \\ g_2 & z_i \end{vmatrix} = 0, \quad i = 1, 2, \cdots, (n - m + 1)_2, \tag{3}$$

then $\alpha_{ij} = 0$ *for all values of i and j from* 1 to λ.

That is *if in a determinant A of order n all the coaxial minors of order* $(m-1)$, *except*

$$\begin{vmatrix} g_1 & m \\ g_1 & m \end{vmatrix},$$

of which

$$\begin{vmatrix} g_1 \\ g_1 \end{vmatrix}$$

is a first minor, are different from zero, and if all coaxial minors of orders m and $(m+1)$ *of which*

$$M_1 \equiv \begin{vmatrix} g_2 \\ g_2 \end{vmatrix}$$

is a minor vanish, then all minors of order m vanish.

PROOF. Let us observe in the first place that every minor of order $m+1$ of which any minor in (2) is a coaxial first minor is found in (3).

Considering the coaxial element

$$\alpha_{11} \equiv \begin{vmatrix} g_2 & m \\ g_2 & m \end{vmatrix}$$

in the first row of α, we observe it is zero by (2), and every element in this row which contains $M_1[\neq 0$ by (1)$]$ as a minor vanishes,[1] and hence all elements in this row vanish.[2] In general consider the coaxial element

$$\begin{vmatrix} g_2 & m+h \\ g_2 & m+h \end{vmatrix}$$

in the $(h+1)$th row of α, it is given zero by (2) and every element in this row which contains $M_{h+1}[\neq 0$ by (1)$]$ as a minor vanishes and hence all the elements in this row vanish.

We have now every element of α which contains M_1 as a minor vanishing, and therefore all elements of α vanish.[3]

A theorem akin to the one here considered was given by Kantor.[4]

ON A PROBLEM IN THE ELEMENTARY THEORY OF NUMBERS

By PAUL ERDÖS and PAUL TURÁN, Budapest, Hungary

1. The subject of this note is the following problem, proposed orally by G. Grünwald and D. Lázár. Let $p_1, p_2, \cdots, p_k$ be any prime numbers. We may say that N is *composed of* the primes $p_1, p_2, \cdots, p_k$ when every prime factor of N is one of these primes. Can we find an infinite set of different positive integers $a_1, a_2, \cdots$ so that every sum $a_i+a_j (i\neq j)$ is composed of $p_1, p_2, \cdots, p_k$? The answer that no such set exists was given by the proposers. Their proof depends on a theorem of Mr. Pólya asserting that if we denote by $q_1<q_2<\cdots<q_n<q_{n+1}<\cdots$ the numbers composed of the primes $p_1, p_2, \cdots, p_k$ then $q_{n+1}-q_n$ tends to infinity. But the proof of Pólya's theorem is not elementary; it seems therefore desirable to show the above result in an elementary way. On the other hand Pólya's theorem does not allow any further deductions in the following direction. Let $a_1, a_2, \cdots, a_n$ be a finite set of positive integers such that the sums a_i+a_j contain no prime factors other than $p_1, p_2, \cdots, p_k$; can we find an upper bound for the number n of such integers, depending on $p_1, p_2, \cdots, p_k$ or on k only? (Plainly we can suppose that $p_1=2$, because if the $p_1, p_2, \cdots, p_k$

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[1] *The Theory of Determinants.* By Muir and Metzler §§176 and 178.

[2] Muir and Metzler, loc. cit., §235.

[3] Muir and Metzler, loc. cit., §234.

[4] *Ein Theorem über Determinanten.* Nachrichten Ges. d. Wiss. (Göttingen) 1899. His theorem is: *If all coaxial minors of orders m and $m+1$ vanish, but none of those of order $(m-1)$ vanish, then all minors of order m vanish.*

are all odd, we find $n \leqq 2$. Indeed, otherwise at least one of a_1+a_2, a_1+a_3, a_2+a_3 would be even.)

We present an answer to the last question containing also the original problem. We show in an elementary way that $3\cdot 2^{k-1}-1$ is an upper bound for n, i.e.

Theorem I. The two-term sums formed of $3\cdot 2^{k-1}$ *positive integers cannot all be composed of k given prime numbers.*

From this we deduce as a corollary

Theorem II.

$$\pi(n) > \log_2\left(\frac{n}{3}\right)$$

where $\pi(n)$ denotes the number of primes $<n$.

The bound given in theorem I is probably not exact. The order of the maximum $n(k)$ of n belonging to a given number k of primes is probably[1]

$$n(k) = O(k^{1+\epsilon}) \text{ for any } \epsilon > 0$$

but actually we cannot prove this relation.

In the same way we may treat the analogous problem:

Is it possible to find two infinite sets of positive integers

$$a_1 < a_2 < \cdots$$
$$b_1 < b_2 < \cdots$$

so that every sum a_i+b_j shall be composed of the given primes $p_1, p_2, \cdots, p_k$? The answer is negative. The proof will show even more. We shall prove

Theorem III. The sums (a_i+b_j) formed of the two sets

$$a_1 < a_2 < \cdots < a_{k+1}$$
$$b_1 < b_2 < \cdots < b_\nu$$

cannot be composed of only k primes if one of the b's is greater than a_{k+1}^k. (This surely occurs if $\nu > a_{k+1}^k$.)

2. Before proving theorem I we shall prove the following

LEMMA: *Let $a_1<a_2<\cdots<a_n$ be a set of positive integers and $p>2$ a prime number. It is always possible to select out of this set at least[2] $\{n/2\}=N$ integers $a_{i_1}, a_{i_2}, \cdots, a_{i_N}$ with the following property: if a_{i_ν} is divisible exactly by p^{α_ν}, a_{i_μ} by p^{α_μ} and $a_{i_\nu}+a_{i_\mu}$ by $p^{\beta_{\mu\nu}}$, then*

[1] $f(x)=Og(x)$ means that there exists a B and an A such that for all $x \geqq B$ it is true that $|f(x)| < Ag(x)$; see Landau, *Primzahlen*, vol. 1, p. 31.

[2] The symbol $\{x\}$ denotes the smallest integer $\geqq x$.

$$\beta_{\mu\nu} = \min(\alpha_\mu, \alpha_\nu),$$

where min (α_μ, α_ν) *means the smaller of* α_μ *and* α_ν.

We divide every member of the set $a_1, a_2, \cdots, a_n$ by the highest possible power of p; thus we obtain the integers $a_1^1, a_2^1, \cdots, a_n^1$ (some of them being possibly equal). No member of this new set is divisible by p. We divide the members of this set into two classes according as their smallest positive residue, mod p, is less than or greater than $p/2$. At least one of these two classes must contain N of the a_ν^1. We retain only these; it is clear that the two-term sums formed of these are not divisible by p. The integers a corresponding to these a_ν^1 satisfy the requirement of our lemma. (The lemma is trivial except when some of the a's are divisible by the same power of p.)

3. We can now prove theorem I. Let $n=3\cdot 2^{k-1}$ and $a_1, a_2, \cdots, a_n$ be any positive integers. Suppose that all two-term sums of these are composed of k primes $p_1=2, p_2, \cdots, p_k$; we shall prove that this supposition leads to a contradiction.

We apply our lemma with $p=p_k$; we obtain then $3\cdot 2^{k-2}$ integers a_ν with the property in the lemma. Repeat the same process with $p=p_{k-1}$ upon this system of $3\cdot 2^{k-2}$ integers and so on. Finally we obtain three numbers a_1, a_2, a_3 of the same property with respect to the primes $p_2, p_3, \cdots, p_k$. Let

$$a_1 + a_2 = 2^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k} \tag{1}$$

$$a_1 + a_3 = 2^{\beta_1} p_2^{\beta_2} \cdots p_k^{\beta_k} \tag{2}$$

$$a_2 + a_3 = 2^{\gamma_1} p_2^{\gamma_2} \cdots p_k^{\gamma_k}; \tag{3}$$

then a_1 and a_2 are divisible by $p^{\alpha_2}, \cdots, p_k^{\alpha_k}$; therefore a_1 and a_2 cannot be divided by 2^{α_1}. Hence by (1) a_1 and a_2 must contain the same power of 2. This evidently holds for a_1 and a_3 also. Let us denote this common exponent by γ. Then dividing (1), (2) and (3) by 2^γ, and denoting $a_i/2^\gamma$ by b_i we have

$$b_1 + b_2 = 2^{\delta} p_2^{\alpha_2} \cdots p_k^{\alpha_k} \tag{4}$$

$$b_1 + b_3 = 2^{\epsilon} p_2^{\beta_2} \cdots p_k^{\beta_k} \tag{5}$$

$$b_2 + b_3 = 2^{\theta} p_2^{\gamma_2} \cdots p_k^{\gamma_k}. \tag{6}$$

Here b_1, b_2 and b_3 are odd and each member of the left side of (4) (5) and (6) is divisible by the odd prime-powers on the respective right side. Dividing (4) by $p_1^{\alpha_1}, \cdots, p_{k-1}^{\alpha_{k-1}}$ we get a number >2, for the members on the left side are *different* odd numbers. By this $\delta \geqq 2$ and by analogous reasoning $\epsilon \geqq 2$ and $\theta \geqq 2$. Thus from (4), (5) and (6) it follows that the two-term sums formed of three different odd numbers are all divisible by 4, which is impossible.

4. In order to obtain the inequality of theorem II, let $a_\nu = \nu$ for $\nu = 1, 2, \cdots, \{n/2\}$. Then the prime divisors of the sums $a_i + a_j$ are the primes $\leqq n$. Hence by theorem I, $n/2 < 3\cdot 2^{\pi(n)-1}$, from which we immediately obtain the inequality stated in the introduction.

5. Finally we will prove our theorem III. Let

$$a_1 < a_2 < \cdots < a_{k+1},$$
$$b_1 < b_2 < \cdots < b_\nu$$

be given integers, $b_\nu > a_{k+1}^k$ and suppose that the sums $a_i + b_l$ are all composed of k prime factors $p_1, p_2, \cdots, p_k$. Let us consider the sums

$$a_1 + b_\nu,\ a_2 + b_\nu,\ \cdots,\ a_{k+1} + b_\nu.$$

We next show that one of these $a_l + b_\nu$ contains a power of one of the given primes, say $p^{\alpha_l}_{i_l}$, so that

$$p_{i_l}^{\alpha_l} > a_{k+1} \qquad (l = 1, 2, \cdots, k+1).$$

This we deduce from the fact that $a_l + b_\nu > b_\nu > a_{k+1}^k$ and that $(a_l + b_\nu)$ can have only k different prime factors. We call this prime p_{i_l} (or if there are several, any one of them) "the prime belonging to a_l." We assert that the primes belonging to different a_l are different. For if the same p should belong to a_{l_1} and a_{l_2}, then $(a_{l_1} - a_{l_2})$ would be divisible by p^m, where m is the smaller of α_{l_1} and α_{l_2}; but according to what has been said before, $p^m > a_{k+1}$, whereas both of the numbers a_{l_1} and a_{l_2} are positive and $< a_{k+1}$. Since the same prime can not belong to two integers, it is impossible that k primes shall belong to $(k+1)$ integers. Hence the supposition that all the sums $a_i + b_l$ are composed of the k primes must be false.

STUDENT PLACEMENT IN SECONDARY MATHEMATICS[1]

By W. L. HART, University of Minnesota

In the course of the last twenty-five years, the enrollment in the high schools[2] of the United States has increased at practically a constant rate from 1,100,000 students in 1910 to approximately 6,000,000 students today. Thus, in contrast to earlier conditions, a larger percentage of the population is receiving instruction through the twelfth grade, and the resulting student body in the secondary schools presents a much wider range than before with respect to intelligence, cultural background, and future interests. It would be reasonable to agree that such conditions, particularly in the aggravated form they have assumed during the economic depression, might justify departures from the mathematical viewpoint which existed in the secondary field during the preceding generation. However, it is highly questionable whether some of the observed alterations with respect to mathematics in the high schools can be justified on the grounds of sound educational policy. The major changes which we shall consider debatable are due to two related attitudes which are widely prevalent in the secondary field in certain sections of the country. The first attitude presumes that

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[1] Presented to the Mathematics Section of the Minnesota Educational Association, Nov. 5, 1934.

[2] We include only grades nine to twelve under the term *high school* or *secondary school*.

mathematics beyond the ninth grade has practically no merit in a curriculum for students who do not intend to enter college.[1] The second and more recently developed attitude assumes that mathematics beyond the ninth grade is relatively unimportant also in the training of a high school student who will enter college but who will not aim at one of the so-called mathematical fields—engineering, mathematics itself, and the physical sciences. In the present paper we shall aim to indicate various evils arising from these attitudes. Also, we shall describe in general terms a different point of view which we believe would be appropriate for determining the proper emphasis to be given to mathematics in the high school training of any individual student. It is not claimed that the opinions and recommendations which will be presented are immune from objection or are particularly novel, but it is hoped that their formal reiteration may be of some advantage in connection with the problems in view.

A large part of our remarks will relate to the position of mathematics in a secondary curriculum for students who are preparing for college. As a background for this discussion, it is appropriate to recall certain facts about the position of mathematics in the college curriculum. It is well known that at one time most American colleges of the first rank required not only a large amount of high school mathematics for entrance but, also, a moderate amount of college mathematics for graduation, regardless of the major field of study involved. The diversification of the curricula of some colleges and the growth of the elective system created a tendency to eliminate the general requirement of college mathematics, although very properly this tendency has had no effect in those colleges where the mathematical nature of the curricula or a special educational viewpoint causes mathematics to assume unique importance for all students. In the extremely large number of colleges where there now exists no blanket requirement of mathematics for graduation, it is left for mathematics, as well as for most other subjects in the curricula, to attempt to justify itself objectively in the case of each student. In recent years, it has become increasingly easy to demonstrate this justification for college training in mathematics on both cultural and practical grounds in the cases of large numbers of students. For, at present we are witnessing an increased emphasis on the essential nature of advanced mathematics in the mathematical sciences and of at least elementary college mathematics as a tool for the development of many other disciplines not commonly referred to as mathematical fields. In particular, the tremendous growth in the use of statistical methods and other mathematical devices in the natural sciences, the social sciences, and education has led many students from these fields to study mathematics. As a result of the various tendencies toward expanded uses of mathematics, we believe that representative statistics would show that in many colleges the registration in courses in intermediate and advanced college mathematics is increasing relatively compared to the registration in the colleges as a whole. For example, in the College of Science, Literature, and

[1] The unqualified word *college* will refer to any institution granting a bachelors degree.

the Arts at the University of Minnesota the average registration in mathematics in advance of elementary calculus during the years 1930–34 was approximately twice as large as the similar average for the years 1922–26, whereas the average registration in this college during 1930–34 was approximately the same as in 1922–26. As a generalization, we consider it reasonable to say that today mathematics is in a stronger and healthier position on the grounds of its importance both culturally and practically as a part of the college curriculum than at any previous time during the development of American institutions of higher learning.

The validity of the preceding opinion would not be questioned by anyone so far as the remarks apply to the importance of mathematics in the physical sciences and the various branches of engineering. However, we might expect a non-mathematician to be more hesitant about accepting our opinion concerning the utility of mathematics in many of the fields not originally cultivated by use of mathematical devices. Interesting evidence to present in this connection is furnished by a committee report[1] on the "Collegiate Mathematics Needed in the Social Sciences," prepared for presentation to the Social Science Research Council in December, 1931. This committee report is particularly worthy of renewed notice in view of recent tendencies to make the social sciences exceptionally prominent in all secondary and college curricula. The committee consisted of H. R. Tolley, Holbrook Working, and Mordecai Ezekiel, economists, Charles H. Titus, a political scientist, and F. L. Griffin, a mathematician. The committee listed certain topics in college mathematics which might serve as a basis for a three hour course for two or three semesters, uniquely fitted for a student majoring in the social sciences, even though he might not intend to proceed with graduate work in his major field. The committee suitably qualified its statements so that there could be no inference that the specified three-hour course was recommended as a *requirement* for all students of the social sciences, or as the *maximum* amount of mathematics which might be useful to certain students. We take the liberty of quoting certain passages from this report.

". . . the Committee is of the opinion . . . that all students of the social sciences would find some work in collegiate mathematics very helpful."

"Some students specializing in the social sciences will need a considerably more extensive knowledge of mathematics than is obtainable from the foregoing course."[2]

"Training in mathematics is useful to those who take undergraduate courses in economic theory. . . . "

"Mathematics courses covering the above topics would be very helpful as a preliminary to a first course in statistics. . . . The statistics course could then utilize the usual amount of time to carry the student much farther in the knowl-

[1] Discussed by the chairman of the committee, Professor H. R. Tolley, at a meeting of the Mathematical Association of America, in September, 1932. Cf. this Monthly, vol. XXXIX, 1932, p. 503; pp. 569–577.

[2] Cf. this Monthly, loc. cit.

edge of statistical methods, their possibilities and limitations, or, the customary ground could be covered in a much shorter time."

"The Committee believes that at institutions where mathematics courses are available in which students can obtain a knowledge of the fundamental concepts of differential and integral calculus and the elements of probability, with the necessary preliminary training, in a total of six to nine semester hours, as in the special courses outlined, these courses should be made prerequisite to any course in statistics and to upper division courses in economic theory, and the courses in statistics and economic theory should be organized so as to take advantage of the knowledge gained in the mathematics courses. Perhaps certain other courses in economics, and in the other disciplines as well, would on further consideration be included in the group for which the mathematics courses should be prerequisite."

Even if the committee report just referred to were considered too enthusiastic concerning the usefulness of mathematics, we could state that this report gives ample support to the conservative claim that a strong foundation at least in secondary mathematics should be a prerequisite for major work in any statistically inclined field, in particular in the fields of education and psychology, as well as in important parts of other natural sciences and social sciences. We believe, moreover, that intelligent opinion from these fields supports our contention. It is hard to appreciate, for instance, how undergraduate students who are not presumed to have had mathematics beyond the ninth grade or, perhaps, the tenth grade, can be expected to obtain an intelligent comprehension of the theory and practice of correlation and probability in courses in statistics. Even some elementary facts about the most simple norms met in statistics are beyond the realm of proper description to a student whose algebraic training stopped in the ninth grade. To such a person the normal probability curve remains a piece of inexplicable hocus pocus and correlation coefficients are merely magical numbers with miraculous predictive powers which may justify all sorts of weird and unexpected conclusions. In short, a poor mathematical foundation for the students in the numerous undergraduate courses in statistics lays the basis for a distressing lack of the scientific attitude in future work by the students.

In view of the preceding remarks, we shall now make the very conservative assumption that well informed authorities in various fields of learning recognize the desirability of a good foundation at least in secondary mathematics and perhaps also in elementary college mathematics as a basis for the efficient presentation of undergraduate work in the sciences and in vital parts of many social sciences and other fields which rely on statistical methods. In the second place, we shall assume that two and one half years of secondary mathematics is the *minimum* preparation which gives an efficient basis for college training in mathematics, for the use of elementary algebra and geometry in outside fields, and for the appreciation of logical methods of proof. We stipulate two and one half years instead of two years of secondary mathematics as the minimum because, other conditions being equal, the student who completes a second course in

high school algebra takes long strides in advance with respect to the ability to use mathematics, as compared to the student who has taken only two years of secondary mathematics. As supporting evidence on this point, we may quote the following results concerning placement tests in mathematics given to certain freshmen in the College of Science, Literature, and the Arts at the University of Minnesota in 1926. The Training Test in Mathematics of the University of Iowa Series of placement tests was employed. The test was given to 313 students who had had two years, and to 281 students who had had two and one half years of secondary mathematics. It was found that only thirteen per cent of the first group obtained scores on the test which fell above the median score for the second group.

Under the previous assumptions, let us discuss some consequences of the decreased emphasis on mathematics which we are witnessing in many of the secondary schools. As a first problem, consider the resulting extreme situation which exists in a large number of junior colleges and normal schools and in some colleges, and which is threatened in various other colleges. The condition in mind is the omission of any recommendation or requirement that entering students should have taken mathematics beyond the ninth grade in high school. If our assumption about the utility of mathematics is justified, then in the collegiate institutions where we meet this extreme situation there will be a tendency to present vital parts of the curriculum on a low plane. We condemn such a situation even in junior colleges where there is no aim to prepare students for senior college work.

As a second situation, consider the effects of a reduced emphasis on secondary mathematics from the standpoint of a student with at least the minimum ability necessary for completing two and one half years of mathematics, who intends to enter a college where a first rate faculty does not hesitate to employ secondary mathematics in introductory courses and college mathematics in more advanced courses in the appropriate fields. If a high school adviser suggests or acquiesces in the elimination of mathematics beyond the ninth grade in the program of such a student, the adviser is admitting that, at the age of fifteen years, a child is qualified to rule himself out of (1) easy access to any college of engineering and (2) the efficient pursuit of any science, many social sciences, and any other field which employs modern statistical theory. Undeniably, this miraculous ability for the estimation of his future desires is not possessed by a child who is only fifteen years of age. Even though mathematics might appear hard to this student, his best interests would justify his continued study of the subject at least through part of the eleventh grade, at which time a more exact diagnosis of his future intellectual activities can be made. The taking of mathematics for one and one half years beyond the ninth grade is very cheap insurance against the strong possibility of future disappointments and handicaps in college which can result from insufficient mathematical training. This insurance appears particularly cheap when we contrast its cost and its value, even when college applications of mathematics are disregarded, with the

advantages of the material which might be substituted for any omitted mathematics. Moreover, it creates only a spurious saving of effort if the student with collegiate intentions economizes on his mathematical study in high school by omitting a second course in algebra after having studied tenth grade mathematics. If he enters college after such an omission and takes courses in a mathematical field, he will continue under a handicap with respect to the appropriately prepared students until he has remedied his situation by additional mathematical study in college but on a high school level.

In connection with our discussion about students who will enter college, it is pertinent to ask at what stage in a student's progress through high school can an accurate decision be made that he will *not* go to college? We believe that, particularly under present conditions, it would be very difficult to make the implied accurate diagnosis in the case of a student of reasonable intelligence at any time before his graduation from high school. This belief is based on our consideration of possible future changes in the economic picture and visible trends toward the formation of low cost city colleges, in addition to our conviction that an early vocational diagnosis for any student is relatively unreliable. Hence, we conclude that the viewpoint of the preceding paragraph should be applied to all students possessing reasonable ability and not merely to those who at the moment are avowed candidates for college entrance. We admit that such action might result in many students studying two and one half years or more of high school mathematics without later entering college. However, even under the extreme assumption, *contrary to our belief*, that the advanced secondary mathematics taken by these students might involve a great amount of wasted effort, we assert that the suggested system is educationally defensible. Wasted effort of only moderate duration on the part of some students who will not continue their training is of little importance if this waste results in more efficient training for many future leaders of our social order.

Our remarks about the desirability of at least two and one half years of high school mathematics as preparation for college entrance may appear to conflict with tendencies in many universities of the first rank to make entrance requirements extremely flexible. We offer no argument against permitting loopholes for college entrance by students of proper ability who for various reasons may have omitted taking certain specified work in high school, where mathematics might be one of the forgotten items. However, we believe that whenever such a loophole exists, the college should take two precautions. First, it should advertise prominently in its bulletins the necessity for a strong mathematical foundation as preparation for efficient work in a major portion of the curriculum, with specific mention of the fields relying on mathematics. Second, the college should make certain that the introductory course work in these fields is pitched at the level of students who have the proper mathematical preparation, and is not throttled down to the pace of students who enter without this training. The latter students could be permitted to remedy their deficiency by taking subcollegiate work in mathematics under college supervision.

The previous discussion in this paper has been devoted largely to problems associated with the entrance of high school graduates into college work. We may now ask what related point of view in regard to mathematical training should be adopted for students entering the ninth grade if it is certain that they will not enter college. Various previous remarks indicate that we consider such certainty rather difficult to justify in the case of students possessing reasonable intelligence, but let us proceed as if the diagnosis as to future college entrance could be made with accuracy in all cases. For convenience, let us divide all high school students who are not prospects for college entrance into two categories. In a first category we place those who would reach their mathematical ceiling at or below the level of a first course in high school algebra. In the second category we group the remaining students, who have at least the minimum ability necessary for the completion of a standard course in demonstrative geometry and a second course in algebra. For the practical application of our subsequent remarks, it would be appropriate to have at hand a procedure for dividing students into these categories. This would demand a statistically determined scale of mathematical aptitude for high school freshmen and corresponding norms of mathematical achievement through the eleventh grade, where these norms of achievement would be based on *selected well taught classes.* It might be found on investigation that pupils very far below the median in mathematical aptitude can comfortably learn standard mathematics through the level of a second course in algebra when class room conditions are satisfactory and teachers are well prepared.

The high school students of the first category we have described would receive little benefit from standard courses in elementary algebra and demonstrative geometry. We believe, however, that a curriculum for these students should include a course in ninth grade mathematics, where the content is on a low plane so far as algebraic methods are concerned and is heavily weighted with applications. Also, we consider it justifiable for this curriculum to include a course in tenth grade mathematics whose logical and algebraic methods remain within reach of the students' powers and whose content is not dictated solely by recollections of the classical course in demonstrative geometry. It would be fatal from the standpoint of the better students if the limited capabilities of students of the first category caused the inauguration of courses suited to them but given to *all* students. We believe that, regardless of the types of courses employed, students of the first category should be taught mathematics in classes segregated from the other students in order to avoid low standards of achievement and lack of interest on the part of the better students.

The high school students of our second category, who have reasonable mathematical ability but no apparent prospects for entering college, were referred to implicitly in connection with our remarks about training for college entrance. However, regardless of college possibilities, we would include a considerable amount of mathematics in a curriculum for these students, and we would not differentiate the presentation of this mathematics from that which is

intended for pre-college students. Standard courses in ninth grade mathematics and the second course in high school algebra are centered not only around the manipulative aspects of algebra but also around its numerous applications in modern life. Well planned courses in secondary mathematics involve useful and interesting material from elementary statistics, the various sciences, the social sciences, and the theory of interest. The demonstrative geometry met in the tenth grade, or perhaps later, presents the student with evidence of the high intelligence of the ancients and gives him training in logical thinking, besides extending his geometrical intuitions and knowledge of applications of geometry in various fields. Moreover, if a student of the second category completes a second course in algebra, he is prepared to appreciate many interesting applications which could be organized into a course in applied mathematics in the twelfth grade. Such a course could involve material from advanced arithmetic, applications of statistics in economics and other social sciences, and the mathematics of investment and insurance, treated in a dignified and efficient algebraic fashion. Also, we should recall that the students under consideration have the intellectual qualifications appropriate for studying the physical sciences as offered at the high school level. The most defensible educational policy demands that these sciences should be presented without submerging the possibilities for the application of secondary mathematics. Hence, the obvious desirability of courses in these sciences even for a student who will not enter college adds to the advantages which he might gain from the study of mathematics. These facts about secondary mathematics should be in mind for contrast with the possible advantages of course work which might be proposed to replace mathematics in a curriculum for students of our second category. We recommend that they should study at least two and one half years of secondary mathematics and, if possible, a semester course in applied mathematics which has advanced algebra as a prerequisite.

Summary

We have emphasized the viewpoint that a strong mathematical foundation is necessary for efficient college work in vital parts of the social sciences and in any other fields which rely heavily on statistical methods, as well as in the sciences where the usefulness of mathematics is beyond question. This viewpoint led us to conclude that a student should offer *at least* two and one half years of secondary mathematics for entrance to college in order to avoid restrictions and handicaps in his college work. We expressed our belief in the fundamental inaccuracy of an early diagnosis that any particular high school student of reasonable intelligence would *not* enter college. These opinions, together with our conviction that secondary mathematics, considered by itself, is both culturally and practically useful, caused us to make recommendations concerning the placement of students in secondary mathematics which may be summarized as follows.

A. Students entering the ninth grade should be divided into two categories

on the basis of their mathematical ability, *regardless of their apparent prospects for college entrance.* The category of low rank would contain those who would reach the limit of their mathematical powers in attempting to learn first year high school algebra.

B. Students of the lower category should be segregated from those of the upper category in classes in mathematics and should study a concrete variety of mathematics in the ninth grade, and possibly also in the tenth grade, where any algebraic methods and logical procedures involved are pitched on a low plane.

C. Students of the upper category should study mathematics at least through a second course in algebra. Students who then foresee possibilities for college entrance could profit by the study of additional mathematics, which should be as extensive as possible for those who plan college work in any of the intensively mathematical fields. All students of this category, regardless of their prospects for college entrance, would receive benefit from a semester course in applied mathematics having advanced algebra as a prerequisite.

QUESTIONS, DISCUSSIONS, AND NOTES

EDITED BY R. E. GILMAN, Brown University, Providence, Rhode Island

The department of Questions and Discussions in the Monthly is open to all forms of activity in collegiate mathematics, including the teaching of mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.

APPLICATION OF VECTOR FORMULAE IN SPHERICAL TRIGONOMETRY

By L. RICHARDSON, University of British Columbia

The object of this paper is to indicate the use to which elementary vector analysis can be put in solving certain types of spherical trigonometry problems. In most cases the vector method is very much shorter than the usual method.

In Gibbs's Vector Analysis there are derived two fundamental formulae of Spherical Trigonometry.

$$\cos a = \cos b \cos c + \sin b \sin c \cos A. \tag{1}$$

$$\sin A/\sin a = \sin B/\sin b = \sin C/\sin c. \tag{2}$$

A third fundamental formula[1] is

$$\sin C \cot A = \sin b \cot a - \cos b \cos C. \tag{3}$$

It can be derived vectorially by using the identity

$$(\mathbf{c} \times \mathbf{a})^2 \mathbf{b}\cdot\mathbf{c} = (\mathbf{a} \times \mathbf{c})\cdot(\mathbf{b} \times \mathbf{c})\mathbf{a}\cdot\mathbf{c} + (\mathbf{b} \times \mathbf{a})\cdot(\mathbf{c} \times \mathbf{a}),$$

[1] This formula and the examples appearing later in the paper are taken from Todhunter's *Spherical Trigonometry*.

where $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ are unit vectors drawn from the centre of the sphere to the points A, B, C on the sphere. The identity gives

$$\sin^2 b \cos a = \sin a \sin b \cos b \cos C + \sin c \sin b \cos A,$$

or

$$\sin b \cos a = \sin a \cos b \cos C + \sin c \cos A.$$

Using (2) it follows at once that

$$\sin b \cos a = \cos b \cos C + \sin C \cot A.$$

The next ten formulas are those usually derived by the use of Napier's rule. We use a right spherical triangle ABC in which $C=90°$ for the development of these relations, and the vectorial identities are valid under this restriction.

We have

$$(\boldsymbol{a} \times \boldsymbol{c})\cdot(\boldsymbol{b} \times \boldsymbol{c}) = 0, \text{ or } \boldsymbol{a}\cdot\boldsymbol{b}\boldsymbol{c}\cdot\boldsymbol{c} = \boldsymbol{a}\cdot\boldsymbol{c}\boldsymbol{b}\cdot\boldsymbol{c},$$

hence

(4) $$\cos c = \cos a \cos b.$$

Again, since

$$(\boldsymbol{a} \times \boldsymbol{c}) \times (\boldsymbol{c} \times \boldsymbol{b}) = (\boldsymbol{a}\boldsymbol{c}\boldsymbol{b})\boldsymbol{c},$$

we can take the dot product with $\boldsymbol{c}$ and cancel out common factors and obtain

(5) $$\sin b = \sin c \sin B.$$

Interchanging $\boldsymbol{a}$ and $\boldsymbol{b}$ we obtain

(6) $$\sin a = \sin c \sin A.$$

By the use of (4) and the identity

$$(\boldsymbol{a} \times \boldsymbol{b})\cdot(\boldsymbol{b} \times \boldsymbol{c}) = \boldsymbol{a}\cdot\boldsymbol{b}\boldsymbol{b}\cdot\boldsymbol{c} - \boldsymbol{a}\cdot\boldsymbol{c},$$

we get

(7) $$\cos B = \cot c \tan a.$$

Interchanging $\boldsymbol{a}$ and $\boldsymbol{b}$ we obtain

(8) $$\cos A = \cot c \tan b.$$

By the use of the following identity

$$[(\boldsymbol{a} \times \boldsymbol{b}) \times (\boldsymbol{c} \times \boldsymbol{b})]\cdot[(\boldsymbol{a} \times \boldsymbol{c}) \times (\boldsymbol{c} \times \boldsymbol{b})] = (\boldsymbol{a} \times \boldsymbol{b})\cdot(\boldsymbol{a} \times \boldsymbol{c})(\boldsymbol{c} \times \boldsymbol{b})^2,$$

we can derive

(9) $$\sin B \cos a = \cos A.$$

Interchanging $\boldsymbol{a}$ and $\boldsymbol{b}$ we obtain

(10) $$\sin A \cos b = \cos B.$$

The identity

$$[(\boldsymbol{a} \times \boldsymbol{c}) \times (\boldsymbol{a} \times \boldsymbol{b})]\ [(\boldsymbol{a} \times \boldsymbol{b}) \times (\boldsymbol{c} \times \boldsymbol{b})] = (\boldsymbol{a} \times \boldsymbol{c})\cdot(\boldsymbol{a} \times \boldsymbol{b})(\boldsymbol{a} \times \boldsymbol{b})\cdot(\boldsymbol{c} \times \boldsymbol{b}),$$

gives, after the removal of factors,

$$\cos c = \cot A \cot B. \tag{11}$$

Again, the vector identity first given reduces, when $C=90°$, to

$$(\boldsymbol{c} \times \boldsymbol{a})^2\boldsymbol{b}\ \boldsymbol{c} = (\boldsymbol{b} \times \boldsymbol{a})\ (\boldsymbol{c} \times \boldsymbol{a}),$$

from which we can obtain

$$\sin b = \tan a \cot A. \tag{12}$$

By a similar method we get

$$\sin a = \tan b \cot B. \tag{13}$$

The following eight examples may be of interest to readers who are teaching elementary vector analysis. All of them refer to spherical trigonometry.

1. If $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$, $\boldsymbol{d}$ are position vectors of the corners $ABCD$ of a quadrilateral on a unit sphere we can interpret the formula

$$(\boldsymbol{b} \times \boldsymbol{c})\cdot(\boldsymbol{a} \times \boldsymbol{d}) + (\boldsymbol{c} \times \boldsymbol{a})\cdot(\boldsymbol{b} \times \boldsymbol{d}) + (\boldsymbol{a} \times \boldsymbol{b})\cdot(\boldsymbol{c} \times \boldsymbol{d}) = 0$$

as follows

$$\sin(\boldsymbol{a}, \boldsymbol{b}) \sin(\boldsymbol{c}, \boldsymbol{d}) \cos P = \sin(\boldsymbol{a}, \boldsymbol{d}) \cos(\boldsymbol{b}, \boldsymbol{c}) \cos Q = \sin(\boldsymbol{a}, \boldsymbol{c}) \sin(\boldsymbol{b}, \boldsymbol{d}) \cos R,$$

where P is the intersection of AB, DC; Q is the intersection of AD, BC; R is the intersection of AC, BD; P, Q, R are also used to indicate the angles of the spherical triangle PQR. Also if $P=Q=90°$, it follows that $R=90°$. This result is due to Joachimsthal.

2. If P, Q are the midpoints of AC, and BD respectively then

$$\boldsymbol{p} = \frac{(\boldsymbol{a}+\boldsymbol{c})}{2} \sec \frac{(\boldsymbol{a}, \boldsymbol{c})}{2}$$

$$\boldsymbol{q} = \frac{(\boldsymbol{b}+\boldsymbol{d})}{2} \sec \frac{(\boldsymbol{b}, \boldsymbol{d})}{2}$$

and if we take the dot product of these vectors we can rearrange the result and get

$$4 \cos(\boldsymbol{p}, \boldsymbol{q}) \cos\left(\frac{\boldsymbol{a}, \boldsymbol{c}}{2}\right) \cos\left(\frac{\boldsymbol{b}, \boldsymbol{d}}{2}\right) = \cos(\boldsymbol{a}, \boldsymbol{b}) + \cos(\boldsymbol{b}, \boldsymbol{c}) + \cos(\boldsymbol{c}, \boldsymbol{d}) + \cos(\boldsymbol{d}, \boldsymbol{a}).$$

3. In a spherical triangle ABC let P, Q, R be the mid-points of AB, AC, BC respectively and let these points be joined by great circles.
Since

$$\boldsymbol{p} = \frac{(\boldsymbol{a}+\boldsymbol{b})}{2} \sec\frac{c}{2}, \qquad \boldsymbol{q} = \frac{(\boldsymbol{a}+\boldsymbol{c})}{2} \sec\frac{b}{2},$$

the dot product is

$$\boldsymbol{p}\cdot\boldsymbol{q} = \frac{1}{4}[1 + \boldsymbol{a}\cdot\boldsymbol{b} + \boldsymbol{b}\cdot\boldsymbol{c} + \boldsymbol{c}\cdot\boldsymbol{a}] \sec\frac{b}{2} \sec\frac{c}{2};$$

whence we can get

$$4\cos(\boldsymbol{p}, \boldsymbol{q}) = [1 + \cos a + \cos b + \cos c] \sec\frac{b}{2} \sec\frac{c}{2}.$$

By symmetry we can get two other similar expressions and from these three we can derive

$$\cos(\boldsymbol{q}, \boldsymbol{p}) \sec\frac{a}{2} = \cos(\boldsymbol{p}, \boldsymbol{r}) \sec\frac{b}{2} = \cos(\boldsymbol{r}, \boldsymbol{q}) \sec\frac{c}{2}.$$

4. Using the two triangles of example 3, we can take vector products as follows:

$$\boldsymbol{p}\times\boldsymbol{q} = \frac{1}{4}[(\boldsymbol{b}+\boldsymbol{c})\times(\boldsymbol{c}+\boldsymbol{a})] \sec\frac{a}{2} \sec\frac{b}{2},$$

$$\boldsymbol{q}\times\boldsymbol{r} = \frac{1}{4}[(\boldsymbol{c}+\boldsymbol{a})\times(\boldsymbol{a}+\boldsymbol{b})] \sec\frac{b}{2} \sec\frac{c}{2},$$

and then take the dot product of the vectors $(\boldsymbol{p}\times\boldsymbol{q})$ and $(\boldsymbol{q}\times\boldsymbol{r})$. Since

$$[(\boldsymbol{b}+\boldsymbol{c})\times(\boldsymbol{c}+\boldsymbol{a})]\ [(\boldsymbol{c}+\boldsymbol{a})\times(\boldsymbol{a}+\boldsymbol{b})]$$
$$= (\cos a + \cos b + \cos c + 1)(\cos a - \cos b + \cos c - 1),$$

and

$$(\boldsymbol{p}\times\boldsymbol{q})\cdot(\boldsymbol{q}\times\boldsymbol{r}) = -\sin(\boldsymbol{p}, \boldsymbol{q}) \sin(\boldsymbol{q}, \boldsymbol{r}) \cos Q,$$

we can combine these results and obtain

$$-\sin(\boldsymbol{p}, \boldsymbol{q}) \sin(\boldsymbol{q}, \boldsymbol{r}) \cos Q$$
$$= \frac{1}{16}(\cos a + \cos b + \cos c + 1)(\cos a - \cos b + \cos c - 1) \sec\frac{a}{2} \sec^2\frac{b}{2} \sec\frac{c}{2}.$$

From problem 3 we take

$$\cos(\boldsymbol{p}, \boldsymbol{q}) \cos(\boldsymbol{q}, \boldsymbol{r}) = \frac{1}{16}(\cos a + \cos b + \cos c + 1)^2 \sec\frac{a}{2} \sec^2\frac{b}{2} \sec\frac{c}{2}$$

and by the use of these relations we can write

$$\cos Q \tan (\boldsymbol{p}, \boldsymbol{q}) \tan (\boldsymbol{q}, \boldsymbol{r}) = -\frac{(\cos a - \cos b + \cos c - 1)}{\cos a + \cos b + \cos c + 1}.$$

It easily follows that

$$\sum \cos Q \tan (\boldsymbol{p}, \boldsymbol{q}) \tan (\boldsymbol{q}, \boldsymbol{r}) = \frac{4}{\cos a + \cos b + \cos c + 1} - 1,$$

where the summation is for P, Q, R.

5. Let $A, B, C, \cdots, M$ be the points on the surface of a unit sphere and let $a, b, c, \cdots, m$ be given constants. If X be a point on the sphere such that

$$a \cos (\boldsymbol{a}, \mathrm{x}) + b \cos (\boldsymbol{b}, \mathrm{x}) + \cdots + m \cos (\boldsymbol{m}, \mathrm{x}) = \text{constant},$$

then the locus of X is a circle. Since

$$\begin{aligned} a \cos (\boldsymbol{a}, \mathrm{x}) + b \cos (\boldsymbol{b}, \mathrm{x}) + \cdots + m \cos (\boldsymbol{m}, \mathrm{x}) &= a\boldsymbol{a}\cdot \mathrm{x} + b\boldsymbol{b}\cdot \mathrm{x} + \cdots + m\boldsymbol{m}\cdot \mathrm{x} \\ &= (a\boldsymbol{a} + b\boldsymbol{b} + \cdots + m\boldsymbol{m})\cdot \mathrm{x} = \text{constant}, \end{aligned}$$

the angle between the vectors x and $(a\boldsymbol{a}+b\boldsymbol{b}+ \cdots +m\boldsymbol{m})$ is constant and the locus of X is a circle.

6. The vector formula

$$\boldsymbol{p} = \frac{[\boldsymbol{pbc}]\boldsymbol{a} + [\boldsymbol{pca}]\boldsymbol{b} + [\boldsymbol{pab}]\boldsymbol{c}}{[\boldsymbol{abc}]},$$

where $\boldsymbol{p}$ is expressed in terms of $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ can be used to advantage. For example if all vectors are unit vectors and we take the dot product of each side with $\boldsymbol{p}$ we have

$$[\boldsymbol{abc}] = [\boldsymbol{pbc}] \cos (\boldsymbol{a}, \boldsymbol{p}) + [\boldsymbol{pca}] \cos (\boldsymbol{b}, \boldsymbol{p}) + [\boldsymbol{pab}] \cos (\boldsymbol{c}, \boldsymbol{p}).$$

Now

$$[\boldsymbol{pbc}] = \sin BPC \sin (\boldsymbol{b}, \boldsymbol{p}) \sin (\boldsymbol{c}, \boldsymbol{p}),$$

therefore

$$[\boldsymbol{pbc}] \cos (\boldsymbol{a}, \boldsymbol{p}) = \sin (\boldsymbol{a}, \boldsymbol{p}) \sin (\boldsymbol{b}, \boldsymbol{p}) \sin (\boldsymbol{c}, \boldsymbol{p}) \cot (\boldsymbol{a}, \boldsymbol{p}) \sin BPC.$$

The other terms on the right give similar expressions. Also $[\boldsymbol{abc}]$ represents the product of sines of two sides and the sine of the included angle of triangle ABC. Hence we obtain:

THEOREM. If P be any point within a spherical triangle ABC the product of the sines of any two sides and the sine of the included angle is equal to

$$\begin{aligned} \sin (\boldsymbol{a}, \boldsymbol{p}) \sin (\boldsymbol{b}, \boldsymbol{p}) \sin (\boldsymbol{c}, \boldsymbol{p}) \{ &\cot (\boldsymbol{a}, \boldsymbol{p}) \sin BPC + \cot (\boldsymbol{b}, \boldsymbol{p}) \sin CPA \\ &+ \cot (\boldsymbol{c}, \boldsymbol{p}) \sin APB \}. \end{aligned}$$

7. Again using the formula

$$p = \frac{[pbc]a + [pca]b + [pab]c}{[abc]}:$$

let P be any point on the unit sphere and let D be any point in the side BC of ABC. If we let P coincide with D then $[dbc]$ is equal to zero, and we can obtain

$$d = \frac{[dca]b + [dab]c}{[abc]}.$$

Taking the dot product of each side with a we have

$$a \cdot d = \frac{[dca]a \cdot b + [dab]a \cdot c}{[abc]}$$

or

$$\cos(a, d) = \frac{[dca]\cos(a, b) + [dab]\cos(a, c)}{[abc]}.$$

Cancelling out common factors we obtain

$$\cos(a, d)\sin(b, c) = \sin(c, d)\cos(a, b) + \cos(a, c)\sin(b, d).$$

8. Using the same formula for p we have, if each side of ABC is a quadrant

$$p \cdot a = \frac{[pbc]}{[abc]}, \quad p \cdot b = \frac{[pca]}{[abc]}, \quad p \cdot c = \frac{[pab]}{[abc]}:$$

and since in this case $[abc] = 1$ we have

(14) $$\cos(a, p)\cos(b, p)\cos(c, p) = [pbc][pca][pab].$$

We also take

$$\begin{aligned}(p \times a) \cdot (p \times b) &= \sin(a, p)\sin(p, c)\cos APC, \\ &= -p \cdot c\, a \cdot p = -\cos(c, p)\cos(a, p),\end{aligned}$$

which gives us

(15) $$\cos(c, p)\cos(a, p) = -\sin(a, p)\sin(c, p)\cos APC = -[pca]\cot APC.$$

Similarly we can obtain

(16) $$\cos(a, p)\cos(b, p) = - \; pbc]\cot CPB,$$

(17) $$\cos(b, p)\cos(c, p) = -[pab]\cot BPA.$$

Taking the product of (15), (16), (17) and using (14) we have

$$\cos(a, p)\cos(b, p)\cos(c, p) + \cot APC \cot BPC \cot BPA = 0.$$

RECENT PUBLICATIONS

EDITED BY R. A. JOHNSON, Brooklyn College of the City of New York

All books for review should be sent directly to the editor of this department, at Brooklyn College, 66 Court Street, Brooklyn, N. Y., and not to any of the other editors or officers of the Association.

REVIEWS

Higher Mathematics for Engineers and Physicists. By I. S. and E. S. Sokolnikoff. New York, McGraw-Hill Book Co., 1934. xiv+482 pages. $4.00.

This volume gives in printed form the substance of a course offered for many years to third-year students of engineering and the physical sciences at the University of Wisconsin. The object of this course is to give to students who are chiefly interested in the applications of mathematics, a brief account of those parts of mathematics which are indispensable in their further study of the physical sciences.

Hence the authors have attempted to make "practical utility" the keynote of the book. This has affected their choice of topics and their treatment of the topics selected. They have not hesitated to state a theorem without proof, when they desired to make use of the theorem but found that its proof would require too far an excursion into mathematical theory. In such cases, references to other texts are given for the benefit of those students who wish to inquire further into the underlying theory. As is natural in a text of this type, there are numerous illustrative examples taken from various branches of engineering. Many of the problems are also concerned with engineering situations. In the appendix, answers are given to all problems.

The book contains most of the topics covered in the usual course in advanced calculus, the calculus of variations being one exception, and in addition certain topics from higher algebra. The headings of the fifteen chapters are as follows: Elliptic Integrals; Solution of Equations; Determinants and Matrices; Infinite Series; Partial Differentiation; Fourier Series; Multiple Integrals; Line Integrals; Improper Integrals; Ordinary Differential Equations; Partial Differential Equations; Vector Analysis; Probability; Empirical Formulas and Curve Fitting; Conformal Representation.

The contents of the various chapters are for the most part indicated by the titles. Comments on certain chapters follow.

It seems strange that the first chapter in such a textbook is concerned with elliptic integrals. The chapter begins with Newton's laws, and continues with discussions of freely falling bodies, simple harmonic motion, and the simple pendulum. This leads to a brief treatment of elliptic integrals and elliptic functions. Neither of these topics is mentioned thereafter. It should be noted that a complete understanding of the contents of this chapter involves some knowledge of infinite series and of differential equations, two topics with which the student is presumably not familiar, inasmuch as they are discussed in great detail in later chapters.

The second and third chapters are concerned with certain topics from algebra, which are also contained in Dickson's "Theory of Equations" and Bôcher's "Higher Algebra." One topic which has been omitted and might well have been included under the solution of equations, is the isolation of the real roots of an equation by locating the bend points and vertical asymptotes of its graph. The method is more general than the method used in §11, namely the determination of the value of the discriminant, in that it may be applied to a larger class of equations.

The two chapters on Differential Equations are chiefly concerned with applications to engineering problems. There is a particularly good introductory section showing how various physical problems give rise to differential equations. The authors might have given a clearer warning to the student that he cannot always depend upon the efficacy of the method used for solving partials in chapter 11, namely assuming that the solution is the product of two functions each involving a single variable.

In the chapter on vectors, various topics previously treated, such as line integrals, Green's theorem, and Stokes's theorem are considered anew using vector methods.

Chapter 14, Empirical Formulas and Curve Fitting, is concerned with the computational aspects of mathematics. Topics included in this chapter are: least squares, method of moments, harmonic analysis, interpolation formulas, and numerical integration.

The concluding chapter, Conformal Representation, is a reprint of a lecture by Dr. Warren Weaver published in this MONTHLY for October 1932. Because it was originally composed for a different purpose, this chapter of necessity contains less detailed exposition than the other chapters of the book.

The chief criticism to be directed against this book is that the chapters do not seem to be arranged in any logical order. For instance, the chapter on partial differentiation is sandwiched between the chapters on infinite series and Fourier series. The authors state that each chapter has been made as nearly as possible an independent unit, in order to enhance the availability of the book for reference purposes. For the most part, they have been careful to preserve the proper sequence of topics from chapter to chapter, but in several cases they have not. While this may not detract from the usefulness of this volume as a textbook, it does detract from its usefulness as a reference book.

The following errors and misprints were noted: On page 148, line 5, replace x_{n-1} by x_n. On page 161, figure 46 is incomplete, as it does not show the element of volume. On page 168, line 7 from bottom, it would be better to let $\Delta s_i = P_{i-1}P_i$. Then the summation limits for W would be uniform with the notation of Chapter 7 and §54. On page 195, the proof of Test 1 does not take care of the case $A=0$, which requires further consideration. On page 404, in figure 107, the point of intersection should be at (1, 3).

This volume seems to be an excellent one for the purpose for which it was

written. There are a sufficient number of topics treated in sufficiently great detail to keep any ordinary class of third-year engineers occupied for a full year.

H. M. GEHMAN

Plane and Spherical Trigonometry. By W. L. Hart. New York, D. C. Heath and Company, 1934. Price $2.12.

This book contains 5 pages of preface and table of contents, 216 pages of text, 20 pages of answers to exercises, 124 pages of logarithmic and other tables. For the plane trigonometry the answers are given only for odd-numbered exercises. Answers to even-numbered problems are furnished free in a separate pamphlet when requested by the instructor. This last fact ought to appeal to many teachers.

The author is careful to note in his collection of answers those obtained by using four-place tables and those by using five-place tables, also he inserts other explanatory remarks among the answers. How many of us have had to explain to a class that the answers given in a trigonometry book (when these answers differed from ours) were probably worked out by someone hired to do so, who used a table with fewer or with more places than the tables in the book possessed!

The author gives the answers to all the problems in spherical trigonometry. The tables are especially good and complete and well-printed, also they include both four- and five-place tables, even tables of secants and cosecants, of squares and square roots, and of natural logarithms of numbers from 1 to 10 at intervals of tenths, as well as a table for computing compound interest.

Other good features of the book include a complete chapter on logarithms, review exercises at intervals throughout the book, a summarizing review at the end of the treatment of plane trigonometry, a fine index, an interesting appendix on plane trigonometry, and a good chapter on some applications of spherical trigonometry. The starred supplementary sections and examples can be omitted without disturbing the continuity of the text. The problems containing applications of the subject seem to be very interesting. The historical notes are very good. Numerous helpful hints are given, so that the text almost teaches itself and could be used without an instructor.

This is one of the most complete and carefully written textbooks on trigonometry that have appeared in print. So many questions that students often raise are answered in the text. Such careful definitions, proofs, and discussions are given that the students should know just what they are doing at all times. The author has apparently noticed during his years of teaching trigonometry just what are the students' difficulties with the subject.

One criticism that might be made of the book is that the pages seem crowded with material. Also on the subject of identities it would seem best to emphasize their training in reducing complicated trigonometric expressions to simpler forms, and so to discourage the students' working with both sides of the identity.

The part of the book that deals with plane trigonometry is the same as Professor Hart's text on plane trigonometry that was published in 1933. The addition of the part on spherical trigonometry makes the text much more valuable for the student to keep as a reference book. It appears now as though spherical trigonometry will come back into favor and be taught oftener because of its beauty and disciplinary value and also because of its increasing number of appearances in advanced mathematics as well as in applications in other fields.

A. D. CAMPBELL

MATHEMATICS CLUBS

EDITED BY F. M. WEIDA, The George Washington University, Washington, D.C.

All reports of club activities, suggestions and topics for club programs, and material of interest to clubs should be sent to F. M. Weida, The George Washington University, Washington, D.C. All manuscript should be typewritten, with double spacing, and with margins at least one inch wide.

CLUB ACTIVITIES

1933–1934

THE PI MU EPSILON MATHEMATICAL FRATERNITY

Pi Mu Epsilon is an academic fraternity in institutions of university grade. Its primary aim is the advancement of mathematics and scholarship. It is a living, active, working fraternity of scholars in which the members are actively engaged in study and research and the exchange of ideas in the field of mathematical science.

Pi Mu Epsilon of Hunter College of the City of New York

During the past year the Hunter College Chapter of Pi Mu Epsilon has held eight program meetings, two business meetings and three social functions. At the first three program meetings of the Fall Semester a synthetic treatment of "Hyperbolic Non-Euclidean geometry" was given. Nine student members gave papers on this subject. At the fourth program meeting the chapter was extremely fortunate in having as speaker Professor Jesse Douglas of Massachusetts Institute of Technology who discussed "Space curves which bound no finite area." During the Spring Semester the program meetings have been devoted to various aspects of "Functions of a complex variable" and to "Linear algebras over a real field." Eleven students reported on these two topics.

The chapter initiated twenty-two new members, making the total active membership for the year, including faculty members, fifty-seven.

In October the new members were initiated at a banquet at which Professor Arnold Dresden of Swarthmore College was our guest and speaker. In March the initiation was held at a tea. Both these functions were attended by a large number of alumnae as well as by undergraduate and faculty members.

The officers for the academic year 1933–1934 were: Professor Mina S. Rees, Director; Florence Josephson, Vice Director; Miriam Gold, Corresponding Secretary; Mildred Frey, Recording Secretary; Mary Dever, Treasurer.

MIRIAM GOLD, *Corresponding Secretary*

PROBLEMS AND SOLUTIONS

ELEMENTARY PROBLEMS

EDITED BY OTTO DUNKEL, H. L. OLSON, AND W. F. CHENEY, JR.

Send all communications about Elementary Problems and Solutions to W. F. Cheney, Jr., Dept. Box 35, Storrs, Connecticut.

The Department of Elementary Problems welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. Problems may be submitted unaccompanied by their solutions.

PROBLEMS FOR SOLUTION

E 125. *Proposed by Elmer Schuyler, Bay Ridge High School, Brooklyn, N. Y.*

Construct the triangle ABC, given the vertex A and the points of contact of BC produced with each of the escribed circles corresponding to sides AC and AB respectively.

E 126. *Proposed by M. C. Holmes, West Virginia University.*

A series of events, $e_1, e_2, \cdots e_n$, are given, with constant probabilities $p_1, p_2, \cdots p_n$ respectively of happening in any given trial. Show that in a series of repeated trials the chance that event e_i will happen before any of the others is given by

$$u_i = p_i/(p_1 + p_2 + \cdots + p_n)$$

if the events are mutually exclusive, and by

$$v_i = \frac{p_i(1 - p_1)(1 - p_2) \cdots (1 - p_{i-1})(1 - p_{i+1}) \cdots (1 - p_n)}{1 - (1 - p_1)(1 - p_2) \cdots (1 - p_n)}$$

if the events are independent of each other. (Note that the result of any trial may be something different from any of the e's.)

E 127. *Proposed by E. P. Starke, Rutgers University.*

There is a number with three like middle digits, and such that its square has for its digits a permutation of the digits from one through nine. Find it and prove it unique.

E 128. *Proposed by J. A. van Groos, Oregon State College.*

If PQ is a chord of the parabola, $y^2 = ax$, and the ordinates of P and Q are p and q respectively, with $p < q$, show that the area of the segment cut from the parabola by PQ is given by $(q-p)^3/6a$.

E 129. *Proposed by Leon Battig, University of Wisconsin.*

In the parallelogram, $ABCD$, points E and F are in sides AB and CD respectively. AF intersects ED in G. EC intersects FB in H. GH produced intersects AD in L and BC in M. Prove by high school geometry that $DL = BM$.

E 130. *Proposed by Wm. F. Cheney, Jr., Connecticut State College.*

In the following two sums, each different letter represents a different digit. Identify them and show that the solution is unique.

$$U\ S\ A+F\ D\ R=N\ R\ A, \qquad U\ S\ A+N\ R\ A=T\ A\ X.$$

SOLUTIONS

E 93 [1934, 327]. *Proposed by H. T. R. Aude, Colgate University.*

Find the locus of the centers of the circles in a plane which pass through a given point and are orthogonal to a given circle.

Solution by Sidney Kaplan, Brooklyn, New York.

This is a special case of the more general problem of finding the locus of the centers of the circles in a plane orthogonal to two given circles, which problem appears in R. A. Johnson's text on Modern Geometry. In the general problem, the locus is known to be the radical axis of the two given circles.

In the given problem, the two circles of the general problem are the given circle and the given point, the latter being treated as a circle of zero radius. The straight line which bisects the tangents from the given point to the given circle is the locus sought, for the radical axis of two circles is known to be also the locus of points from which all the tangents to the two circles are equal.

Analytically, if the given circle is $(x-a)^2+(y-b)^2=r^2$ and the given point is the origin, the locus sought is $2ax+2by=a^2+b^2-r^2$. As a check we notice that this radical axis is perpendicular to the line of centers, $ay-bx=0$.

Also solved by E. F. Allen, W. E. Buker, Melvin Dresher, Hansraj Gupta, Margaret L. Harris, L. M. Kelly, Olga Larson, Theodore Lindquist, Roy MacKay, A. V. Richardson, E. P. Starke, M. J. Turner, Simon Vatriquant, Margaret Young-Woodbridge and the proposer.

E 94 [1934, 327]. *Proposed by E. P. Starke, Rutgers University.*

If three integers, a, b and c, satisfying $a+b=c$, together contain each of the nine digits from one to nine just once, show that c is a multiple of nine such that $450<c<1000$. Also show that, with three exceptions, if c satisfies this necessary condition (and is made up of three distinct, non-zero digits) there exist values of a and b satisfying the original hypothesis.

Solution by W. E. Buker, Leetsdale, Pa.

1. a, b and c each contain three digits. For, if a contained three, b two and c four digits, then the second digit of c would be zero. Or, if a contained four digits, b one and c four, some digits of c would repeat those of a.

2. Since a, b and c together contain all the digits from one to nine, inclusive, without duplication,

$$a+b+c\equiv 0 \text{ (mod. 9)},$$

$$2c\equiv 0 \text{ (mod. 9)},$$

whence c is a multiple of nine.

3. To establish a minimum value for c, suppose $a=1xx$, $b=2xx$, $c=4pq$. The only possible values of pq satisfying (2) and not repeating 1, 2 or 4, are 59, 68, 86 and 95. Hence, under this hypothesis, 459 is a minimum value for c. However, if $a=1xx$ and $b=2xx$, it might be that $c=3pq$. In such case, pq would have to equal 69, 78, 87 or 96, but trial of these cases shows them to be impossible. Since any other combination of initial digits for a and b make c larger than 459, it is a minimum, and so of course $450<c$.

4. Trial of all values of c which are multiples of 9, which have no zeros or repeated digits, and which are between 450 and 1000 shows that there exist values of a and b satisfying the original hypothesis, except for the following seven values of c: 513, 531, 612, 621, 684, 756 and 765.

Editorial Note. In the proposer's solution of this problem, the condition that c be a multiple of nine is replaced by the condition that the digits of c add to eighteen, and under this hypothesis only the last three of the above seven exceptions subsist.

Also solved by C. W. Trigg and Simon Vatriquant.

E 95 [1934, 327]. *Proposed by Harry Langman, Cooper Union, New York.*

Show that $(2n)!/(2^n n!)$ is a positive integer if n is.

Solution by Hansraj Gupta, Government College, Hoshiarpur, India.

Since

$$2^n n! = 2^n \cdot 1 \cdot 2 \cdot 3 \cdot 4 \cdots n = 2 \cdot 4 \cdot 6 \cdot 8 \cdots (2n),$$

we have

$$(2n)!/(2^n n!) = 1 \cdot 3 \cdot 5 \cdot 7 \cdots (2n-1) = \text{an integer}.$$

Editorial Note. In his solution, Sidney Kaplan points out that this problem is a special case of problem 9, page 28, of Carmichael's Monograph on the Theory of Numbers, that if m and n are positive integers, $(mn)!/n!(m!)^n$ is an integer, which reduces to the present problem when $m=2$.

Also solved by W. E. Buker, W. B. Clarke, Melvin Dresher, Meyer Karlin, Roy MacKay, E. D. Rainville, A. V. Richardson, E. P. Starke, C. W. Trigg, Simon Vatriquant, Maud Willey, B. C. Zimmerman and the proposer.

E 96 [1934, 327]. *Proposed by R. A. Johnson, Brooklyn College,*

The captain of a man-of-war saw, one dark night, a privateersman crossing his path at right angles, and at a distance ahead of c miles. The privateersman was making a miles an hour, while the man-of-war could only make b miles an hour. The captain's only hope was to come as close to the privateersman as possible, and to disable him by one or two well-directed shots; so the ship's lights were put out and her course altered in accordance with this plan. Through what angle must the ship's course have been changed in order to secure the nearest approach, and when and where did it occur? (This is a modification of a problem appearing on page 126 of Osgood's *Introduction to the Calculus.*)

Solution by A. V. Richardson, Bishop's College, Lennoxville, Quebec.

If the ship's course is changed through an angle θ, then after time t the square of the distance between the two boats will be

$$f(\theta, t) = (at - bt \sin \theta)^2 + (c - bt \cos \theta)^2,$$

which can not be a minimum unless the partial derivatives of f with respect to t and θ are each zero. This gives the two equations,

$$(a^2 + b^2)t - 2abt \sin \theta - bc \cos \theta = 0 \tag{1}$$

$$\tan \theta = at/c. \tag{2}$$

Equation (2) shows at once that the initial and final positions of the man-of-war are collinear with the final position of the privateersman.

On substituting $t = (c/a) \tan \theta$ in (1), a reduction leads to the equation, $ab \sin^2 \theta - (a^2+b^2) \sin \theta + ab = 0$, whence $\sin \theta = a/b$ or b/a. Since $b < a$ by hypothesis, $\sin \theta = b/a$, which gives the angle through which the course of the man-of-war had to be changed. Substitution into (2) gives the time of closest approach as $t = bc/[a(a^2-b^2)^{1/2}]$. The minimum distance between the two ships is then found, by a final substitution, to be $(c/a)(a^2-b^2)^{1/2}$.

Also solved by W. E. Buker, Daniel Finkel, E. L. Harp, Jr., H. K. Hewitt, E. P. Starke, Maud Willey and the proposer.

E 97 [1934, 327]. *Proposed by Churchill Eisenhart and H. N. Russell, Jr., Princeton University.*

In a certain hypothetical state, no one-digit license numbers are issued, and no license number begins with a zero. Assuming that the numbers are issued in numerical order, and that the number 29,000 has just been given out, show that if a palindromic number is seen, it is more likely to be a five-digit number than not.

Solution by L. J. Adams, Santa Monica, California.

1. There are 9 two digit palindromic numbers, such as 11, 22, 33, etc.
2. In the case of three digit numbers, there are nine choices for the first digit, ten choices for the second digit, and but one possibility for the remaining digit, so that there are 90 palindromic three digit numbers.
3. For four digit numbers there are nine choices for the first digit, ten for the second, and but one possibility for each of the remaining digits. Consequently, there are 90 palindromic four digit numbers.
4. For the five digit numbers, consider first those beginning with the digit 1. There are ten choices for the second digit and ten for the third, with but one possibility for each of the remaining digits in any palindromic number. Hence there are 100 palindromic numbers in the ten thousands. Now in the twenty thousands, the second digit is less than nine, by hypothesis, so there are nine choices for the second digit and ten for the third. Consequently, there are ninety palindromic numbers between 20,000 and 29,000. It is thus seen that

there are 190 palindromic numbers of five digits to be considered, and only 189 palindromic numbers of two, three and four digits.

Also solved by W. E. Buker, W. B. Clarke, F. L. Manning, A. V. Richardson, E. P. Starke, C. W. Trigg, Simon Vatriquant, B. C. Zimmerman and the proposer.

E 98 [1934, 327]. *Proposed by Maud Willey, Long Beach, Miss.*

$B_0, A_1, A_2, A_3, \cdots$ are equally spaced points on a line, q. A circle is drawn with radius $B_{i-1}A_i$ and center A_i, and the ends of its diameter perpendicular to q are labeled B_i and C_i ($i=1, 2, 3, \cdots$). What is the curve of lowest degree through the points B_i and C_i?

Solution by Simon Vatriquant, A. R. d'Ixelles, Brussels, Belgium.

Taking the line q as the X-axis and the perpendicular to it at B_0 as the Y-axis, and denoting by a the common distance $A_{i-1}A_i$, the coordinates of B_i may be computed as follows: $x_i=ai$ and $y_i=A_iB_i=A_iB_{i-1}$. Since A_iB_{i-1} is the hypotenuse in triangle $A_{i-1}A_iB_{i-1}$, we have the recurring relation, $y_i^2=y_{i-1}^2+a^2$. This proves that the squares of the ordinates are in arithmetic progression, and thus $y_i^2=a^2i$. Eliminating i between the coordinates of B_i gives, dropping the subscripts, $y^2=ax$, which represents a parabola passing through all the points B_i and C_i.

Also solved by W. B. Clarke, A. V. Richardson, E. P. Starke, C. W. Trigg, and the proposer.

E 99 [1934, 328]. *Proposed by Arnold Dresden, Swarthmore College.*

In the following problem, each letter represents one of the digits from zero through nine, and it is required to so identify them that they will constitute a correctly worked out problem in long division.

$$
\begin{array}{rrcccccccl}
a & b) & c & d & d & e & f & g & (h\,i\,f\,j \\
 & & c & c & h & & & & \\
\cline{3-5}
 & & & & e & e & & & \\
 & & & & a & b & & & \\
\cline{5-6}
 & & & & h & d & f & & \\
 & & & & d & e & c & & \\
\cline{5-8}
 & & & & & h & a & g & \\
 & & & & & h & h & j & \\
\cline{6-8}
 & & & & & & & c & \\
\end{array}
$$

This problem is from the Haagsche Post, with appreciative acknowledgment.

Solution by Margaret Young-Woodbridge, Brooklyn College of the City of N.Y.

It is obvious that the second digit in the quotient is 1, giving the divisor

identically in the second multiplication. In the third step we see that h must be one greater than d, so that when h is subtracted from d in the first step, the remainder e must be 9. Likewise, in the last subtraction, we notice that a must exceed h by 1, and therefore, since in the second step a subtracted from 9 leaves h, a must be 5 and h 4, whence d is 3. Again in the first subtraction, d is seen to be 1 greater than c, so that c is 2. In the second step, b subtracted from 9 leaves 3, making b equal 6. In the third step, 2 from f leaves 5, so f is 7. Finally, in the last step, j subtracted from the sum of 10 and g leaves 2, so that j is 8 and g 0. The restored division then shows that 233,970 divided by 56 gives 4178, with a remainder of 2.

Also solved by L. J. Adams, W. E. Buker, W. B. Clarke, M. L. Constable, Daniel Finkel, Hansraj Gupta, D. W. Hall, E. L. Harp, Jr., Sidney Kaplan, Elmer Latshaw, Theodore Lindquist, Gertrude I. McCain, A. V. Richardson, E. P. Starke, Ruth W. Stokes, C. W. Trigg, M. J. Turner, Simon Vatriquant, B. C. Zimmerman and the proposer.

Advanced Problems

Send all communications about Advanced Problems and Solutions to Otto Dunkel, Washington University, St. Louis, Mo. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide.

Problems containing results believed to be new or extensions of old results are especially sought. The editorial work would be greatly facilitated, if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known textbooks or results found in readily accessible sources, will not be proposed as problems for solution in this department. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

PROBLEMS FOR SOLUTION

3711. *Proposed by J. H. M. Wedderburn, Princeton University.*

If $f(x)=a_0x^n+a_1x^{n-1}+\cdots+a_n$ is a given polynomial, and $f_r(x)=a_0x^r+a_1x^{r-1}+\cdots+a_r$, express the product f_rf_s in the form

$$\sum_{i=0}^{r+s}\alpha_if_i.$$

See Weber's *Algebra*, vol. 1, §§74, 78.

3712. *Proposed by J. R. Musselman, Western Reserve University.*

The necessary and sufficient condition that the six points of intersection of two parallel equilateral triangles, sides produced if necessary, lie on an equilateral hyperbola is that the sides of the two triangles be equal.

3713. *Proposed by R. E. Gaines, University of Richmond.*

Determine the position of a normal chord of a conic which forms a segment of minimum area. Find the area of such a segment of an ellipse.

3714. *Proposed by J. M. Feld, New York City.*

Prove that, if the functions $x_i(t)$, $i=1, 2, 3$, possess second derivatives, and, if

$$(1) \qquad (x_1x_2' - x_2x_1')^2 + (x_2x_3' - x_3x_2')^2 + (x_3x_1' - x_1x_3')^2 = 0,$$

$$(2) \qquad x_1^2 + x_2^2 + x_3^2 \neq 0, \qquad x_2x_3' - x_3x_2' \neq 0,$$

then $(x_1x_2' - x_2x_1')/(x_2x_3' - x_3x_2') = \text{constant}$, and $(x_3x_1' - x_1x_3')/(x_2x_3' - x_3x_2') = \text{constant}$.

3715. *Proposed by Harriet B. Herbert, New York City.*

In the sequence of square, cube, tesseract, continued to n dimensions, what angles are formed between diagonals for each figure?

3716. *Proposed by J. D. Hill and H. J. Hamilton, Brown University.*

Let $E_1, E_2, \cdots, E_n, \cdots$ be an infinite sequence of measurable sets in the interval (a, b) such that $mE_n \geqq k > 0$ for $n=1, 2, \cdots$. Does there necessarily exist some infinite sequence of indices $1 \leqq r_1 < r_2 < \cdots < r_i < \cdots$ for which the measure of $E_{r_1} \cdot E_{r_2} \cdots E_{r_i} \cdots$ is greater than zero?

SOLUTIONS

3611 [1933, 243]. *Proposed by J. Rosenbaum, Milford, Conn.*

If each of the face angles at a vertex of a tetrahedron is a right angle, the square of the area of the face opposite that vertex is equal to the sum of the squares of the areas of the faces adjacent to that vertex.

Is the converse true?

Solution by Roy MacKay, Eastern New Mexico Junior College.

Let A be the vertex of the three right angles of the tetrahedron $ABCD$ and let a, b, c, p, q, r represent the lengths of the edges AB, AC, AD, BC, CD, DB, respectively. Then $p^2 = a^2 + b^2$; $q^2 = b^2 + c^2$; and $r^2 = c^2 + a^2$.

Now Heron's formula for the area of the triangle BCD can be rearranged so that the square of this area is

$$(2p^2q^2 + 2q^2r^2 + 2r^2p^2 - p^4 - q^4 - r^4)/16.$$

If we now substitute for p, q, and r, we obtain that the square of the area of the face BCD is

$$(a^2b^2 + b^2c^2 + c^2a^2)/4,$$

which is the desired result.

The converse is not in general true, for equating the square of the area of the face BCD to the sum of the squares of the areas of the other three faces and solving for p^2 we find that $p^2 = a^2 + b^2$ is only a particular solution and hence the angle BAC is not necessarily a right angle. In this case, however, if two of the angles at A are right angles, the third one must be right also.

Solved also by A. D. Bradley, J. H. Butchart, A. G. Clark, J. W. Clawson, W. V. Parker, L. S. Shively, F. Underwood, S. Vatriquant, Maud Willey, and the proposer.

Editorial Note. Let $A_1A_2A_3A_4$ be the tetrahedron with face areas S_1, S_2, S_3, S_4, and let A_4 be taken as the origin for the vector edges $\boldsymbol{a}_1$, $\boldsymbol{a}_2$, $\boldsymbol{a}_3$. Then the vector areas are given by

$$\begin{aligned}(1)\qquad & 2S_1\boldsymbol{n}_1 = \boldsymbol{a}_2 \times \boldsymbol{a}_3, \qquad 2S_2\boldsymbol{n}_2 = \boldsymbol{a}_3 \times \boldsymbol{a}_1, \qquad 2S_3\boldsymbol{n}_3 = \boldsymbol{a}_1 \times \boldsymbol{a}_2,\\ & 2S_4\boldsymbol{n}_4 = (\boldsymbol{a}_2 - \boldsymbol{a}_1) \times (\boldsymbol{a}_3 - \boldsymbol{a}_2),\end{aligned}$$

where $\boldsymbol{n}_1$, $\boldsymbol{n}_2$, $\boldsymbol{n}_3$, $\boldsymbol{n}_4$, are unit vectors of which the first three are directed toward the interior and the last toward the exterior of the tetrahedron. The development of the last equation gives

$$\begin{aligned}(2)\qquad 2S_4\boldsymbol{n}_4 &= \boldsymbol{a}_2 \times \boldsymbol{a}_3 - \boldsymbol{a}_1 \times \boldsymbol{a}_3 + \boldsymbol{a}_1 \times \boldsymbol{a}_2\\ &= 2S_1\boldsymbol{n}_1 + 2S_2\boldsymbol{n}_2 + 2S_3\boldsymbol{n}_3.\end{aligned}$$

Taking the scalar square of each side, we obtain

$$(3)\qquad S_1^2 + S_2^2 + S_3^2 - S_4^2 = 2(S_1S_2 \cos\theta_{12} + S_2S_3 \cos\theta_{23} + S_3S_1 \cos\theta_{31}),$$

where θ_{12} is the dihedral angle between the faces S_1 and S_2, etc. If no dihedral angle of the trihedral angle A_4 may exceed 90°, then the converse is true. For the right side can be zero in this case only if each of the three dihedral angles is 90°.

Since in the problem there is no restriction upon the angles, the converse is not true. This will be shown by indicating how many tetrahedrons may be formed in which the right side of the last equation is zero and for which the trihedral angle is not trirectangular. Let a_1, a_2, a_3 be the lengths of the edges at A_4, and ϕ_1 the angle between a_2 and a_3, etc.; then the right side may be written

$$\begin{aligned}(4)\qquad -\frac{1}{2}\, a_1a_2a_3[a_3 \sin\phi_1 \sin\phi_2 \cos\theta_{12} &+ a_1 \sin\phi_2 \sin\phi_3 \cos\theta_{23}\\ &+ a_2 \sin\phi_3 \sin\phi_1 \cos\theta_{31}].\end{aligned}$$

Now select for θ_{12}, θ_{23}, ϕ_2, any angles such that $90° < \theta_{12} < 180°$, $0 < \theta_{23} < 90°$, $0 < \phi_2 < 180°$. Then the remaining parts of A_4 are determined. For a_1 and a_2 select any values such that $a_1 > 0$, $a_2 > 0$, and so that

$$a_1 \sin\phi_2 \cos\theta_{23} + a_2 \sin\phi_1 \cos\theta_{31}$$

is positive. Then a_3 may be found so that it is greater than zero and (4) is zero. Thus there are ∞^5 ways in which the converse fails to be true.

3639 [1933, 561]. *Proposed by N. A. Court, University of Oklahoma.*

The two planes passing through the circumcenter of a tetrahedron and perpendicular to two bimedians (i.e., lines joining the mid-points of pairs of opposite edges) divide the third bimedian harmonically.

Solution by A. S. Householder, Washburn College.

Let the vectors from a given origin to the vertices $A_i (i=1, 2, 3, 4)$ be denoted by $\boldsymbol{a}_i$. The perpendicular bisector of the edge A_iA_j has the equation

$$(1) \qquad \beta_{ij}(\mathbf{x}) \equiv 2(\boldsymbol{a}_i - \boldsymbol{a}_j)\cdot\mathbf{x} - \boldsymbol{a}_i^2 + \boldsymbol{a}_j^2 = 0,$$

and all such planes pass through the circumcenter O. Hence

$$(2) \ \beta_{ij}(\mathbf{x}) + \beta_{kl}(\mathbf{x}) \equiv 2(\boldsymbol{a}_i + \boldsymbol{a}_k - \boldsymbol{a}_j - \boldsymbol{a}_l)\cdot\mathbf{x} - \boldsymbol{a}_i^2 - \boldsymbol{a}_k^2 + \boldsymbol{a}_j^2 + \boldsymbol{a}_l^2 = 0$$

is the equation of the plane through O whose vector normal is parallel to the bimedian for the edges A_iA_k and A_jA_l.

It will be shown that the planes $\beta_{12}+\beta_{34}=0$ and $\beta_{13}+\beta_{24}=0$ divide harmonically the bimedian for A_1A_4 and A_2A_3. If $\mathbf{x}$ is the vector of a point on this bimedian, the ratio $\lambda_1:\lambda_2$ in which it divides the bimedian is given by

$$(3) \qquad 2(\lambda_1 + \lambda_2)\mathbf{x} = \lambda_1(\boldsymbol{a}_1 + \boldsymbol{a}_4) + \lambda_2(\boldsymbol{a}_2 + \boldsymbol{a}_3).$$

The intersection of (3) with $\beta_{12}+\beta_{34}=0$ is determined by

$$\lambda_1 : \lambda_2 = (\boldsymbol{a}_1 - \boldsymbol{a}_4)\cdot(\boldsymbol{a}_1 - \boldsymbol{a}_2 - \boldsymbol{a}_3 + \boldsymbol{a}_4) : (\boldsymbol{a}_3 - \boldsymbol{a}_2)\cdot(\boldsymbol{a}_1 - \boldsymbol{a}_2 - \boldsymbol{a}_3 + \boldsymbol{a}_4).$$

By interchange of the subscripts 2 and 3, which merely changes the sign of the ratio, we obtain the ratio for its intersection with $\beta_{13}+\beta_{24}=0$; and this completes the proof.

Solved also by J. W. Clawson, W. C. Janes, and W. P. Udinski.

Editorial Note. In all the other solutions the vector or the coordinates of the circumcenter were used; also one vertex was taken as the origin leading to unsymmetric equations. Udinski used vectors in a manner quite different from the above solution, while the others used coordinates.

The latter half of the reasoning in the above solution may be put in a different form. We easily find from (1)

$$(\beta_{13} + \beta_{24}) + (\beta_{12} + \beta_{34}) \equiv 2\beta_{14},$$
$$(\beta_{13} + \beta_{24}) - (\beta_{12} + \beta_{34}) \equiv 2\beta_{23}.$$

Thus the planes $\beta_{13}+\beta_{24}=0$, $\beta_{12}+\beta_{34}=0$, $\beta_{14}=0$, $\beta_{23}=0$, form a harmonic set with the axis determined by the first two; and they pass through the two division points and the two end points of the bimedian A_1A_4, A_2A_3. This concludes the proof.

Very little advantage results from the use of vectors, if we take for the origin of rectangular coordinates the circumcenter O. We may then write for the equation of the perpendicular bisector of A_iA_j

$$\beta_{ij} \equiv (x_i - x_j)x + (y_i - y_j)y + (z_i - z_j)z = 0,$$

where x_i, y_i, z_i are the coordinates of A_i. The proof then follows precisely as above.

3642. [1933, 561]. *Proposed by H. Halperin, A. and M. College of Texas.*

Show that the caustic curve by reflection from a circle, of the rays issuing from a point on the circle, can be generated as follows:

Let A be the point source on the circle O; B a point on the diameter AOB, such that OB equals one third of AO; M a variable point on the circle. Then the point P of intersection of the reflection MP of the ray AM with the line BP parallel to OM generates the caustic curve.

Solution by S. Vatriquant, Brussels, Belgium.

Let us take the diameter through A and the tangent at the same point, respectively, as Ox and Oy-axes. If t is the angle xAM, the parametric equations of the circle are

$$x = 2R\cos^2 t, \quad y = 2R\sin t\cos t.$$

Angle xOM equals $2t$ and angle xQM (Q denoting the point where the reflection MP meets the Ox-axis) equals $3t$. Hence the equation of BP is

$$(1) \qquad y = (x - 4R/3)\tan 2t,$$

and that of MP is

$$(2) \qquad y - 2R\sin t\cos t = (x - 2R\cos^2 t)\tan 3t.$$

The equation of the caustic curve may be obtained by taking the derivative of (2) with respect to t and eliminating t between (2) and this derivative, which may be written, after reductions, under the form

$$(3) \qquad 3x - 4R\cos^2 t\,(3 - 2\cos^2 t) = 0.$$

Equations (1) and (2) may be written

$$(1) \qquad x\sin 2t - y\cos 2t - \frac{4R}{3}\sin 2t = 0,$$

$$(2) \qquad x\sin 3t - y\cos 3t - 2R\sin 2t\cos t = 0.$$

By eliminating y, we get precisely (3) after reductions.

Solved also by C. T. Allen.

Editorial Note. The construction may be derived geometrically. Let M' be a variable point on the circle near the fixed point M; and let the reflected rays from M and M' meet in P'. Denote the angles OAM and MAM' by θ and ϵ; then the angles MOM' and $MP'M'$ are 2ϵ and 3ϵ, respectively. From the triangle $MP'M'$ we have

$$\frac{MP'}{\sin P'M'M} = \frac{MM'}{\sin 3\epsilon} = \frac{2R\epsilon}{\sin 3\epsilon}, \qquad R = OA.$$

As $M' \to M$, $P' \to P$; and we have in the limit

$$MP = 2R\cos\theta/3.$$

Constructions for P may be obtained by drawing within (O) a circle upon the diameter $MN=2R/3$. This circle cuts the reflected ray from M in P. Draw DP, where D is the center of this last circle. Draw PB parallel to MO cutting AO in B. The trapezoid $OBPD$ has equal angles at O and D. Hence $OB=DP$ $=R/3$. From this we see that the caustic is generated by the fixed point P on the circle (D) as the latter rolls upon a fixed equal circle with center at O. The curve is a cardioid with its cusp at B.

The expression for MP may also be obtained from the formula for the general caustic given in this Monthly 1924, 306 in the solution of problem 2966.

3643 [1933, 561]. *Proposed by H. G. Green, University College, Nottingham, England.*

Show, that, if a and b are positive numbers with b greater than a, and $e\xi=(b^b/a^a)^{1/(b-a)}$, then ξ lies between a and b. Given that $\log_{10} e=0.4343$ to four decimal places, deduce that $\log_{10} 99$ lies between 1.99565750 and 1.99561262, and explain theoretically the close value of the mean of these numbers to the true value of the log.

Solution by the Proposer.

Consider the function $x \log x$, the base being e. Then

$$a \log a = b \log b - (b-a)(1+\log \xi), \text{ where } b > \xi > a,$$

or ξ lies between a and b where

$$e\xi = (b^b/a^a)^{1/b-a}.$$

We have also

$$a \log a < b \log b - (b-a)(1+\log a), \quad \text{or} \quad \log a < \log b - \frac{b-a}{b}\cdot$$

Similarly

$$a \log a > b \log b - (b-a)(1+\log b), \quad \text{or} \quad \log a > \log b - \frac{b-a}{a}\cdot$$

If $a=99$, $b=100$ and $\log_{10} e>0.43425$ and <0.43435 we have, on transferring to base 10,

$$\log_{10} 99 < 2 - \frac{0.43425}{100} < 1.99565750,$$

and

$$> 2 - \frac{0.43435}{99} > 1.99561262.$$

We now explain the close approximation of the mean of these quantities to the true value.

Inspection shows at once that the mean, 1.99563506, is affected only in the 7th decimal place by the margin in the value of $\log_{10} e$; varying in fact by $\mp 5\times 10^{-7}$ in accord with the possible excess, $\pm 5\times 10^{-5}$, of $\log_{10} e$ over 0.4343 (actually the excess is $< -0.6\times 10^{-5}$).

Now to base e,

$$a \log a = b \log b - (b-a)\left[1 + \log \{b - (b-a)\vartheta\}\right]$$

where $0 < \vartheta < 1$,

$$\log a = \log b - \frac{b-a}{a} - \frac{b-a}{a} \log \left\{1 - \left(\frac{b-a}{b}\right)\vartheta\right\}.$$

The mean of the inequalities is

$$\log b - \frac{b-a}{2}\left(\frac{1}{b} + \frac{1}{a}\right),$$

or the error is

$$\frac{b-a}{a}\left[1 + \log\left\{1 - \left(\frac{b-a}{b}\right)\vartheta\right\} - \frac{b+a}{2b}\right]$$

$$\doteq \frac{b-a}{a}\left[\frac{b-a}{2b} - \left\{\left(\frac{b-a}{b}\right)\vartheta + \frac{1}{2}\left(\frac{b-a}{b}\right)^2\vartheta^2 + \cdots\right\}\right]$$

$$= \frac{1}{99\times 10^2}\left[\frac{1}{2} - \left\{\vartheta + \frac{1}{2\times 10^2}\vartheta^2 \cdots\right\}\right]$$

when $a=99$, $b=100$. But when $f(x) = x \log x$, $f''(x) = 1/x$, $f'''(x) = -1/x^2$,

$$\vartheta \doteq \frac{1}{2} + \frac{1}{24\times 10^2}$$

and the difference, on conversion, is approximately

$$-\frac{1}{99\times 10^2}\left[\frac{1}{24\times 10^2} + \frac{1}{8\times 10^2}\right]0.43 \doteq -\frac{.43}{6\times 10^6} \doteq -7\times 10^{-8}.$$

Editorial Note. The computation of the error by series in the solution abo' assumes that the same value for $\log_{10} e$, correct as far as is necessary, is used the upper and lower bounds for $\log_{10} 99$. This is not the case in the mean 1.99563506. The error in the mean in the first case is $-10^{-8}\times 7.3$, which agrees very closely with the approximate computation of the solution; in the second case the error is $-10^{-8}\times 13$.

NEWS AND NOTICES

Readers are invited to contribute to the general interest of this department by sending items to Professor J. H. Weaver, Ohio State University, Columbus, Ohio.

The next general meeting of the International Astronomical Union will be held in Paris, July 10–17, 1935.

An international Congress for Problems of Vector and Tensor Analysis was held at the University of Moscow, May 17–23, 1934. It was organized by Professor B. Kagan of the University of Moscow, and Professor J. A. Schouten of the Technische Hochschule of Delft.

The Lamme Medal, given each year to the teacher of Engineering who is judged to have done outstanding pioneer work, was awarded at the Ithaca meeting of the Society for the Promotion of Engineering Education to Edward Rose Maurer, Professor of Mechanics at the University of Wisconsin.

At the Mathematics Conference of the Society for the Promotion of Engineering Education held at Ithaca June 19–20, 1934, an unofficial committee made the following recommendation to the council of the Society:

"Whereas the widespread and increasing interest in advanced courses in mathematics for engineering students in America has hitherto expressed itself in a diversity of courses and topics taken up for special study—and

"Whereas no survey of practice in this field has yet been undertaken;

"Therefore, we recommend with a view to future improvements—and to serve as a basis for future intelligent discussion—that *a fact-finding survey* be undertaken by a national committee appointed for this purpose, to determine what is being done and is now being recommended in collegiate and industrial organizations giving such advanced mathematical work."

The following committee has been appointed to carry on the work suggested in the above recommendation: J. H. Weaver, Chairman, W. J. Berry, W. E. Brooke, F. W. Bubb, R. S. Burington, B. F. Dostal, A. M. Dudley, H. A. Fisher, T. C. Fry, H. K. Fulmer, E. J. Oglesby, H. W. March, L. O'Shaughnessy, H. L. Rietz, W. P. Webber.

The university award granted by Rutgers University for distinguished services was presented on September 18, 1934, to A. E. Meder, associate professor of mathematics at the New Jersey College for Women.

Yeshiva College conferred the degree of Doctor of Humane Letters on Professor Albert Einstein of the Institute for Advanced Study at Princeton, at the opening exercises of the college on October 8. Professor Einstein and Professor David Eugene Smith gave addresses on that occasion.

Professor Georg Gamov, head of the department of physics and mathematics at the Polytechnic Institute of Leningrad, has been appointed visiting professor at the George Washington University for the year 1934–35. He will give a seminar in theoretical physics and continue his research on the atomic nucleus.

Dr. R. A. Millikan of the California Institute of Technology attended a joint conference of the International Union of Pure and Applied Sciences and the British Physical Society which met in London, England. Dr. Millikan and Lord Rayleigh were joint presidents of the conference.

Dr. H. A. Compton, professor of physics at the University of Chicago, is visiting professor at the University of Oxford where he will spend the first six weeks of the present academic year.

Professor Merle Bishop of Brooklyn College is on leave of absence for one year to study at the University of Chicago. Aubrey Landers returns to the department after one year's absence spent at the University of Chicago, and Vincent Dillon has been appointed tutor.

Dr. Marjorie Leffler of Ohio State University has been appointed tutor in Mills College, Oakland, California.

Dr. Rufus Oldenburger was appointed lecturer for the Mathematics Section of the Hall of Science at the Chicago Century of Progress.

Dr. R. F. Rinehart of Ohio State University has been appointed professor of mathematics in Ashland College, Ashland, Ohio.

Dr. W. J. Robinson of Ohio State University has been appointed assistant professor of mathematics in Washington College, Chestertown, Maryland.

Dr. C. F. Roos, chief of the research and planning division of the N.R.A., has been appointed professor of economics at Colorado College.

Dr. J. W. T. Suckau of Ohio State University, has been appointed teacher of mathematics at Stony Brook School, Stony Brook, L. I.

Dr. C. C. Weidemann has been appointed associate professor of mathematics in University Laboratory High School at the Ohio State University.

The following appointments to instructorships in mathematics are announced:

Albany College (Portland, Oregon): C. A. Keeler.
Brooklyn College: Dr. Samuel Borofsky.
University of Buffalo: Dr. E. R. Ott.
Gooding College (Wesleyan, Idaho): Dr. Mary E. Haller.
John Burroughs School (Clayton, Missouri): Dr. M. F. Rosskopf.
State Teachers College (Montclair, New Jersey): Dr. E. H. C. Hildebrandt.

Arthur Latham Baker, formerly head of the department of mathematics at Manual Training High School, Brooklyn, N. Y., died August 13, aged eighty-one years.

Associate Professor V. B. Teach of the Armour Institute of Technology died September 8, 1934.

CORRIGENDA

Volume XLI, 1934.

P. 25, line 8 from bottom, for "(12)," read "(17)."
P. 406, last line of abstract 2, for "*l*," read "1."
P. 484, line 7, for "Bryne," read "Byrne."
P. 485, line 20, for "Baussinesq," read "Boussinesq."
P. 530, line 9, for "Raudebush," read "Raudenbush."
P. 547, line 22, for "180," read "160."

At the close of the treasurer's report in the issue for March 1934, page 137, the item "Initiation fees due to sections . . . $900.00" was omitted from the "Bills Payable." The last paragraph of the report should therefore read as follows:

The estimated surplus, $7,435, of Jan. 1, 1933 was depleted by transfer of $4,885 to the General Endowment Fund, leaving the estimated surplus as $2,550. If to the balance on 1933 business shown in the report, $6,868.16, there be added the bills receivable, $250.00, and there be subtracted the estimated bills payable, $5,138.32, there results an estimated final balance on 1933 business of approximately $2,000, which represents the accumulated surplus in current funds. This decrease in surplus of about $550 is more than offset by the item of $760 covering the supply of paper for the MONTHLY for the year 1934, the Association having had a chance to purchase the supply at a favorable figure.

W. D. CAIRNS, *Secretary-Treasurer*

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DIRECTORY

EDITORIAL CORRESPONDENCE should be addressed to the EDITOR-IN-CHIEF, W. B. CARVER, White Hall, Cornell University, Ithaca, N.Y.

BOOKS FOR REVIEW should be addressed to R. A. JOHNSON, Brooklyn College, 66 Court Street, Brooklyn, N.Y.

BUSINESS CORRESPONDENCE should be addressed to the SECRETARY-TREASURER of the Association, W. D. CAIRNS, 33 Peters Hall, Oberlin, Ohio.

CHANGE OF ADDRESS: Members should send notice of any change of address to the SECRETARY-TREASURER, W. D. CAIRNS, Oberlin, Ohio, before the 10th of each month.

MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Nineteenth Annual Meeting of the Association, Pittsburgh, Pa., Dec. 28, 1934—Jan. 1, 1935

The following is a list of the sections of the Association, with dates of those section meetings which have been scheduled for 1935 and reported to the Secretary.

ALLEGHENY MOUNTAIN.
ILLINOIS.
INDIANA, Hanover, May.
IOWA.
KANSAS.
KENTUCKY.
LOUISIANA-MISSISSIPPI, Pineville, La., March.
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, May 11.
MICHIGAN, Ann Arbor, March.
MINNESOTA.
MISSOURI.
NEBRASKA.
OHIO, Columbus, Apr. 4.
OKLAHOMA, Tulsa, Feb.
PHILADELPHIA, Easton, Pa., Mar. 30.
ROCKY MOUNTAIN.
SOUTHEASTERN, Decatur, Ga., March.
SOUTHERN CALIFORNIA, Los Angeles, Mar. 2.
TEXAS.
WISCONSIN.

AFFILIATED ORGANIZATIONS: THE NEW ENGLAND ASSOCIATION OF TEACHERS OF MATHEMATICS, THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS.

The Chauvenet Prize

In the year 1925, the MATHEMATICAL ASSOCIATION OF AMERICAN established a prize of one hundred dollars for the best expository paper published in English during successive periods of five years by a member of the Association.

The purpose of the prize is to stimulate expository contributions in mathematical journals. The award does not apply to books, although the CARUS MONOGRAPHS are expository in character and on this score might be included. They carry their own reward in the form of a cash honorarium to each author.

It is believed that clear expositions of mathematical subjects are greatly needed in this country and that the CHAUVENET PRIZE will tend to stimulate such production.

The prize will be awarded hereafter every three years. The last award was in December, 1932.

Note that the prize is to be awarded only to a *member* of the ASSOCIATION—one more of the many good reasons for membership.

The Carus Mathematical Monographs

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THE CARUS MONOGRAPH COMMITTEE is pleased to announce that the first edition of Number Four is well advanced in sales and that each of the others has gone into a second edition; also that a German Edition of Number One is being brought out by the firm of Teubner in Leipzig and Berlin. The titles of the monographs are: (1) "Calculus of Variations" by Professor GILBERT A. BLISS; (2) "Analytic Functions of a Complex Variable" by Professor DAVID R. CURTISS; (3) "Mathematics of Statistics" by Professor HENRY L. RIETZ; (4) "Projective Geometry," by Professor JOHN W. YOUNG; (5) " - -

The price of these Monographs is $1.25 per copy to institutional and individual members of the Association when ordered directly through the Secretary, one copy to each member; this is the bare cost of production. The price to all non-members of the Association and for all quantity orders for class use is $2.00 per copy, obtained only through the Open Court Publishing Company, 149 E. Huron Street, Chicago, Illinois, distributors to the general public of Association publications.

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